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Dependency of the Solution of a Class of Quartic Partial Differential Quasilinear Equation With Periodic Boundary Condition on ε

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Abstract

In this paper, we study the multidimensional mixed problem with periodic boundary condition for quasilinear Euler-Bernoulli equation $\frac{\partial^2 u}{\partial t^2} - \varepsilon b^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t,x,u)$*. We also consider the mixed problem* $\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 v}{\partial x^4} = f(t,x,v)$*. Finally we prove that the solution function* $u(t,x,\varepsilon)$ *of the quasilinear Euler-Bernoulli equation is convergent to the solution function* $v(t,x)$ *of the quasi-linear quartic equation, as* $\varepsilon \to 0$*.*

Keywords: *Mixed problem, Periodic boundary condition, Quasi-linear quartic partial differential equation.*

1 Introduction

In recent years, mathematical modeling of sound wave distribution problems and also the vibration, buckling and dynamic behavior of various building elements widely used in nano-technology are formulated with following Euler-Bernoulli equations

$$\frac{\partial^2 u}{\partial t^2} - \varepsilon b^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t,x,u) \tag{1}$$

Due to the new and exceptionally its electronic and mechanical properties, carbon nanotubes are considered to be one of the most useful material in future. Nowadays, nanotubes are used as atomic force microscopy, nanofillers for composite materials, nanoscale electronic devices and even frictionless nanoactuators, nanomotors, nanobearings and nanosprings [4,9,13,16,17]. These elements are tackled by different boundary conditions depending on different loading conditions. Therefore, investigation of existence and uniqueness of the solution of Euler-Bernoulli equations with different boundary conditions used in the mathematical modeling of the structural components of nano-materials continues to be a focus of interest amongst mathematicians.The reader is refereed to [2,6,7,8] for some relevant previous work on linear and quasi-linear equations. The textbooks [3,11,12,14,15] also contain important results.

In mathematics, the classical statement of Euler-Bernoulli equation

$$\frac{\partial^2 v}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0 \tag{2}$$

which is taken $\varepsilon = 0$ from (1) is used for beam vibration equation. As well as the homogeneous equation, quasilinear and non-linear equations can be handled in this case. Various problems for equations of this type were investigated and many results have been obtained in different ways.

We first discuss the following mixed problem for the Euler-Bernoulli equation with the nonlinear source term $f = f(t,x,u)$:

$$\frac{\partial^2 u}{\partial t^2} - \varepsilon b^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t,x,u), \quad (t,x) \in D\{0 < t < T,\ 0 < x < \pi\} \tag{3}$$

$$u(0,x,\varepsilon) = \varphi(x),\ u_t(0,x,\varepsilon) = \psi(x),\ (0 \le x \le \pi) \tag{4}$$

$$\begin{gathered} u(t,0,\varepsilon) = u(t,\pi,\varepsilon),\ u_x(t,0,\varepsilon) = u_x(t,\pi,\varepsilon), \\ u_{x^2}(t,0,\varepsilon) = u_{x^2}(t,\pi,\varepsilon),\ u_{x^3}(t,0,\varepsilon) = u_{x^3}(t,\pi,\varepsilon),\ (0 \le t \le T) \end{gathered} \tag{5}$$

and also we consider the following mixed problem for a quasilinear quartic equation with the nonlinear source term $f = f(t,x,v)$:

$$\frac{\partial^2 v}{\partial t^2} + a^2 \frac{\partial^4 v}{\partial x^4} = f(t,x,v), \quad (t,x) \in D\{0 < t < T,\ 0 < x < \pi\} \tag{6}$$

$$v(0,x) = \varphi(x),\ v_t(0,x) = \psi(x),\ (0 \le x \le \pi) \tag{7}$$

$$\begin{aligned} &v(t,0)=v(t,\pi),\ v_x(t,0)=v_x(t,\pi),\\ &v_{x^2}(t,0)=v_{x^2}(t,\pi),\ v_{x^3}(t,0)=v_{x^3}(t,\pi),\ (0\le t\le T) \end{aligned} \tag{8}$$

where $\varphi(x),\psi(x)$ and $f(t,x,u)$ are given functions defined on $[0,\pi]$ and $\bar{D}\{0\le t\le T,\ 0\le x\le\pi\}\times(-\infty,\ \infty)$, respectively, and the functions $u(t,x,\varepsilon)$ and $v(t,x)$ are solutions of the problems(3)-(5) and (6)-(8).

