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On Lacunary Strongly Convergent Difference Sequence Spaces Defined by a Sequence of φ-Functions

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Abstract

In this paper, we introduce the new sequence spaces with lacunary strong convergence using by a sequence of modulus functions and a sequence of φ-functions. We also study some connections between lacunary $(A, \varphi_k, \Delta_u^m)$ - statistically convergence and lacunary strong $(A, \varphi_k, \Delta_u^m)$ - convergence .

Keywords: *Difference sequence, modulus function, φ- function, lacunary sequence, statistical convergence.*

1 Introduction

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be, respectively, the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. In [1], the space of lacunary strongly convergent sequences N_θ was defined as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

A modulus function f is a function from acting $[0,\infty)$ to $[0,\infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f increasing,
(iv) f is continuous from at the right zero.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in N$, from condition (ii) and so

$$f(x) = f(nx\frac{1}{n}) \leq nf(\frac{x}{n}).$$

Hence, for all $n \in N$

$$\frac{1}{n}f(x) \leq f(\frac{x}{n}).$$

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, for $0 < p \leq 1$ is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded. Ruckle [9] and Maddox [10], used a modulus f to construct some sequence spaces.

Furthermore, modulus function has been discussed in [5], [11], [12], [13] and [14] and many others.

The difference sequence space $X(\Delta)$ was first introduced by Kızmaz [2] as follows:

$$X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\}$$

for $X = l_\infty$, c and c; where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

The notion of difference sequence spaces was further generalized by Et and Colak [3] as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in X\}$$

for $X = l_\infty$, c and c; where $\Delta^m x_k = \Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in N$. Taking $X = l_\infty(p)$, $c(p)$ and $c_0(p)$, these sequence spaces has been generalized by Et and Başarır [4].

The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

for all $k \in N$.

Subsequently, difference sequence spaces have been discussed by several authors [18], [14], [6] and [7].

By a φ-function we understood a continuous non-decreasing function $\varphi(v)$ defined for $v \geq 0$ and such that $\varphi(0) = 0, \varphi(v) > 0$ for $v > 0$ and $\varphi(v) \to \infty$ as $v \to \infty$.

In [15], [16], [17] and [19]; some sequence spaces was studied using by φ-function.

Let $\varphi = (\varphi_k)$ and $\psi = (\psi_k)$ be sequences of φ-functions. A sequence of φ-functions φ is called non weaker than a sequence of φ-function ψ and we write $\psi \prec \varphi$ (or $\psi_k \prec \varphi_k$ for all k) if there are constants $c, b, n, l > 0$ such that $c\psi_k(lv) \prec b\varphi_k(nv)$ (for all, large or small v, respectively).

Two sequences of φ-functions φ and ψ are called equivalent and we write $\varphi \sim \psi$ (or $\psi_k \prec \varphi_k$ for all k) if there are positive constants b_1, b_2, c, k_1, k_2, l such that $b_1\varphi_k(k_1 v) \leq c\psi_k(lv) \leq b_2\varphi_k(k_2 v)$ (for all, large or small v, respectively).

A sequence of φ-functions φ is said to satisfy the Δ_2-condition (for all, large or small v, respectively) if for some constant $l > 1$ there is satisfied the inequality $\varphi_k(2v) \leq l\varphi_k(v)$ for all k. For a φ-function satisfying the Δ_2-condition, there is $L > 0$ such that

$$\varphi_k(cv) \leq L\varphi_k(v) \tag{1}$$

for v large enough. Indeed, for every $c > 0$ there is an integer s such that $c \leq 2^s$ and

$$\varphi_k(cv) \leq \varphi_k(2^s v) \leq l^s \varphi_k(v) \tag{2}$$

for v large enough.

Let $A = (a_{nk})$ be an infinite matrix such that;

a) A is non-negative, i.e. $a_{nk} \geq 0$ for $n, k = 1, 2, ...$,

b) for an arbitrary positive integer n (or k) there exists a positive integer k_0 (or n_0) such that $a_{nk} \neq 0$ (or $a_{n_0 k} \neq 0$), respectively,

c) there exists $\lim_n a_{nk} = 0$ for $k = 1, 2, \ldots,$,

d) $\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$,

e) $\sup_n a_{nk} \to 0$ as $k \to \infty$.

