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Conditional Full Support and No Arbitrage

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Abstract

In this paper we establish conditions that imply the conditional full support (CFS) property, introduced by Guasoni et al. (2008) [3], for two processes the Ornstein Uhlenbeck, Stochastic integral in which the Brownian Bridge is the integrator and build the absence of arbitrage opportunities without calculating the risk-neutral probability.

Keywords: *Conditional full support, Ornstein Uhlenbeck process, The absence of arbitrage opportunities, Brownian Bridge.*

1 Introduction

Condition full support (CFS) is a simple condition on asset prices which specified that from any time, the asset price path can continue arbitrarily close to any given path with positive conditional probability. The CSF's notion was introduced by Guasoni et al. (2008) [3] where it was proved that the fractional Brownian motion with arbitrary Hurst parameter has a desired property. This later was generalized by Cherny (2008) [1] who proved that any Brownian moving average satisfies the conditional full support condition. Then, the (CSF) property was established for Gaussian processes with stationary increments by Gasbarra (2011) [2].
Let's note that, by the main result of Guasoni et al. (2008) [3] the CFS generated the consistent price systems which admits a martingale measure.

Pakkanan (2009) [5] presented conditions that imply the conditional full support for the process $Z := R + \phi * W$, where W is a Brownian motion, R is a continuous process.
In this paper, we enjoy this property by thinking of the problems of no arbitrage for asset prices on the one hand process Ornstein Uhlenbeck and Stochastic integral in which the Brownian Bridge is the integrator on the other hand.
The layout of the paper is as follows. Section 2 we present some basic concepts from stochastic portfolio theory and the result on consistent price system. In section 3 we present conditions that imply the conditional full support (CFS) property, for processes Z := H + K * W. In section 4 we establish our main result on the conditional full support for the processes the Ornstein Uhlenbeck, stochastic integral sauch that the Brownian Bridge is the integrator and build the absence of arbitrage opportunities without calculating the risk-neutral probability by the existence of the consistent price systems.

2 Reminder

2.1 Markets and Portfolios

We shall place ourselves in a model M for a financial market of the form

$$dB(t) = B(t)r(t)dt \qquad B(0) = 1$$

$$dS_i(t) = S_i(t)\Big(b_i(t)dt + \sum_{v=1}^{d} \sigma_{iv}(t)dW_v(t)\Big) \qquad S_i(0) = s_i > 0;\ i = 1, \ldots, n \quad (1)$$

consisting of a money-market $B(.)$ and of n stocks, whose prices $S_1(.); \ldots; S_n(.)$ are driven by the $d-dimensional$ Brownian motion $W(.) = W_1(.); \ldots; W_d(.))'$ with $d \geq n$.
We shall assume that the interest-rate process $r(.)$ for the money-market, the vector-valued process $b(.) = (b_1(.); \ldots; b_n(.))'$ of rates of return for the various stocks, and the (n*d)-matrix-valued process $\sigma(.) = (\sigma_{iv}(.))_{1<i<n, 1\leq v\leq d}$ of stock-price volatilities.

Definition 2.1 *A portfolio $\pi(.) = (\pi_1(.), \ldots, \pi_n(.))'$ is an F-progressively measurable process, bounded uniformly in (t, w), with values in the set*

$$\bigcup_{k\in N} (\pi_1, \ldots, \pi_n) \in R^n | \pi_1^2 + \ldots + \pi_n^2 \leq k^2, \pi_1 + \ldots + \pi_n = 1$$

2.1.1 The Market Portfolio

The stock price $S_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$S(t) = S_1(t) + \ldots + S_n(t) \qquad and \qquad \mu_i(t) = \frac{S_i(t)}{S(t)}, \quad i = 1, \ldots, n \qquad (2)$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Clearly $0 < \mu_i(t) < 1$, $\forall i = 1, \ldots, n$ and $\sum_{i=1}^n \mu_i(t) = 1$.
The resulting wealth process $V^{w,\mu}(.)$ satisfies

$$\frac{dV^{w,\mu}(t)}{V^{w,\mu}(t)} = \sum_{i=1}^n \mu_i(t) \frac{dS_i(t)}{S_i(t)} = \sum_{i=1}^n \frac{dS_i(t)}{S(t)} = \frac{dS(t)}{S(t)}$$