In [7], the authors have studied the solution of the problem (3)-(5) for $\varepsilon>0$. The same authors have considered the solution of the problem (6)-(8) for $\varepsilon=0$ in [8]. In the present paper, we discuss the solution of the problem (3)-(5) and (6)-(8) and show that the solution function of $u(t,x,\varepsilon)$ of the quasilinear Euler-Bernoulli equation is convergent to the solution function $v(t,x)$ of the quasilinear quartic equation, as $\varepsilon\to 0$ (i.e., $\lim_{\varepsilon\to 0}u(t,x,\varepsilon)=v(t,x)$, $(t\in[0,t])$).

2 Preliminaries

Definition 2.1 *[1, 10] The function $w(t,x)\in C(\bar{D})$ is called* test function *if it has continuous partial derivatives of order contained in equation (1) and satisfies both following conditions*

$$w(T,x)=w_t(T,x)=w_{x^2}(T,x)=w_{tx^2}(T,x)=0$$

and the boundary condition;

$$w(t,0)=w(t,\pi),\ w_x(t,0)=w_x(t,\pi),\ w_{x^2}(t,0)=w_{x^2}(t,\pi),\ w_{x^3}(t,0)=w_{x^3}(t,\pi)\blacksquare$$

Definition 2.2 *The function $u(t,x,\varepsilon)\in C(\bar{D}\times[0,\varepsilon_0])$ satisfying the integral identity*

$$\int_0^T\int_0^\pi\left\{u\left[\frac{\partial^2 w}{\partial t^2}-\varepsilon b^2\frac{\partial^4 w}{\partial t^2\partial x^2}+a^2\frac{\partial^4 w}{\partial x^4}\right]-f(t,x,u)w\right\}dx\,dt+$$
$$\int_0^\pi\varphi(x)\left[v_t(0,x)-\varepsilon b^2\frac{\partial^4 w}{\partial t^2\partial x^2}\right]dx-\int_0^\pi\psi(x)\left[v(0,x)-\varepsilon b^2\frac{\partial^4 w}{\partial t^2\partial x^2}\right]dx=0 \tag{9}$$

for an arbitrary test function $w(t,x)$ is called weak generalized solution *of problem (3)-(5).*

Definition 2.3 *The function $v(t,x)\in C(\bar{D})$ satisfying the integral identity*

$$\begin{aligned} &\int_0^T\int_0^\pi\left\{v\left[\frac{\partial^2 w}{\partial t^2}+a^2\frac{\partial^4 w}{\partial x^4}\right]-f(t,x,u)w\right\}dx\,dt+\\ &\int_0^\pi\varphi(x)v_t(0,x)\,dx-\int_0^\pi\psi(x)v(0,x)\,dx=0 \end{aligned} \tag{10}$$

for an arbitrary test function $w(t,x)$ is called weak generalized solution *of problem (6)-(8).*

3 Solution of Problem

The set

$$\{\bar{u}(t,\varepsilon)\} = \{\frac{1}{2}u_0(t,\varepsilon), u_{c1}(t,\varepsilon), u_{s1}(t,\varepsilon), \dots, u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon), \dots\}$$

of functions satisfying the condition

$$\frac{1}{2}\max_{t\in[0,T]}|u_0(t,\varepsilon)| + \sum_{k=1}^{\infty}\left[\max_{t\in[0,T]}|u_{ck}(t,\varepsilon)| + \max_{t\in[0,T]}|u_{sk}(t)|\right] < \infty.$$

for all $\varepsilon \in [0, \varepsilon_0]$ is denoted by B_T. Let

$$\|\overline{u}(t,\varepsilon)\|_{B_T} = \frac{1}{2}\max_{t\in[0,T]}|u_0(t,\varepsilon)| + \sum_{k=1}^{\infty}\left[\max_{t\in[0,T]}|u_{ck}(t,\varepsilon)| + \max_{t\in[0,T]}|u_{sk}(t,\varepsilon)|\right]$$

be the norm in B_T. It can be shown that B_T is Banach space.

