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Connectedness in (Ideal) Bitopological Ordered Spaces

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Abstract

The aim of the present paper is to study connectedness in bitopological ordered spaces and in ideal bitopological ordered spaces.

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1 Introduction

In 1963 Kelly [14] was introduced a bitopological space (X, τ_1, τ_2) as a richer structure than topological space. A study of bitopological space is a generalization of the study of general topological space as every bitopological space (X, τ_1, τ_2). can be regarded as a topological space (X, τ). if $\tau_1 = \tau_2 = \tau$.

In 1971 Singal and Singal [23] were studied the bitopological ordered space (X, τ_1, τ_2, R). which is a generalization of the study of general topological space, bitopological space and topological ordered space. Every bitopological ordered space (X, τ_1, τ_2, R) can be regarded as a bitopological space (X, τ_1, τ_2) if R is the equality relation "Δ" and every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Also, every bitopological ordered space (X, τ_1, τ_2, R) can be regarded as a topological ordered space (X, τ, R) if $\tau_1 = \tau_2 = \tau$.

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [15] and Vaidyanathaswamy [25]. An ideal is a nonempty collection of subsets closed under heredity and finite additivity. The study of ideal bitopological spaces was initiated by Jafari and Rajesh [9].

The notion of connectedness in bitopological spaces has been studied by Pervin [20], Reily [21] and Swart [24]. In 2014 S. A. El-Sheikh and M. Hosny [5], Mandira Kar and Thakur [16] have been studied the notion of connectedness in ideal bitopological spaces.

Many authors [1, 4, 12, 13, 22, 23] have already been studied the bitopological ordered spaces, but the studying of the notion of connectedness in bitopological ordered spaces has not been considered.

The purpose of this paper is to introduce and study the notion of connectedness in bitopological ordered spaces. We study the notions of pairwise connected ordered spaces, pairwise separated ordered sets and pairwise connected ordered sets in bitopological ordered spaces. Moreover, comparisons between the current study and the previous one [20, 21] are presented. Furthermore, we introduce the notion of ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ which is a generalization of the study of bitopological ordered spaces (X, τ_1, τ_2, R) and bitopological space (X, τ_1, τ_2). Every ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ can be regarded as a bitopological ordered space (X, τ_1, τ_2, R) if $\mathcal{I} = \{\phi\}$ and can be regarded as bitopological space (X, τ_1, τ_2) if $\mathcal{I} = \{\phi\}, R$ is the equality relation "Δ". In addition, the notion of pairwise $*$-connected ordered spaces, pairwise $*$-separated ordered sets, pairwise $*$-connected ordered sets, pairwise $*s$-connected ordered sets in ideal bitopological ordered spaces has introduced. Some examples are given to illustrate the concepts. Furthermore, the relationship between these types of connectedness and the previous one [16, 20, 21] has obtained. Its therefore shown that the current work are more generally.

2 Preliminaries

In this section, we collect the relevant definitions and results from bitopological ordered spaces.

Definition 2.1. *[19] Let (X, R) be a poset. A set $A \subseteq X$ is said to be*

1. *Decreasing if for every $a \in A$ and $x \in X$ such that xRa, then $x \in A$.*
2. *Increasing if for every $a \in A$ and $x \in X$ such that aRx, then $x \in A$.*

Theorem 2.1. *[7] Let (X, R) be a poset. Let A be an increasing and B be a decreasing subsets of X. Then $X \setminus A = A'$ is a decreasing and $X \setminus B$ is an increasing subset of X.*

Definition 2.2. *[19] Let (X, R) be a poset, $x \in X$ and $A \subseteq X$. We define:*

1. *$D(A) = \{x \in X : xRa$ for some $a \in A\}$.*
2. *$I(A) = \{x \in X : aRx$ for some $a \in A\}$.*
3. *$C(A) = D(A) \cap I(A)$.*

Definition 2.3. *[19]Let (X, R) and (Y, R^*) be two posets. Then, a mapping $f : (X, R) \to (Y, R^*)$ is called an increasing (a decreasing) if $\forall x_1, x_2 \in X$ such that $x_1 R x_2 \Rightarrow f(x_1) R^* f(x_2) (f(x_2) R^* f(x_1))$.*