Definition 2.2 *Let $\mathcal{O} \subset R^n$ be open set and $(S(t))_{t\in[0,T]}$ be a continuous adapted process taking values in $\mathcal{O}$.
We say that S has conditional full support in $\mathcal{O}$ if for all $t \in [0,T]$ and open set $G \subset C([0,T], \mathcal{O})$*

$$P(S \in G|F_t) > 0, \qquad \text{a.s. on the event } S|_{[0,t]} \in \{g|_{[0,t]} : g \in G\} \tag{3}$$

We will also say that S has full support in $\mathcal{O}$, or simply full support when $\mathcal{O} = R^n$, if 3 holds for $t = 0$ and for all open subset of $C([0,T], \mathcal{O})$.

Recall also, the notion of consistent price system.

Definition 2.3 *Let $\varepsilon > 0$. An $\varepsilon -$ consistent price system to S is a pair $(\widetilde{S}, \mathbf{Q})$, where $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ and $\widetilde{S}$ is a $\mathbf{Q}-$ martingale in the filtration F, such that*

$$\frac{1}{1+\varepsilon} \le \frac{\widetilde{S}_i(t)}{S_i(t)} \le 1 + \varepsilon, \quad \text{almost surely for all } t \in [0,T] \text{ and } i = 1, \ldots, n.$$

Note, that $\widetilde{S}$ is a martingale under $\mathbf{Q}$, hence we may asuume that it is càdlàg, but it is not required in the definition that $\widetilde{S}$ is continuous.

Theorem 2.4 *[4] Let $\mathcal{O} \subset (0,\infty)^n$ be the open set defined by*

$$\mathcal{O} = \mathcal{O}(\delta) = \left\{ x \in (0,\infty)^n : \max_j \frac{x_j}{x_1 + \ldots + x_n} < 1 - \delta \right\} \tag{4}$$

and assume that the price process takes values and has conditional full support in $\mathcal{O}$. Then for any $\varepsilon > 0$ there is an $\varepsilon -$ consistent price system $(\widetilde{S}, \mathbf{Q})$ such that $\widetilde{S}$ takes values in $\mathcal{O}$

To check the condition of Theorem 2.4 we apply the next Theorem. To compare it with existing results we mention that it seems to be new in the sense, that we do not assume that our process solves a stochastic differential equation as it is done in Stroock and Varadhan [6] and it is not only for one dimensional processes as it is in Pakkanen [5].

Theorem 2.5 *[4] Let X be a n-dimensional Itô proces on $[0,T]$, such that*

$$dX_i(t) = \mu_i(t)dt + \sum_{v=1}^{n} \sigma_{iv}(t)dW_v(t)$$

Assume that $|\mu|$ is bounded and σ satisfies

$$\varepsilon|\xi|^2 \leq |\sigma^{'}(t)\xi|^2 \leq M|\xi|^2, \quad a.s. \ for \ all \ t \in [0,T] \ and \ \xi \in R^n \ and \ \varepsilon, M > 0.$$

Then X has conditional full support.

2.2 Consistent Price System and Conditional Full Support

Theorem 2.6 *[4] Let $\mathcal{O} \subset R^n$ be an open set and $(S(t))_{t\in[0,T]}$ be an $\mathcal{O}$-valued, continuous adapted process having conditional full support in $\mathcal{O}$. Besides, let $(\varepsilon_t)_{t\in[0,T]}$ be a continuous positive process, that satisfies*

$$|\varepsilon_t - \varepsilon_s| \leq L_s \ sup_{s\leq u\leq t}|S(u) - S(s)|, \quad for \ all \ 0 \leq s \leq t \leq T$$

with some progressively measurable finite valued $(L_s)_{s\in[0,T]}$.

Then S admits an $\varepsilon -$ *consistent* price system in the sense that, there is an equivalent probability Q on F_t, a process $(\widetilde{S}(t))_{t\in[0,T]}$ taking values in $\mathcal{O}$, such that $\widetilde{S}$ is Q martingale, bounded in $L^2(\mathrm{Q})$ and finally $|S(t) - \widetilde{S}(t)| \leq \varepsilon_t$ almost surely for all $t \in [0,T]$.