Looking for the weak solutions of (3)-(5) and (6)-(8) of which existence and uniqueness are proven under certain conditions as following respectively;

$$u(t,x,\varepsilon) = \frac{1}{2}u_0(t,\varepsilon) + \sum_{k=1}^{\infty}\left[u_{ck}(t,\varepsilon)\cos 2kx + u_{sk}(t,\varepsilon)\sin 2kx\right] \tag{11}$$

and

$$v(t,x) = \frac{1}{2}u_0(t) + \sum_{k=1}^{\infty}\left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right], \tag{12}$$

we get the following infinite system of integral equations for the unknown functions $u_0(t,\varepsilon), u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon)$ and $v_0(t), v_{ck}(t), v_{sk}(t)$, $(k = \overline{1,\infty})$

$$v_0(t) = \varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t \int_0^\pi (t-\tau) f\big\{\tau, \xi, \frac{1}{2}u_0(\tau,\varepsilon) + \sum_{n=1}^{\infty} [u_{cn}(\tau,\varepsilon)\cos 2n\xi + u_{sn}(\tau,\varepsilon)\sin 2n\xi]\big\}\ d\xi\, d\tau,$$

$$u_{ck}(t,\varepsilon) = \varphi_{ck}\cos\alpha_k t + \frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t + \frac{2}{\pi\alpha_k}\int_0^t \int_0^\pi f\big\{\tau, \xi, \frac{1}{2}u_0(\tau,\varepsilon) + \sum_{n=1}^{\infty} [u_{cn}(\tau,\varepsilon)\cos 2n\xi + u_{sn}(\tau,\varepsilon)\sin 2n\xi]\big\}\times \cos 2k\xi \sin\alpha_k(t-\tau)\ d\xi\, d\tau,$$

$$u_{sk}(t,\varepsilon) = \varphi_{sk}\cos\alpha_k t + \frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t + \frac{2}{\pi\alpha_k}\int_0^t \int_0^\pi f\big\{\tau, \xi, \frac{1}{2}u_0(\tau,\varepsilon) + \sum_{n=1}^{\infty} [u_{cn}(\tau,\varepsilon)\cos 2n\xi + u_{sn}(\tau,\varepsilon)\sin 2n\xi]\big\}\times \sin 2k\xi \sin\alpha_k(t-\tau)\ d\xi\, d\tau,$$

$$\alpha_k = \frac{a(2k)^2}{1+\varepsilon(2k)^2},\ k=\overline{1,\infty}. \quad (13)$$

and

$$v_0(t) = \varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t \int_0^\pi (t-\tau) f\big\{\tau, \xi, \frac{1}{2}v_0(\tau) + \sum_{n=1}^{\infty} [v_{cn}(\tau)\cos 2n\xi + v_{sn}(\tau)\sin 2n\xi]\big\}\ d\xi\, d\tau,$$

$$v_{ck}(t) = \varphi_{ck}\cos\widetilde{\alpha_k} t + \frac{\psi_{ck}}{\widetilde{\alpha_k}}\sin\widetilde{\alpha_k} t + \frac{2}{\pi\widetilde{\alpha_k}}\int_0^t \int_0^\pi f\big\{\tau, \xi, \frac{1}{2}v_0(\tau) + \sum_{n=1}^{\infty} [v_{cn}(\tau)\cos 2n\xi + v_{sn}(\tau)\sin 2n\xi]\big\}\times \cos 2k\xi \sin\widetilde{\alpha_k}(t-\tau)\ d\xi\, d\tau,$$