In the present paper, we introduce and study some properties of the following difference sequence space that is defined by using a sequence of φ-functions and a sequence of modulus functions.

2 Main Results

Let $\theta = (k_r)$ be a lacunary sequence, $\varphi = (\varphi_k)$ and $f = (f_n)$ be given a sequence of φ-functions and a sequence of modulus functions, respectively, m

be a positive integer and $u = (u_k)$ be any sequence such that $u_k \neq 0$ for all k. Moreover, let a matrix $A = (a_{nk})$ be given in the above. Then we define,

$$V_\theta^0\left((A, \varphi_k, \Delta_u^m), f_n\right) = \left\{x = (x_k) \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right) = 0\right\}$$

where $\Delta_u^m x_k = u_k \Delta^m x_k = (u_k \Delta^{m-1} x_k - u_{k+1}\Delta^{m-1} x_{k+1})$ such that $\Delta_u^m x_k = \sum_{n=0}^{m} (-1)^n \begin{pmatrix} m \\ n \end{pmatrix} u_{k+n} x_{k+n}$, $\Delta_u^0 x_k = (u_k x_k)$ and $\Delta_u x_k = (u_k x_k - u_{k+1} x_{k+1})$.

Throughout this paper, the sequence of modulus functions $f = (f_n)$ satisfy the condition $\lim_{v \to 0^+} \sup_n f_n(v) = 0$.

If $x \in V_\theta^0\left((A, \varphi_k, \Delta_u^m), f_n\right)$ then the sequence x is said to be lacunary strongly $(A, \varphi_k, \Delta_u^m)$- convergent to zero with respect to a sequence of modulus f.

If we take $\theta = (2^r)$ then we have

$$V^0\left((A, \varphi_k, \Delta_u^m), f_n\right) = \left\{x \in w : \lim_k \frac{1}{k} \sum_{n=1}^{k} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right) = 0\right\}.$$

When $\varphi_k(x) = x$ for all x and k, $u_k = 1$ for all k, we obtain

$$V_\theta^0\left((A, \Delta^m), f_n\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\left(|\Delta^m x_k|\right)\right) = 0\right\}.$$

If $f_n(x) = x$ for all x and n, $u_k = 1$ for all k, we write

$$V_\theta^0\left(A, \varphi_k, \Delta^m\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta^m x_k|\right)\right) = 0\right\}.$$

When $A = I$ and $u_k = 1$ for all k, we get the following sequence space,

$$V_\theta^0\left((I, \varphi_k, \Delta^m), f_n\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\varphi_n\left(|\Delta^m x_n|\right)\right) = 0\right\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k and $u_k = 1$ for all k then we have

$$V_\theta^0\left((I, \Delta^m), f_n\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(|\Delta^m x_n|\right) = 0\right\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k and $u_k = 1$ for all k, $f_n(x) = f(x)$ for all x and n

$$V_\theta^0\left(\left(I, \Delta^m\right), f\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f\left(\left|\Delta^m x_n\right|\right) = 0\right\}.$$

If we take $A = I$, $\varphi_k(x) = x$ for all x and k, $f_n(x) = x$ for all x and n and $u_k = 1$ for all k then we have

$$V_\theta^0\left(I, \Delta^m\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \left(\left|\Delta^m x_n\right|\right) = 0\right\}.$$

If we define the matrix $A = (a_{nk})$ as follows:

$$a_{nk} = \frac{1}{n} \text{ for } n \geq k \text{ and } a_{nk} = 0 \text{ for } n < k$$

then we have the sequence space,

$$V_\theta^0\left(\left(C, \varphi_k, \Delta_u^m\right), f_n\right) = \left\{x \in w : \lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\frac{1}{n} \sum_{k=1}^{n} \varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right) = 0\right\}.$$

Now we have,

Theorem 2.1 *Let us suppose that $\varphi = (\varphi_k)$ and $\psi = (\psi_k)$ be two sequences of φ-functions and $\psi = (\psi_k(v))$ satisfies the Δ_2-condition for large v.*

(i) If $\psi \prec \varphi$ then $V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right) \subset V_\theta^0\left(\left(A, \psi_k, \Delta_u^m\right), f_n\right)$.