Lemma 2.7 *[4] Under the assumption of theorem 2.6 there is a sequence of stopping times $(\tau_n)_{n\geq 1}$, a sequence of random variables $(X_n)_{n\geq 0}$ and an equivalenty Q such that*

1. *$\tau_0 = 0$, (τ_n) is increasing and $\bigcup_n\{\tau_n = T\}$ has full probability,*
2. *$(X_n)_{n\geq 0}$ is a Q martingale in the discrete time filtration $(g_n = F_{\tau_n})_{n\geq 0}$, bounded in $L^2(\mathrm{Q})$,*
3. *if $\tau_n \leq t \leq \tau_{n+1}$ then $|S_t - X_{n+1}| \leq \varepsilon_t$.*

Corollary 2.8 *[4] Assume that the continuous adapted process S evoling in $\mathcal{O}$ has conditional full support in $\mathcal{O}$. Let τ be a stopping time and denote by $\mathrm{Q}_{S|F_\tau}$ the regular version of the conditional distribution of S given F_τ. Then the support of the radom measure $\mathrm{Q}_{S|F_\tau}$ is*

$$supp \, \mathrm{Q}_{S|F_\tau} = \Big\{ g \in C([0,T], \mathcal{O}) : g|_{[0,\tau]} = S|_{[0,\tau]} \Big\}, \qquad almost \ surely.$$

3 Conditional Full Support for Stochastic Integrals

We shall establish the CFS for processes of the form

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0,T]$$

where H is a continuous process, the integrator W is a Brownian motion, and the integrand k satisfies some varying assumptions (to be clarified below). We focus on three cases, each of which requires a separate treatment (see [5]). First, we study the case:

1. **Independent Integrands and Brownian Integrators:**

Theorem 3.1 *[5] Let us define*

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0,T]$$

Suppose that

- $(H_t)_{t\in[0,T]}$ *is a continuous process*
- $(k_t)_{t\in[0,T]}$ *is a measurable process s.t.* $\int_0^T K_s^2 ds < \infty$
- $(W_t)_{t\in[0,T]}$ *is a standard Brownian motion independent of H and k.*

If we have

$$meas(t \in [0,T] : k_t = 0) = 0 \qquad \mathbf{P} - a.s$$

then Z has CFS.

As an application of this result, we show that several popular stochastic volatility models have the CFS property.

Application to Stochastic Volatility Model:

Let us consider price process $(P_t)_{t\in[0,T]}$ in R_+ given by :

$$dP_t = P_t(f(t,V_t)dt + \rho g(t,V_t)dB_t + \sqrt{1-\rho^2}g(t,V_t)dW_t,$$

$P_0 = p_0 \in R_+$ where

(a) $f, g \in C([0,T] \times R^d, R)$,

(b) (B,W) is a planar Brownian motion,

(c) $\rho \in (-1, 1)$,

(d) V is a (measurable) process in R^d s.t. $g(t, V_t) \neq 0$ a.s. for all $t \in [0, T]$,

(e) (B,V) is independent of W,

write using Itô's formula:

$$logP_t = \underbrace{logP_0 + \int_0^t (f(s, V_s) - \frac{1}{2} g(s, V_s)^2) ds + \rho \int_0^t g(s, V_s) dB_s}_{=H_t}$$
$$\underbrace{+\sqrt{1-\rho^2} \int_0^t g(s, V_s) dW_s}_{=K_s}$$

Since W is independent of B and V, the previous Theorem implies that $logP$ has CFS, and from the next remark which it follows that P has CFS.