$$v_{sk}(t) = \varphi_{sk}\cos\widetilde{\alpha_k} t + \frac{\psi_{sk}}{\widetilde{\alpha_k}}\sin\widetilde{\alpha_k} t + \frac{2}{\pi\widetilde{\alpha_k}}\int_0^t \int_0^\pi f\big\{\tau, \xi, \frac{1}{2}v_0(\tau) + \sum_{n=1}^{\infty} [v_{cn}(\tau)\cos 2n\xi + v_{sn}(\tau)\sin 2n\xi]\big\}\times \sin 2k\xi \sin\widetilde{\alpha_k}(t-\tau)\ d\xi\, d\tau,$$

$$\widetilde{\alpha_k} = a(2k)^2,\ k=\overline{1,\infty}. \quad (14)$$

For simplicity, let

$$Au(\tau,\xi,\varepsilon)=\frac{1}{2}u_0(t,\varepsilon)+\sum_{k=1}^{\infty}\left[u_{cn}(\tau,\varepsilon)\cos 2kx+u_{sn}(\tau,\varepsilon)\sin 2nx\right]$$

and

$$Av(\tau,\xi)=\frac{1}{2}u_0(t)+\sum_{k=1}^{\infty}\left[u_{cn}(\tau)\cos 2kx+u_{sn}(\tau)\sin 2nx\right].$$

To examine the difference of the systems (13) and (14), we write as following;

$$u_0(t,\varepsilon)-v_0(t)=\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\,d\xi\,d\tau,$$

$$u_{ck}(t,\varepsilon)-v_{ck}(t)=\varphi_{ck}(\cos\alpha_k t-\cos\widetilde{\alpha_k}t)+\psi_{ck}\left(\frac{\sin\alpha_k t}{\alpha_k}-\frac{\sin\widetilde{\alpha_k}t}{\widetilde{\alpha}_k}\right)+$$
$$\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau-$$
$$\frac{2}{\pi\widetilde{\alpha_k}}\int_0^t\int_0^{\pi}f\{\tau,\xi,Av(\tau,\xi)\}\cos 2k\xi\sin\widetilde{\alpha_k}(t-\tau)\,d\xi\,d\tau$$

$$u_{sk}(t,\varepsilon)-v_{sk}(t)=\varphi_{sk}(\cos\alpha_k t-\cos\widetilde{\alpha_k}t)+\psi_{sk}\left(\frac{\sin\alpha_k t}{\alpha_k}-\frac{\sin\widetilde{\alpha_k}t}{\widetilde{\alpha}_k}\right)+$$
$$\frac{2}{\pi\alpha_k}\int_0^t\int_0^{\pi}f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\sin 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau-$$
$$\frac{2}{\pi\widetilde{\alpha_k}}\int_0^t\int_0^{\pi}f\{\tau,\xi,Av(\tau,\xi)\}\sin 2k\xi\sin\widetilde{\alpha_k}(t-\tau)\,d\xi\,d\tau$$

then we have

$$u_0(t,\varepsilon)-v_0(t)=\frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\,d\xi\,d\tau,$$

$$u_{ck}(t,\varepsilon)-v_{ck}(t)=\varphi_{ck}(\cos\alpha_k t-\cos\widetilde{\alpha_k}t)+\psi_{ck}\left(\frac{\sin\alpha_k t}{\alpha_k}-\frac{\sin\widetilde{\alpha_k}t}{\widetilde{\alpha}_k}\right)+$$
$$\left(\frac{1}{\alpha_k}-\frac{1}{\widetilde{\alpha_k}}\right)\frac{1}{\pi}\int_0^t\int_0^{\pi}f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau+$$
$$\frac{2}{\pi\widetilde{\alpha_k}}\int_0^t\int_0^{\pi}f\{\tau,\xi,Av(\tau,\xi)\}\cos 2k\xi[\sin\alpha_k(t-\tau)-sin\widetilde{\alpha_k}(t-\tau)]\,d\xi\,d\tau+$$

$$\frac{2}{\pi\widetilde{\alpha_k}}\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau,$$