Theorem 2.2. *[1] Let $f : (X, R) \to (Y, R^*)$ be a mapping. Then, the following statements are equivalent:*

1. *f an increasing mapping.*
2. *If $B \subseteq Y$ is an increasing (a decreasing), then $f^{-1}(B)$ is an increasing (a decreasing) subset of X.*

Definition 2.4. *[6] Let X be a non-empty set. A class τ of subsets of X is called a topology on X iff τ satisfies the following axioms.*

1. *$X, \phi \in \tau$.*
2. *An arbitrary union of the members of τ is in τ.*
3. *The intersection of any two sets in τ is in τ.*

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A' is an open set.

Definition 2.5. *[10] A non-empty collection $\mathcal{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions*

1. $A \in \mathcal{I}$ *and* $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
2. $A \in \mathcal{I}$ *and* $B \subseteq A \Rightarrow B \in \mathcal{I}$.

Definition 2.6. *[10] Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. Then*

$$A^*(\mathcal{I}, \tau)\ (or\, A^*) := \{x \in X : O_x \cap A \notin \mathcal{I}\ \forall O_x\}$$

is called the local function of A with respect to $\mathcal{I}$ and τ, where O_x is an open set containing x.

Theorem 2.3. *[10] Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. Then, the operator $cl^* : P(X) \to P(X)$ defined by:*

$$cl^*(A) = A \cup A^* \tag{1}$$

satisfies Kuratwski's axioms and induces a topology $\tau^*(\mathcal{I})$ on X given by:

$$\tau^*(\mathcal{I}) = \{A \subseteq X :\ cl^*(A') = A'\}. \tag{2}$$

Proposition 2.1. *[10] Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. Then, $\tau \subseteq \tau^*(\mathcal{I})$, i.e., $\tau^*(\mathcal{I})$ is finer than τ.*

Lemma 2.1. *[11] Let (X, τ, I) be an ideal topological space and $B \subseteq A \subseteq X$. Then, $B^*(\tau_A, I_A) = B^*(\tau, I) \cap A$.*

Lemma 2.2. *[8] Let (X, τ, I) be an ideal topological space and $B \subseteq A \subseteq X$. Then, $cl^*_A(B) = cl^*(B) \cap A$.*

If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a subset of X, then $(A, \tau_A, \mathcal{I}_A)$, where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological subspace [3].

Lemma 2.3. *[3] Let (X, τ, I) be an ideal topological space, $A \subseteq Y \subseteq X$ and $Y \in \tau$. Then, A is $*$-open in Y is equivalent to A is $*$-open in X, i.e $(\tau_Y)^* = (\tau^*)_Y$.*

Definition 2.7. *[14] A bitopological space (bts, for short) is a triple (X, τ_1, τ_2), where τ_1, τ_2 are arbitrary topologies for a set X.*

Definition 2.8. *[9] An ideal bitopological space has the form $(X, \tau_1, \tau_2, \mathcal{I})$, where (X, τ_1, τ_2) is a bts and $\mathcal{I}$ is an ideal on X.*

Definition 2.9. *[17, 22] A function $f:(X_1,\tau_1,\tau_2)\to(X_2,\eta_1,\eta_2)$ is called*

1. *p.continuous (respectively p.open, p.closed) if $f:(X_1,\tau_i)\to(X_2,\eta_i), i=1,2$ are continuous (respectively open, closed).*
2. *p.homeomorphism if $f:(X_1,\tau_i)\to(X_2,\eta_i), i=1,2$ are homeomorphism.*

Definition 2.10. *[20, 21] Let (X,τ_1,τ_2) be a bts-space, $A,B\subset X$. Then A and B are said to be P-separated sets if $\overline{A}^i\cap B=\phi, A\cap\overline{B}^j=\phi, i,j=1,2, i\neq j$.*

Definition 2.11. *[20, 21] A bts-space (X,τ_1,τ_2) is said to be P-connected space if X can not be expressed as a union of two non-empty disjoint τ_i-open set A and τ_j-open set B. If X can be so expressed we shall write $X=A|B$ and we call this a separation or disconnection.*

We call (X,τ_1,τ_2) is P-disconnected space if it is not P-connected.