(ii) If two sequences of φ-functions $(\varphi_k(v))$ and $(\psi_k(v))$ are equivalent for large v and they satisfy the Δ_2-condition for large v then $V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right) = V_\theta^0\left(\left(A, \psi_k, \Delta_u^m\right), f_n\right)$.

Proof. *(i) Let $x = (x_k) \in V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$. Then*

$$\lim_r \frac{1}{h_r} \sum_{n \in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk} \varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right) = 0.$$

By assumption, $\psi \prec \varphi$, we have

$$\psi_k\left(\left|x_k\right|\right) \leq b \varphi_k\left(c\left|x_k\right|\right) \tag{3}$$

for $b, c > 0$, all k, and $|x_k| > v_0$. Let us denotes $x = x' + x''$, where for all m, $x' = \left(\Delta_u^m x_k'\right)$ and $\Delta_u^m x_k' = \Delta_u^m x_k$ for $\left|\Delta_u^m x_k\right| < v_0$ and $\Delta_u^m x_k' = 0$ remaining *values of k. It is easy to see that* $x^{'} \in V_{\theta}^{0}\left(\left(A,\psi_k,\Delta_u^m\right),f_n\right)$. *Furthermore, by the assumptions and the inequality (3) we get*

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\psi_k\left(\left|\Delta_u^m x_k''\right|\right)\right) &\leq \frac{1}{h_r}\sum_{n\in I_r} f_n\left(b\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(c\left|\Delta_u^m x_k''\right|\right)\right)\\ &\leq \frac{1}{h_r}\sum_{n\in I_r} f_n\left(bL\sum_{k=1}^{\infty} a_{nk}\psi_k\left(\left|\Delta_u^m x_k''\right|\right)\right)\\ &\leq \frac{K}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k''\right|\right)\right)\end{aligned}$$

where the constants K and L *are connected with properties of* f *and* φ *functions. We recall that a* $\varphi-$*function satisfying the* Δ_2-*condition implies (1) and (2).*

Finally, we obtain $x^{''} = \left(x_k^{''}\right) \in V_{\theta}^{0}\left(\left(A,\psi_k,\Delta_u^m\right),f_n\right)$ *and in consequence* $x \in V_{\theta}^{0}\left(\left(A,\psi_k,\Delta_u^m\right),f_n\right)$.

(ii) The identity $V_{\theta}^{0}\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right) = V_{\theta}^{0}\left(\left(A,\psi_k,\Delta_u^m\right),f_n\right)$ *is proved by using the same argument.*

Theorem 2.2 *Let the sequence* $\varphi = \left(\varphi_k\left(v\right)\right)$ *of* φ*-functions satisfies the* Δ_2*-condition for all k and for large v then* $V_{\theta}^{0}\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$ *is linear space.*

Proof. *Firstly we prove that if* $x = (x_k) \in V_{\theta}^{0}\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$ *and* α *is an arbitrary number then* $\alpha x \in V_{\theta}^{0}\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$. *Let us remark that for* $0 < \alpha < 1$ *we get*

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m \alpha x_k\right|\right)\right) \leq \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right).$$

Moreover, if $\alpha > 1$ *then we may find a positive number s such that* $\alpha < 2^s$ *and we obtain*

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m \alpha x_k\right|\right)\right) &\leq \frac{1}{h_r}\sum_{n\in I_r} f_n\left(d^s\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right)\\ &\leq \frac{K}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right)\end{aligned}$$

where d and K are constants connected with the properties of φ *and* f *functions. We recall that a* $\varphi-$*function satisfying the* Δ_2-*condition implies (1) and (2). Hence we obtain* $\alpha x \in V_{\theta}^{0}\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$.