$$u_{sk}(t,\varepsilon)-v_{sk}(t)=\varphi_{ck}(\cos\alpha_k t-\cos\widetilde{\alpha_k}t)+\psi_{ck}\left(\frac{\sin\alpha_k t}{\alpha_k}-\frac{\sin\widetilde{\alpha_k}t}{\widetilde{\alpha}_k}\right)+$$

$$\left(\frac{1}{\alpha_k}-\frac{1}{\widetilde{\alpha_k}}\right)\frac{1}{\pi}\int_0^t\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\sin 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau+$$

$$\frac{2}{\pi\widetilde{\alpha_k}}\int_0^t\int_0^\pi f\{\tau,\xi,Av(\tau,\xi)\}\sin 2k\xi[\sin\alpha_k(t-\tau)-sin\widetilde{\alpha_k}(t-\tau)]\,d\xi\,d\tau+$$

$$\frac{2}{\pi\widetilde{\alpha_k}}\int_0^t\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi\sin\widetilde{\alpha_k}(t-\tau)\,d\xi\,d\tau.$$

Take the absolute values of differences and after making the necessary grouping, we form a sum as following;

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|=\frac{1}{2}|\overline{u}_0(t,\varepsilon)-\overline{v}_0(t)|+$$
$$\sum_{k=1}^{\infty}[|u_{ck}(t,\varepsilon)-v_{ck}(t)|+|u_{sk}(t,\varepsilon)-v_{sk}(t)|]\leq$$
$$\frac{T}{\pi}\int_0^t\int_0^\pi|f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}|\,d\xi\,d\tau+$$
$$\sum_{k=1}^{\infty}(|\varphi_{ck}|+|\varphi_{sk}|)|\cos\alpha_k t-\cos\widetilde{\alpha_k}t|+\sum_{k=1}^{\infty}(|\psi_{ck}|+|\psi_{sk}|)\left|\tfrac{\sin\alpha_k t}{\alpha_k}-\tfrac{\sin\widetilde{\alpha_k}t}{\widetilde{\alpha}_k}\right|+$$
$$\sum_{k=1}^{\infty}\left|\tfrac{1}{\alpha_k}-\tfrac{1}{\widetilde{\alpha_k}}\right|\int_0^t\left(\left|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\cos 2k\xi d\xi\right|+\right.$$
$$\left.\left|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\sin 2k\xi d\xi\right|\right)d\tau+$$
$$\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}}\int_0^t\left(\left|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\cos 2k\xi d\xi\right|+\right.$$
$$\left.\left|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\sin 2k\xi d\xi\right|\right)|\sin\alpha_k(t-\tau)-sin\widetilde{\alpha_k}(t-\tau)|d\tau+$$
$$\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}}\int_0^t\left(\left|\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi\right|+\right.$$
$$\left.\left|\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi d\xi\right|\right)d\tau.\quad(15)$$

Each of the statements $|\cos\alpha_k t - \cos\widetilde{\alpha_k} t|$, $\left|\frac{\sin\alpha_k t}{\alpha_k} - \frac{\sin\widetilde{\alpha_k} t}{\widetilde{\alpha_k}}\right|$, $\left|\frac{1}{\alpha_k} - \frac{1}{\widetilde{\alpha_k}}\right|$, $|\sin\alpha_k(t-\tau) - sin\widetilde{\alpha_k}(t-\tau)|$ of the right side of in the equality (15) are bounded for k, τ and t $(0 \le \tau \le t \le T)$, as $\varepsilon \to 0$. Let us denote these statements by $\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon), \delta_4(\varepsilon)$, respectively, and then write last inequality as following;