Definition 2.12. *[16] An ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is called P-$*$-connected if X cannot be written as a union of a non-empty disjoint τ_i-open set and τ_j^*-open set , $i,j=1,2, i\neq j$.*

Definition 2.13. *[16] Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space, $A,B\subset X$. Then, A and B are said to be P-$*$-separated sets if $\tau_i^*cl(A)\cap B=\phi, A\cap\tau_j cl(B)=\phi$.*

Definition 2.14. *[16] A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is called P-$*s$-connected if A is not the union of two P-$*$-separated sets in $(X,\tau_1,\tau_2,\mathcal{I})$.*

Definition 2.15. *[23] A bitopological ordered space (bto-space, for short) has the form (X,τ_1,τ_2,R), where (X,R) is a poset and (X,τ_1,τ_2) is a bts.*

3 P-Connectedness in Bitopological Ordered Spaces

The aim of this section is to study the notions of pairwise connected ordered bitopological spaces, pairwise separated ordered sets and pairwise connected ordered sets in bitopological ordered spaces. In addition, comparisons between the current work and the previous one [20, 21] are introduced.

Definition 3.1. *Let (X,τ_1,τ_2,R) be a bto-space, $A,B\subset X$. Then, A and B are said to be Pairwise separated ordered sets (P-separated ordered sets) if $\overline{A}^i\cap B=\phi, A\cap\overline{B}^j=\phi$ such that A is a decreasing set and B is an increasing set.*

Example 3.1. *Let* $(\mathbb{R}, \tau_{ul}, \tau_{\mathbb{U}}, R)$ *be a bto-space in which* $\mathbb{R}$ *is the real numbers and* R *is the usual order relation on* $\mathbb{R}$, τ_{ul} *is the upper limit topology and* $\tau_{\mathbb{U}}$ *is the usual topology. Let* $A, B \subseteq \mathbb{R}$ *such that* $A = (-\infty, 0)$ *is a decreasing set,* $B = [1, \infty)$ *is an increasing set. It is clear that* A *and* B *are* P*-separated ordered sets as* $\overline{A}^{ul} = \overline{A}^{\mathbb{U}} = (-\infty, 0], \overline{B}^{ul} = (1, \infty), \overline{B}^{\mathbb{U}} = [1, \infty)$, *and so* $\overline{A}^{ul} \cap B$, $\overline{B}^{\mathbb{U}} \cap A, \overline{A}^{\mathbb{U}} \cap B$ *and* $\overline{B}^{ul} \cap A$ *are empty. On the other hand, let* $(\mathbb{R}, \tau_{\mathbb{U}}, \tau_{\infty}, R)$ *be a bto-space, where* τ_{∞} *is the co-finite topology and let* $A, B \subseteq \mathbb{R}$ *such that* $A = (-\infty, 1)$ *is a decreasing set and* $B = (2, \infty)$ *is an increasing set. It is clear that* A *and* B *are not* P*-separated ordered set as* $\overline{A}^{\mathbb{U}} = (-\infty, 1], \overline{B}^{\infty} = \mathbb{R}$ *and so* $A \cap \overline{B}^{\infty} = (-\infty, 1), \overline{A}^{\mathbb{U}} \cap B = \phi$.

Remark 3.1. *Every* P*-separated ordered sets are a* P*-separated sets.*

The following example shows the converse of Remark 3.1 is not necessarily true.