Remarque 3.2 *If $I \in R$ is an open interval and $f : R \longrightarrow I$ is a homeomorphism, then $g \longmapsto f \circ g$ is a homeomorphism between $C_x([0, T])$ and $C_{f(x)}([0, T], I)$.*
Hence, for $f(X)$, understood as a process in I, we have

$$f(X) \quad has \quad F - CFS \Longleftrightarrow X \quad has \quad F - CFS$$

Next, we relax the assumption about independence, and consider the second case:

2. **Progressive Integrands and Brownian Integrators:**

Remarque 3.3 *The assumption about independence between W and (H,k) cannot be dispensed with in general without imposing additional conditions.*
Namely, if e.g.

$$H_t = 1; k_t := e^{W_t - \frac{1}{2}t}; t \in [0, T]$$

then $Z = k = \xi(W)$, the Dolans exponential of W,
which is stricly positive and thus does not have CFS, if process is consider in R.

Theorem 3.4 *[5]*

Suppose that

- $(X_t)_{t\in[0,T]}$ *and* $(W_t)_{t\in[0,T]}$ *are continuous process*
- h *and* k *are progressive* $[0,T] * C([0,T])^2 \longrightarrow R$,
- ε *is a random variable.*
- *and* $\mathcal{F}_t = \sigma\{\varepsilon, X_s, W_s : s \in [0,t]\}, t \in [0,T]$

If W *is an* $\mathcal{F}_{t\in[0,T]} -$ *Brownian motion and*

- $E[e^{\lambda \int_0^T k_s^{-2} ds}] < \infty$ *for all* $\lambda > 0$
- $E[e^{2 \int_0^T k_s^{-2} h_s^2 ds}] < \infty$ *and*
- $\int_0^T k_s^2 ds \le \overline{K}$ *a.s for some constant* $\overline{K} \in (0,\infty)$

then the process

$$Z_t = \varepsilon + \int_0^t h_s ds + \int_0^t k_s dw_s, \qquad t \in [0,T]$$

has CFS.

3. **Independent Integrands and General Integrators:**

Since Brownian motion has CFS, one might wonder if the preceeding results generalize to the case where the integrator is merely a continuous process with CFS. While the proofs of these results use quite heavily methods speciFIc to Brownian motion (martingales, time changes), in the case independent integrands of finite variation we are able to prove this conjecture.

Theorem 3.5 *[5] Suppose that*

- $(H_t)_{t\in[0,T]}$ *is a continuous process*
- $(k_t)_{t\in[0,T]}$ *is a process of finite variation, and*
- $X = (X_t)_{t\in[0,T]}$ *is a continuous process independent of* H *and* k.

Let us define

$$Z_t := H_t + \int_0^t k_s dX_s, \qquad t \in [0,T]$$

If X *has CFS and*

$$\inf_{t\in[0,T]} |k_t| > 0 \qquad \mathbf{P} - a.s$$

then Z *has CFS.*

4 Main Result

In this part, we will use the following theorem to demonstrate the absence of arbitration without calculating the risk-neutral probability for the two models below.

Theorem 4.1 *[3, Theorem 1.2] Let X_t be an R_+^d-valued, continuous adapted process satisfying (CFS); then X admits an ε-consistent pricing system for all $\varepsilon > 0$*

4.1 Ornstein-Uhlenbeck Process Driven by Brownian Motion

The (one-dimensional) Gaussian Ornstein-Uhlenbeck process $X = (X_t)_t \geq 0$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \qquad t > 0$$

Where we see

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s. \qquad t \geq 0$$

It is readily seen that X_t is normally distributed. We have

$$X_t = \underbrace{X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})}_{H_t} + \int_0^t \underbrace{\sigma e^{\theta(s-t)}}_{K_s}\, dW_s. \qquad t \geq 0 \tag{6}$$

to establish the property of CFS for this process, the conditions of theorem 3.1 will be applied.

The processes (H_s) and (K_s) in (6) satisfie

1. Process (H_s) is a continuous process,
2. (K_s) is a measurable process such that $\int_0^T K_s^2 ds < \infty$, and
3. (W_t) is a standard Brownian motion independent of H and K.

Consequently, the process (X_t) has the property of CFS and there is the consistent price systems which can be seen as generalization of equivalent martingale measures.
This observation we basically say that this price process doesn't admit arbitrage opportunities under arbitrary small transaction, with it we guarantee no-arbitrage without calculating the risk-neutral probability.

4.2 Independent Integrands and Brownian Bridge Integrators

Before giving the application of CFC, we recall some facts on Brownian bridge.