$$
\begin{aligned}
|\overline{u}(t,\varepsilon) - \overline{v}(t)| \le \frac{T}{\pi}\int_0^t\int_0^\pi |f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}|+ \\
\delta_1(\varepsilon)\sum_{k=1}^{\infty}(|\varphi_{ck}| + |\varphi_{sk}|) + \delta_2(\varepsilon)\sum_{k=1}^{\infty}(|\psi_{ck}| + |\psi_{sk}|)+ \\
\delta_3(\varepsilon)\sum_{k=1}^{\infty}\int_0^t\left(|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\cos 2k\xi d\xi| + \right. \\
\left. |\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\sin 2k\xi d\xi|\right) d\tau + \\
\delta_4(\varepsilon)\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}}\int_0^t\left(|\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\cos 2k\xi d\xi| + \right. \\
\left. |\frac{2}{\pi}\int_0^\pi f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\sin 2k\xi d\xi|\right) d\tau + \\
\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}}\int_0^t\left(|\frac{2}{\pi}\int_0^\pi \left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi| + \right. \\
\left. |\frac{2}{\pi}\int_0^\pi \left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi d\xi|\right) d\tau.
\end{aligned}
\tag{16}
$$

Under the assumptions of the theorem given below, we can take $\delta(\varepsilon)$ as the sum of the 2nd, 3rd, 4th, 5th sums in the right side of the inequality (14), then we have

$$
\begin{aligned}
|\overline{u}(t,\varepsilon) - \overline{v}(t)| \le \delta(\varepsilon) + \\
\frac{T}{2}\int_0^t |\frac{2}{\pi}\int_0^\pi \left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}\right] d\xi| d\tau + \\
\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}}\int_0^t\left(|\frac{2}{\pi}\int_0^\pi \left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi| + \right. \\
\left. \frac{2}{\pi}\int_0^\pi \left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\} - f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi d\xi|\right) d\tau.
\end{aligned}
$$

Applying Hölder inequality to third sum in the right side of the inequality above, we obtain

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|\leq\delta(\varepsilon)+$$
$$\frac{T}{2}\int_0^t\frac{2}{\pi}\int_0^\pi|f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}|d\xi d\tau+\left(\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}^2}\right)^{1/2}$$
$$\left\{\sum_{k=1}^{\infty}\left[\int_0^t\left(|\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi\right.\right.\right.+$$
$$\left.\left.\left.\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi d\xi|\right)d\tau\right]^2\right\}^{1/2}. \quad (17)$$

Applying Cauchy inequality to the integrals in the right side of (17), then applying following inequality

$$(a_1+a_2+\ldots+a_n)^2\leq n(a_1^2+a_2^2+\ldots+a_n^2) \quad (18)$$

to third sum for $n=2$, we obtain

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|\leq\delta(\varepsilon)+$$
$$\frac{T^{3/2}}{2}\left\{\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right)d\xi\right]^2d\tau\right\}^{\frac{1}{2}}+$$
$$\left(\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}^2}\right)^{1/2}\sqrt{2}\left\{\sum_{k=1}^{\infty}T\int_0^t\left(\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\right.\right.\right.-$$
$$\left.f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi\Big)^2d\tau+$$
$$\left.\sum_{k=1}^{\infty}T\int_0^t\left(\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right]\sin 2k\xi d\xi\right)^2d\tau\right\}^{1/2} \quad (19)$$

and then,

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|\leq\delta(\varepsilon)+$$
$$\frac{T^{3/2}}{2}\left\{\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}-f\{\tau,\xi,Av(\tau,\xi)\}\right)d\xi\right]^2d\tau\right\}^{\frac{1}{2}}+$$
$$\sqrt{2T}\left(\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}^2}\right)^{1/2}\left\{\int_0^t\left[\sum_{k=1}^{\infty}T\left(\frac{2}{\pi}\int_0^\pi\left[f\{\tau,\xi,Au(\tau,\xi,\varepsilon)\}\right.\right.\right.\right.-$$
$$\left.f\{\tau,\xi,Av(\tau,\xi)\}\right]\cos 2k\xi d\xi\Big)^2+$$

$$\left(\frac{2}{\pi}\int_0^{\pi}\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\sin 2k\xi d\xi\right)^2\Bigg]d\tau\Bigg\}^{1/2}. \quad (20)$$

Taking square and using the inequality (18) to the inequality above for $n=3$, we have