Example 3.2. *Let* $(\mathbb{R}, \tau_{ll}, \tau_{ul}, R)$ *be a bto-pace and* τ_{ll} *is the lower limit topology and* τ_{ul} *is the upper limit topology. Let* $A, B \subseteq \mathbb{R}$ *such that* $A = (1, 2)$, $B = (3, 5)$. *It is clear that* A *and* B *are* P*-separated sets as* $\overline{A}^{ll} = [1, 2), \overline{A}^{ul} = (1, 2], \overline{B}^{ll} = [3, 5), \overline{B}^{ul} = (3, 5]$ *and so* $\overline{A}^{ll} \cap B$, $\overline{B}^{ul} \cap A$ *and* $\overline{A}^{ul} \cap B$, $\overline{B}^{ll} \cap A$ *are empty, but* A *and* B *are not* P*-separated ordered sets as* A *is not decreasing set and* B *is not increasing.*

Definition 3.2. *A bto-space* (X, τ_1, τ_2, R) *is said to be* P*-connected ordered space if* X *can not be expressed as a union of two non-empty disjoint* τ_i*-open set* A *and* τ_j*-open set* B *where* A *is a decreasing and* B *is an increasing sets.*

We call (X, τ_1, τ_2, R) *is* P*-disconnected ordered space if it is not* P*-connected ordered space.*

Remark 3.2. *Each* P*-connected spaces is* P*-connected ordered space.*

The following example shows that (X, τ_1, τ_2, R) is P-connected ordered space, but not P-connected space.

Example 3.3. *Let* (X, τ_1, τ_2, R) *be a bto-space, where* $X = \mathbb{R}, \tau_1 = \{\mathbb{R}, \phi, \mathbb{Q}\}, \tau_2 = \{\mathbb{R}, \phi, \mathbb{Q}^*\}, \mathbb{R}$ *is the set of real numbers,* $\mathbb{Q}$ *is the set of rational number and* $\mathbb{Q}^*$ *is the set of irrational numbers. Then,* X *is not* P*-connected space, but it is* P*-connected ordered space.*

Dvalishvili [2] defined boundary on a bts (X, τ_1, τ_2) for $A \subseteq X$, as, $b_{ij}(A) = \overline{A}^i \cap \overline{A'}^j$, $b_{ji}(A) = \overline{A}^j \cap \overline{A'}^i$, $i, j = 1, 2, i \neq j$ and he proved that $b_{ij}(A) = \phi \Leftrightarrow A$ is τ_i-closed and τ_j-open set, $b_{ji}(A) = \phi \Leftrightarrow A$ is τ_j-closed and τ_i-open set.

Theorem 3.1. *Let (X, τ_1, τ_2, R) be a bto-space. Then, the following are equivalent:-*

1. *X is P-connected ordered space.*
2. *X can not be expressed as a union of two non-empty disjoint sets A and B such that A is a decreasing τ_i-open and B is an increasing τ_j-open .*
3. *X can not be expressed as a union of two non-empty disjoint sets A and B such that A is an increasing τ_i-closed and B is a decreasing τ_j-closed.*
4. *There is no proper subset of X which is a decreasing, τ_i-open and τ_j-closed.*
5. *There is no proper subset of X which is an increasing, τ_i-closed and τ_j-open.*
6. *Every non-empty proper, decreasing (increasing) subset of X has $b_{ji}(A) \neq \phi$ $(b_{ij}(A) \neq \phi)$.*

Proof.
$(1 \Rightarrow 2)$ By Definition 3.2.

$(2 \Rightarrow 3)$ Let (2) holds and $X = A \cup B$ such that A and B are non-empty disjoint, A is an increasing τ_i-closed set and B is a decreasing τ_j-closed set. Then, $A = B'$ and $X = A \cup A'$, where A' is a decreasing τ_i-open set and $A = B'$ is an increasing τ_j-open set. So, we have a contradiction.

$(3 \Rightarrow 4)$ Let (3) holds and let there is a proper subset A of X such that A is a decreasing τ_i-open and τ_j-closed set. Then, A' is an increasing τ_i-closed and τ_j-open set and therefore, $X = A' \cup A$, where A' is an increasing τ_i-closed set and A is a decreasing τ_j-closed set. So, we have a contradiction with (3).

$(4 \Rightarrow 5)$ Let (4) holds and let there is a proper subset A of X such that A is an increasing τ_i-closed and τ_j-open set. Then, there is a proper subset A' of X such that A' is a decreasing τ_i-open and τ_j-closed set. So, we have a contradiction.