The Brownian bridge $(b_t; 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t; t \leq 1 | B_1 = 0)$.

Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1; t \leq 1)$ and the random variable B_1 are independent. Hence

$$(b_t; 0 \leq t \leq 1) =^{law} (B_t - tB_1; 0 \leq t \leq 1).$$

The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t); s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.

Let
$\mathrm{dS}_t = S_t(\mu dt + \sigma db_t)$,
where μ and σ are constants, be the price of a risky asset. Assume that the riskless asset has an constant interest rate r.

The standard Brownian bridge $b(t)$ is a solution of the following stochastic equation.

$$\begin{aligned} db_t &= -\frac{b_t}{1-t}dt + dW_t; \qquad 0 \leq t < 1 \\ b_0 &= 0. \end{aligned} \tag{7}$$

The solution of above equation is

$$b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s,$$

We may now verify that S has CFS.
By positivity of S, It$\hat{o}$'s formula yields

$$logS_t = logS_0 + \left\{ \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\left(1-t\right)\int_0^t \frac{1}{1-s} dW_s \right\}, \qquad 0 \leq t < 1.$$

We have

$$logS_t = \underbrace{logS_0 + \left(\mu - \frac{\sigma^2}{2}\right)t}_{=:H_t} + \int_0^t \underbrace{\sigma\left(1-t\right)\frac{1}{1-s}}_{=:K_s} dW_s, \qquad 0 \leq t < 1.$$

1. (H_t) is a continuous process,

2. $(K_s) = \sigma(1-t)\frac{1}{1-s}$ is a measurable process s.t. $\int_0^t K_s^2 ds < \infty$,

3. (W_t) is a standard Brownian motion independent of H and K,

which clearly satisfie the assumptions of theorem (3.1) and $logS_t$ has CFS, then S has CFS for $0 \leq t < 1$ and there is the consistent price systems which is a martingale. With it we guarantee no-arbitrage without calculating the risk-neutral probability.

5 Conclusion

In this paper we have investigated the conditional Full support for two processes the Ornstein Uhlenbeck, Stochastic integral in which the Brownian Bridge is the integrator, and we have also built the absence of arbitrage opportunities without calculating the risk-neutral probability by the existence of the consistent price systems which admits a martingale measure.

6 Prospects

In mathematical finance, the CoxIngersollRoss model (or CIR model) describes the evolution of interest rates. It is a type of "one factor model" (short rate model) as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model.

The CIR model specifies that the instantaneous interest rate follows the stochastic differential equation, also named the CIR Process:

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t \qquad t > 0$$

where W_t is a Wiener process and θ, μ and σ, are the parameters. The parameter θ corresponds to the speed of adjustment, μ to the mean and σ, to volatility. The drift factor, $\theta(\mu - X_t)$, is exactly the same as in the Vasicek model. It ensures mean reversion of the interest rate towards the long run value μ, with speed of adjustment governed by the strictly positive parameter θ.

As prospects, we establish the condition of CFS for the Cox-Ingersoll-Ross model.

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References

[1] A. Cherny, Brownian moving average have conditional full support, *The Annals of Applied Probability*, 18(2008), 1825-1830.

[2] H. Van Zanten, D. Gasbarra and T. Sottinen, Conditional full support of gaussian processes with stationary increments, *J. Appl. Prob*, 48(2011), 561-568.

[3] P. Guasoni, M. Rasonyi and W. Schachermayer, Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.*, 18(2) (2008), 491-520, MR 2398764 (2009a:91050).

[4] A. Herczegh, V. Prokaj and M. Rasonyi, Diversity and no arbitrage, *Stoch. Anal. Appl.*, 32(5) (2014), 876-888, MR 3254711.

[5] M.S. Pakkanen, Stochastic integrals and conditional full support, *J. Appl. Probab.*, 47(3) (2010), 650-667, MR 2731340 (2011g:60095).

[6] D.W. Stroock and S.R.S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Univ. California, Berkeley, California, 1970/1971, *Probability Theory (Vol. III)*, Univ. California Press, Berkeley, California, (1972), 333-359, MR 0400425 (53 4259).