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta(\varepsilon)^2 + \frac{3T^3}{4}\int_0^t\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right)d\xi\right]^2 d\tau +$$
$$6T\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}^2}\int_0^t\sum_{k=1}^{\infty}\left\{\left(\frac{2}{\pi}\int_0^{\pi}\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\cos 2k\xi d\xi\right)^2+\right.$$
$$\left.\left(\frac{2}{\pi}\int_0^{\pi}\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\sin 2k\xi d\xi\right)^2\right\}. \quad (21)$$

Supposing

$$max\left(\frac{3T^3}{2}, 6T\sum_{k=1}^{\infty}\frac{1}{\widetilde{\alpha_k}^2}\right)=\mu_0,$$

let us combine the integrals of the right side of (21) as follows,

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta(\varepsilon)^2 + \mu_0\int_0^t\left\{\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right)d\xi\right]^2 d\tau +\right.$$
$$\sum_{k=1}^{\infty}\left[\left(\frac{2}{\pi}\int_0^{\pi}\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\cos 2k\xi d\xi\right)^2+\right.$$
$$\left.\left.\left(\frac{2}{\pi}\int_0^{\pi}\left[f\left\{\tau,\xi,Au(\tau,\xi,\varepsilon)\right\}-f\left\{\tau,\xi,Av(\tau,\xi)\right\}\right]\sin 2k\xi d\xi\right)^2\right]\right\}. \quad (22)$$

Applying Bessel inequality to the sum in (22), we obtain

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta^2+\mu_0\frac{2}{\pi}\int_0^t\int_0^{\pi}\left[f\left\{\tau,x,Au(\tau,x,\varepsilon)\right\}-f\left\{\tau,x,Av(\tau,x)\right\}\right]^2 dx.$$

Using Lipshitzs inequality, we have

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta(\varepsilon)^2+\mu_0\frac{2}{\pi}\int_0^t\int_0^{\pi} b^2(t,x)\left[Au(\tau,x,\varepsilon)-Av(\tau,x)\right]^2 dxd\tau$$

or

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta(\varepsilon)^2+\mu_0\frac{2}{\pi}\int_0^t\int_0^\pi b^2(t,x)|\overline{u}(\tau,\varepsilon)-\overline{v}(\tau)|^2dxd\tau. \quad (23)$$

Applying Gronwall inequality to (23), we obtain

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|^2 \le 3\delta(\varepsilon)^2\exp\mu_0\frac{2}{\pi}\int_0^t\int_0^\pi b^2(t,x)dxd\tau,$$

or

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)| \le \sqrt{3}\delta(\varepsilon)\exp\exp\frac{\mu_0}{\pi}\parallel b(t,x)\parallel_{L_2}^2 dxd\tau.$$

In the last inequality, taking into account $\delta(\varepsilon)\to 0$ for $\varepsilon\to 0$, we see that

$$\lim_{\varepsilon\to 0}\overline{u}(t,\varepsilon)=\overline{v}(t).$$

Hence, the following theorem is proved.

Main Theorem: *Suppose the following conditions are satisfied;*

- *a)* $f(t,x,u)$ *is continuous respect to all arguments on* $\overline{D}\times(-\infty,\infty)$, *and satisfies the following condition* $|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$ *where* $b(t,x)\in L_2(D),\ b(t,x)\ge 0$,
- *b)* $[f(t,x,u)]_x\in L_2(D)$,
- *c)* $f(t,x,0)\in L_2(D)$,
- *d) The functions* $\varphi(x),\ \psi(x)$ *with* $\varphi(x)\in C^1[0,\pi],\psi(x)\in C[0,\pi]$ *satisfy the following conditions;*

$$\varphi(0)=\varphi(\pi),\ \varphi'(0)=\varphi'(\pi),\ \psi(0)=\psi(\pi).$$

In this case, $\lim_{\varepsilon\to 0}\overline{u}(t,\varepsilon)=\overline{v}(t)$, *i.e.* $\lim_{\varepsilon\to 0}u(t,x,\varepsilon)=\lim_{\varepsilon\to 0}v(t,x)$ *is true* ∎

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