$(5 \Rightarrow 6)$ Let (5) holds and let there exists a non-empty proper subset A of X, decreasing such that $b_{ji}(A) = \phi$. Then, A is τ_j-closed and τ_i-open set. Hence, there exists a non-empty proper increasing set A' which is τ_i-closed and

τ_j-open. So, we have a contradiction.
In the case of increasing. If we have a non-empty proper subset A of X, increasing such that $b_{ij}(A) = \phi$. Then, A is a non-empty proper increasing subset of X such that A is τ_i-closed and τ_j-open set. So, we have a contradiction.

($6 \Rightarrow 1$) Let (6) holds and let $X = A \cup B$ such that A is a decreasing τ_i-open set and B is an increasing τ_j-open set, $A \neq \phi, B \neq \phi, A \cap B = \phi$. Then, we have $A = B'$, A is a decreasing τ_j-closed and τ_i-open. Then, $b_{ji}(A) = \phi$. So, we have a contradiction. On the other hand, if $B = A'$, B is an increasing τ_i-closed and τ_j-open. Then, $b_{ij}(A) = \phi$. So, we have a contradiction. Hence, the result.

Let $Y \subseteq X$ and R be a relation on X. Then, $R_Y := R \cap (Y \times Y)$ is a relation on Y and is called the relation induced by R on Y. If a relation has any properties of reflexivity, transitivity, symmetry and anti-symmetry, then the properties are inherited by induced relations [18].

Definition 3.3. *Let (X, τ_1, τ_2, R) be a bto-space. Then, $A \subset X$ is a P-disconnected ordered set if $(A, \tau_{1|A}, \tau_{2|A}, R_A)$ is P-disconnected ordered i.e., there exist a decreasing $\tau_{i|A}$-open set $A \cap G, G$ is a decreasing τ_i-open set and an increasing $\tau_{j|A}$-open set $A \cap H, H$ is an increasing τ_j-open set such that $A \cap G$ and $A \cap H$ are disjoint non-empty sets whose union is A. In this case, $G \cup H$ is called a P-disconnection ordered of A. A set is P-connected ordered set if it is not P-disconnected ordered set.*
Observe that
$A = (A \cap G) \cup (A \cap H) \Leftrightarrow A \subseteq G \cup H$ *and*
$\phi = (A \cap G) \cap (A \cap H) \Leftrightarrow H \cap G \subseteq A'$.
Therefore $G \cup H$ is called a P-disconnection ordered set of $A \Leftrightarrow A \cap G \neq \phi, A \cap H \neq \phi, A \subseteq G \cup H, H \cap G \subseteq A'$

Example 3.4. *Let $(\mathbb{R}, \tau_{ul}, \tau_{\mathbb{U}}, R)$ be a bto-space and $A \subseteq \mathbb{R}$ such that $A = [0, 1)$. Then, A is P-disconnected ordered set, since $(A, \tau_{ul|A}, \tau_{\mathbb{U}|A}, R_A)$ is P-disconnected ordered, $G \cup H$ is a P-disconnection ordered of A, such that $G = (-\infty, 0]$ and $H = (0, \infty)$, H is $\tau_{\mathbb{U}}$-open set and G is τ_{ul}-open are non-empty set and $A \cap G = \{0\}$ is a decreasing $\tau_{ul|A}$-open set and $A \cap H = (0, 1)$ is an increasing $\tau_{\mathbb{U}|A}$-open set are disjoint non-empty sets whose union is A.*

Remark 3.3. *Each P-connected set is P-connected ordered set.*

The following example shows that $G \cup H$ is P-disconnection of A, but not P-disconnected ordered of A.

Example 3.5. *Let $(\mathbb{R}, \tau_{ul}, \tau_{\mathbb{U}}, R)$ be a bto space and $A \subseteq \mathbb{R}$ such that $A = (0, 2)$. Then, A is P-disconnected set. For, let $G = (1, 3]$ and $H = (0, 1)$.*

It is clear that G is τ_{ul}-open set and H is $\tau_{\mathbb{U}}$-open set which are non-empty set and $A \cap G = (1,2)$ is a $\tau_{ul|A}$-open set and $A \cap H = (0,1)$ is a $\tau_{\mathbb{U}|A}$-open set which are disjoint non-empty sets whose union is A. Hence, $G \cup H$ is a P-disconnection set of A, but not P-disconnected ordered set as $A \cap G$ is not decreasing set and $A \cap H$ is not increasing set.