Secondly, let $x, y \in V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$ *and* α, β *arbitrary numbers. We will show that* $\alpha x + \beta y \in V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$.

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m\left(\alpha x_k + \beta y_k\right)\right|\right)\right) \leq \frac{K_1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right) + \frac{K_2}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m y_k\right|\right)\right)$$

where the constants K_1 *and* K_2 *are defined as above. In consequence,* $\alpha x + \beta y \in V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$.

Now, we give the following Proposition that is necessary for proof of the Theorem 2.4.

Proposition 2.3 ([5]) *Let* f *be a modulus and let* $0 < \delta < 1$. *Then for each* $v \geq \delta$ *we have* $f(v) \leq 2f(1)\delta^{-1}v$.

Theorem 2.4 *Let* $\varphi = (\varphi_k)$ *and* $f = (f_n)$ *be given a sequence of* φ-*functions and a sequence of modulus functions, respectively and* $\sup_n f_n(1) < \infty$. *Then* $V_\theta^0\left(A, \varphi_k, \Delta_u^m\right) \subset V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$.

Proof. Let $x \in V_\theta^0\left(A, \varphi_k, \Delta_u^m\right)$ *and put* $\sup_n f_n(1) = M$. *For a given* $\varepsilon > 0$ *we choose* $0 < \delta < 1$ *such that* $f_n(x) < \varepsilon$ *for every* $x \in [0, \delta]$ *and for all* n. *We can write*

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right) = S_1 + S_2$$

where

$$S_1 = \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right) \leq \delta$$

and

$$S_2 = \frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right) > \delta.$$

By the definition of the modulus f we have

$$S_1 \leq \frac{1}{h_r}\sum_{n\in I_r} f_n(\delta) < \frac{1}{h_r}(h_r\varepsilon) = \varepsilon$$

and moreover

$$S_2 \leq 2M\frac{1}{\delta}\frac{1}{h_r}\sum_{n\in I_r}\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)$$

by Proposition 2.3. Finally we have $x \in V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$*. This completes the proof.*

Theorem 2.5 *Let* $\varphi = (\varphi_k)$ *and* $f = (f_n)$ *be given a sequence of* φ*-functions and a sequence of modulus functions, respectively. If* $\lim_{v\to\infty}\inf_n \frac{f_n(v)}{v} > 0$ *then*

$$V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right) = V_\theta^0\left(A,\varphi_k,\Delta_u^m\right).$$

Proof. *If* $\lim_{v\to\infty}\inf_n \frac{f_n(v)}{v} > 0$ *then there exists a number* $c > 0$ *such that* $f_n(v) > cv$ *for* $v > 0$ *and* $n \in N$*. Let* $x \in V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$ *. Clearly*

$$\begin{aligned}\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right) &\geq \frac{1}{h_r}\sum_{n\in I_r} c\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right)\\ &= \frac{c}{h_r}\sum_{n\in I_r}\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\right).\end{aligned}$$

Therefore $x \in V_\theta^0\left(A,\varphi_k,\Delta_u^m\right)$ *. By using Theorem 2.4, the proof is completed.*

Theorem 2.6 *Let* $\theta = (k_r)$ *be a lacunary sequence and* $f = (f_n)$ *be a sequence of modulus functions.*

(i) If $\liminf q_r > 1$ *then* $V^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right) \subset V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$.

(ii) If $\limsup q_r < \infty$ *then* $V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right) \subset V^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$.

(iii) If $1 < \liminf q_r \leq \limsup q_r < \infty$ *then* $V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right) = V^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$.

Proof. *This can be proved by using the same techniques in [11] and hence we omit the proof.*

The next result follows from Theorem 2.5 and Theorem 2.6.