Proposition 3.1. *If A and B are P-separated ordered sets, then $A \cup B$ is P-disconnected ordered set.*

Proof.
Since A and B are non-empty P-separated ordered sets, then $\overline{A}^i \cap B = \phi$, $A \cap \overline{B}^j = \phi$, such that A is a decreasing and B is an increasing set. Let $G = (\overline{B}^j)'$ be a τ_j-open set, $H = (\overline{A}^i)'$ be a τ_i-open set, $(A \cup B) \cap G = A$ is a decreasing and $(A \cup B) \cap H = B$ is an increasing set which are disjoint non-empty sets whose union is $A \cup B(A|B)$, and so $A \cup B$ is a P-disconnected ordered set.

Proposition 3.2. *Let $G \cup H$ be a P-disconnection ordered of X and let B be a P-connected ordered subset of X. Then, either $B \subseteq G$ or $B \subseteq H$.*

Proof.
Since $G \cup H$ is a P-disconnection ordered of X. Then, $X = G \cup H$ and $G \cap H = \phi$. But $B \subseteq X$, hence $B \subseteq G \cup H$, $G \cap H \subseteq B'$. If $B \cap H$ and $B \cap G$ are non-empty, then $G \cup H$ forms a P-disconnected ordered of B which is a contradiction. Hence, $B \cap H = \phi$ or $B \cap G = \phi$. It follows that $B \subseteq G$ or $B \subseteq H$.

Proposition 3.3. *Let A be a P-connected ordered set in X and $B \subseteq X$ such that $A \subseteq B \subseteq C(A)$, then B is a P-connected ordered set.*

Proof.
Suppose B is a P-disconnected ordered set and suppose $G \cup H$ be a P-disconnection of B. By Proposition 3.2, $A \subseteq G$ or $A \subseteq H$. Let $A \subseteq G$. Because $B \cap H$ is a non-empty set, there exists a point z such that $z \in B \cap H \subseteq B \subseteq C(A)$. Hence, $z \in H, z \in C(A)$, then there exists $x, y \in A$ such that $x \leq z \leq y$, but H is an increasing set, $z \in H, z \leq y$ it follows that $y \in H$. Hence, $y \in H \cap A$ in contradiction with $A \subseteq G$. Consequently, B is a P-connected ordered set.

Theorem 3.2. *Let $f : (X, \tau_1, \tau_2, R) \to (Y, \eta_1, \eta_2, R^*)$ be a P-continuous, surjective and increasing. If A is a P-connectedness ordered subset of X, then its image $f(A)$ is a P-connectedness ordered subset of Y.*

Proof.
Let $f : (X, \tau_1, \tau_2, R) \to (Y, \eta_1, \eta_2, R^*)$ be a P-continuous, surjective and increasing function. Let B be a decreasing $\eta_{i|f(A)}$-open and $\eta_{j|f(A)}$-closed subset of $f(A)$. Then, $f^{-1}(B)$ is a decreasing $\tau_{i|A}$-open and $\tau_{j|A}$-closed subset of A. Since A is P-connected ordered set, then $f^{-1}(B)$ is either ϕ or A. Hence, $B = f(f^{-1}(B))$ is either ϕ or $f(A)$.