Corollary 2.7 *If* $\lim_{v\to\infty}\inf_n \frac{f_n(v)}{v} > 0$ *and* $1 < \liminf q_r \leq \limsup q_r < \infty$ *then* $V_\theta^0\left(A,\varphi_k,\Delta_u^m\right) = V^0\left(\left(A,\varphi_k,\Delta_u^m\right), f_n\right)$.

3 $S_\theta^0(A,\varphi_k,\Delta_u^m)$ -Statistical Convergence

Let the matrix $A=(a_{nk})$ be given as previously, $\theta=(k_r)$ be a lacunary sequence, the sequence of φ-functions $\varphi=(\varphi_k)$ and a positive number $\varepsilon>0$ be given. We write,

$$K_\theta^r\left(\left(A,\varphi_k,\Delta_u^m\right),\varepsilon\right)=\left\{n\in I_r:\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)\geq\varepsilon\right\}.$$

The sequence x is said to be lacunary (A,φ_k,Δ_u^m)- statistically convergent to a number zero if for every $\varepsilon>0$

$$\lim_r\frac{1}{h_r}\mu\left(K_\theta^r\left(\left(A,\varphi_k,\Delta_u^m\right),\varepsilon\right)\right)=0$$

where $\mu\left(K_\theta^r\left(\left(\left(A,\varphi_k,\Delta_u^m\right),\varepsilon\right)\right)\right)$ denotes the number of element belonging to $K_\theta^r\left(\left(\left(A,\varphi_k,\Delta_u^m\right),\varepsilon\right)\right)$. We denote by $S_\theta^0(A,\varphi_k,\Delta_u^m)$, the set of sequences $x=(x_k)$ which are lacunary (A,φ_k,Δ_u^m)-statistically convergent to a number zero. We write

$$S_\theta^0\left(A,\varphi_k,\Delta_u^m\right)=\left\{x=(x_k):\lim_r\frac{1}{h_r}\mu\left(K_\theta^r\left(\left(A,\varphi_k,\Delta_u^m\right),\varepsilon\right)\right)=0\right\}.$$

When we take $\theta=(2^r)$, $S_\theta^0(A,\varphi_k,\Delta_u^m)$ reduces to $S^0(A,\varphi_k,\Delta_u^m)$.

If we take $A=I$ and $\varphi_k(x)=x$ for all k and x, then $S_\theta^0(A,\varphi_k,\Delta_u^m)$ reduces to $S_\theta^0(\Delta_u^m)$ defined by

$$S_\theta^0\left(\Delta_u^m\right)=\left\{x=(x_k):\lim_r\frac{1}{h_r}\mu\left(\left\{k\in I_r:\left(\left|\Delta_u^m x_k\right|\right)\geq\varepsilon\right\}\right)=0\right\}.$$

Now we have,

Theorem 3.1 *Let $\theta=(k_r)$ be a lacunary sequence, $\varphi=(\varphi_k(v))$ and $\psi=(\psi_k(v))$ are two sequences of φ-functions.*

(i) If $\psi\prec\varphi$ and φ_k satisfies the Δ_2-condition for large v and for all k then $S_\theta^0(A,\psi_k,\Delta_u^m)\subset S_\theta^0(A,\varphi_k,\Delta_u^m)$.

(ii) If $\varphi\sim\psi$ and φ_k and ψ_k satisfy the Δ_2-condition for large v and for all k then $S_\theta^0(A,\psi_k,\Delta_u^m)=S_\theta^0(A,\varphi_k,\Delta_u^m)$.

Proof. *(i) Let $x\in S_\theta^0(A,\psi_k,\Delta_u^m)$. By assumption we have $\psi_k\left(\left|\Delta_u^m x_k\right|\right)\leq b\varphi_k\left(c\left|\Delta_u^m x_k\right|\right)$ and we have for all n and m,*

$$\sum_{k=1}^{\infty}a_{nk}\psi_k\left(\left|\Delta_u^m x_k\right|\right)\leq b\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(c\left|\Delta_u^m x_k\right|\right)\leq K\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(\left|\Delta_u^m x_k\right|\right)$$