Corollary 3.1. *P-connectedness ordered is invariant under a P-continuous, surjective and increasing function.*

Theorem 3.3. *Let (X, τ_1, τ_2, R) be a bto-space, $A \subseteq X$, $(A, \tau_{1|A}, \tau_{2|A}, R_A)$ be a relative bto-space on A, G be a decreasing set, H be a increasing set. Then, A is P-connected ordered set on $(X, \tau_1, \tau_2, R) \Leftrightarrow (A, \tau_{1|A}, \tau_{2|A}, R_A)$ is a P-connected ordered space.*

Proof.
" $\Leftarrow$ " Suppose A is a P-disconnected ordered on (X, τ_1, τ_2, R) and suppose $G \cup H$ is a P-disconnected ordered of A. Then, there exists a decreasing τ_i-open set G and an increasing τ_j-open set H. Accordingly, $A \cap G$ is a decreasing $\tau_{i|A}$-open set and $A \cap H$ is an increasing $\tau_{j|A}$-open set. Hence, $G \cup H$ form a P-disconnection ordered on $(A, \tau_{1|A}, \tau_{2|A}, R_A)$, hence $(A, \tau_{1|A}, \tau_{2|A}, R_A)$ is a P-disconnected ordered space.

" $\Rightarrow$ " Conversely, suppose $(A, \tau_{1|A}, \tau_{2|A}, R_A)$ is a P-disconnected ordered and suppose $G^* \cup H^*$ is a P-disconnection of A. Then, there exists a decreasing $\tau_i|A$-open set $G^* = A \cap G, G$ is a decreasing τ_i-open set and an increasing $\tau_{j|A}$-open set $H^* = A \cap H, H$ is an increasing τ_j-open set. But $A \cap G^* = A \cap A \cap G = A \cap G$ and $A \cap H^* = A \cap A \cap H = A \cap H$.
Hence, $G \cup H$ is a P-disconnection ordered on (X, τ_1, τ_2, R) and so A is a P-disconnection ordered on (X, τ_1, τ_2, R).

Theorem 3.4. *Let A be a τ_i-open-and-τ_j-closed subset of X, and S be a P-connected ordered subset of X. Then, either $S \subset A$ or $S \subset A'$.*

Proof.
Since A is a τ_i-open-and-τ_j-closed, then $A \cap S$ is a $\tau_{i|S}$-open and $\tau_{j|S}$-closed on a relative bto-space on S. But S is P-connected ordered set, then either $A \cap S = S$ or ϕ. Then, either $S \subset A$ or $S \subset A'$.

4 P-$*$-Connectedness in Ideal Bitopological Ordered Spaces

In this section, ideal bitopological ordered spaces are presented by using the concept of ideal. It is a generalization of the study of bitopological space, bitopological ordered space. Moreover, the notion of pairwise $*$-connected ordered spaces, pairwise $*$-separated ordered sets, pairwise $*$-connected ordered sets, pairwise $*s$-connected ordered sets in ideal bitopological ordered spaces has introduced. Furthermore, the relationship between the current notion of connectedness in this section, the notion of connectedness in Section 3 and the previous one in [16] is obtained.

Definition 4.1. *An ideal bitopological ordered space has the form* $(X, \tau_1, \tau_2, R, \mathcal{I})$, *where* (X, R) *is a poset and* (X, τ_1, τ_2) *is a bts and* $\mathcal{I}$ *is an ideal on* X.

Definition 4.2. *An ideal bitopological ordered space* $(X, \tau_1, \tau_2, R, \mathcal{I})$ *is called* P-$*$-*connected ordered if* X *cannot be written as a union of two non-empty disjoint decreasing* τ_i-*open set and a non-empty increasing* τ_j^*-*open set* $i, j = 1, 2, i \neq j$.

Example 4.1. *The system* $(X, \tau_1, \tau_2, R, \mathcal{I})$ *is an ideal bitopological ordered space in which* $X = \{1, 2, 3, 4\}$, $\tau_1 = \{X, \phi, \{1\}, \{4\}, \{1, 4\}\}$, $\tau_2 = \{X, \phi, \{1\}, \{1, 2\}\}$, $R = \Delta \cup \{(2, 1), (2, 4)\}$ *and* $\mathcal{I} = \{\phi, \{1\}\}$.

Remark 4.1. *Every* P-$*$-*connected is* P-$*$-*connected ordered.*

Example 4.1 shows that the converse of Remark 4.1 is not true, i.e., $(X, \tau_1, \tau_2, R, \mathcal{I})$ is P-$*$-connected ordered, but not P-$*$-connected (as $\exists$ a non-empty disjoint τ_1-open set $A = \{1\}$ and $\exists$ τ_2^*-open set $B = \{2, 3, 4\}$ such that $X = A \cup B$. Also, $\exists$ a non-empty disjoint τ_2-open set $A = \{1\}$ and $\exists$ τ_1^*-open set $B = \{2, 3, 4\}$ such that $X = A \cup B$.