for $b, c > 0$, where the constant K is connected with properties of φ functions. Thus the condition $\sum_{k=1}^{\infty} a_{nk}\psi_k\left(|\Delta_u^m x_k|\right) \geq \varepsilon$ *implies the condition* $\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right) \geq \varepsilon$ *and in consequence we get*

$$K_\theta^r\left(\left(A, \varphi_k, \Delta_u^m\right), \varepsilon\right) \subset K_\theta^r\left(\left(A, \psi_k, \Delta_u^m\right), \varepsilon\right)$$

and

$$\lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(A, \varphi_k, \Delta_u^m\right), \varepsilon\right)\right) \leq \lim_r \frac{1}{h_r}\mu\left(K_\theta^r\left(\left(A, \psi_k, \Delta_u^m\right), \varepsilon\right)\right).$$

This completes the proof.

(ii) The identity $S_\theta^0\left(A, \psi_k, \Delta_u^m\right) = S_\theta^0\left(A, \varphi_k, \Delta_u^m\right)$ *is proved by using the same argument.*

Theorem 3.2 *Let $f = (f_n)$ be given a sequence of modulus functions. If* $\inf_n f_n(v) > 0$ *then*

$$V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right) \subset S_\theta^0\left(A, \varphi_k, \Delta_u^m\right).$$

Proof. If $\inf_n f_n(v) > 0$ then there exists a number $\alpha > 0$ such that $f_n(v) \geq \alpha$ for $v > 0$ and $n \in N$. Let $x \in V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$.

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right) \geq \frac{1}{h_r}\sum_{\substack{n\in I_r \\ \sum_{k=1}^{\infty} a_{nk}\varphi_k(|\Delta_u^m x_k|)\geq\varepsilon}} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right)$$
$$\geq \frac{\alpha}{h_r}\left|\left\{n \in I_r : \sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right) \geq \varepsilon\right\}\right|$$

and it follows that $x \in S_\theta^0\left(A, \varphi_k, \Delta_u^m\right)$.

Theorem 3.3 *Let $f = (f_n)$ be given a sequence of modulus functions. If* $\sup_v \sup_n f_n(v) < \infty$ *then* $S_\theta^0\left(A, \varphi_k, \Delta_u^m\right) \subset V_\theta^0\left(\left(A, \varphi_k, \Delta_u^m\right), f_n\right)$.

Proof. We suppose $T(v) = \sup_n f_n(v)$ and $T = \sup_v T(v)$.*Let* $x \in S_\theta^0\left(A, \varphi_k, \Delta_u^m\right)$. Since $f_n(v) \leq T$ for $n \in N$ and $v > 0$, we have

$$\frac{1}{h_r}\sum_{n\in I_r} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right) \geq \frac{1}{h_r}\sum_{\substack{n\in I_r \\ \sum_{k=1}^{\infty} a_{nk}\varphi_k(|\Delta_u^m x_k|)\geq\varepsilon}} f_n\left(\sum_{k=1}^{\infty} a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right)$$

$$+\frac{1}{h_r}\sum_{\substack{n\in I_r\\ \sum_{k=1}^{\infty}a_{nk}\varphi_k(|\Delta_u^m x_k|)<\varepsilon}} f_n\left(\sum_{k=1}^{\infty}a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\right)$$

$$\leq \frac{T}{h_r}\left|\left\{n\in I_r : \sum_{k=1}^{\infty}a_{nk}\varphi_k\left(|\Delta_u^m x_k|\right)\geq\varepsilon\right\}\right| + T(\varepsilon).$$

Taking the limit as $\varepsilon\to 0$, it follows that $x\in V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$.

Corollary 3.4 *Let* $f=(f_n)$ *be given a sequence of modulus functions. If* $\inf_n f_n(v)>0 \quad (v>0)$ *and* $\sup_v\sup_n f_n(v)<\infty$ *then* $S_\theta^0\left(A,\varphi_k,\Delta_u^m\right)=V_\theta^0\left(\left(A,\varphi_k,\Delta_u^m\right),f_n\right)$.

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