Remark 4.2. *Every* P-$*$-*connected ordered is* P-*connected ordered.*

The following example shows that the converse of Remark 4.2 is not true.

Example 4.2. *In Example 4.1, let* R *is the usual order relation on* X. *Then, as* $(X, \tau_1, \tau_2, R, \mathcal{I})$ *is* P-*connected ordered space, but not* P-$*$-*connected ordered as* ($\exists$ *two non-empty disjoint decreasing* τ_1-*open set* $A = \{1\}$ *and increasing* τ_2^*-*open set* $B = \{2, 3, 4\}$ *such that* $X = A \cup B$. *Also,* $\exists$ *two non-empty disjoint decreasing* τ_2-*open set* $A = \{1\}$ *and increasing* τ_1^*-*open set* $B = \{2, 3, 4\}$ *such that* $X = A \cup B$.

Definition 4.3. *A subset* A *of an ideal bitopological ordered space* $(X, \tau_1, \tau_2, R, \mathcal{I})$ *is called* P-$*$-*connected ordered if* $(A, \tau_{1A}, \tau_{2A}, R_A, \mathcal{I}_A)$ *is* P-$*$-*connected ordered.*

Definition 4.4. *Let* $(X, \tau_1, \tau_2, R, \mathcal{I})$ *be an ideal bitopological ordered space,* $A, B \subset X$. *Then,* A *and* B *are said to be* P-$*$-*separated ordered sets if* $\tau_i^* cl(A) \cap B = \phi$, $A \cap \tau_j cl(B) = \phi$ *such that* A *is a decreasing and* B *is an increasing set.*

Remark 4.3. *Every* P-*separated ordered sets are* P-$*$-*separated ordered sets.*

Example 4.2 shows that the converse of Remark 4.3 is not true, as $A = \{1\}, B = \{2, 3, 4\}$ are P-$*$-separated ordered sets, but not P-separated ordered sets as $(\tau_1 cl(A) \cap B = \{2, 3\} \neq \phi$.

Remark 4.4. *Every* P-$*$-*separated ordered sets are* P-$*$-*separated sets.*

Example 4.1 shows that the converse of Remark 4.4 is not true, as $A = \{1\}, B = \{2,3,4\}$ are P-$*$-separated sets, but not P-$*$-separated ordered sets since, $\tau_i^* cl(A) \cap B = \phi, A \cap \tau_j cl(B) = \phi$, but A is not decreasing set and B is not increasing set.

Corollary 4.1. *For an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$, we have the following implications*

$$\begin{array}{ccc} P\text{-separated ordered sets} & \Rightarrow & P\text{-separated sets.} \\ \Downarrow & & \Downarrow \\ P\text{-}*\text{-separated ordered sets} & \Rightarrow & P\text{-}*\text{-separated sets.} \end{array}$$

Theorem 4.1. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. If A, B are P-$*$-separated ordered sets of X and $A \cup B \in \tau_i$(respectively τ_j), then A is τ_j-open and B is τ_i^*-open, $i, j = 1, 2, i \neq j$.*

Proof.
Since, A and B are are P-$*$-separated ordered sets in X, then $B = (A \cup B) \cap (X \setminus \tau_i^* cl(A))$. Since $A \cup B \in \tau_i$ and $\tau_i^* cl(A)$ is τ_i^*-closed in X, B is τ_i^*-open in X. Similarly $A = (A \cup B) \cap (X \setminus \tau_j cl(B))$ and we obtain that A is τ_j-open in X.

Theorem 4.2. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space and $A \subseteq B \subseteq Y \subseteq X$. Then, A and B are P-$*$-separated ordered sets in $Y \Leftrightarrow A, B$ are P-$*$-separated ordered sets in X.*

Proof. ($\Leftarrow$) Straightforward.
($\Rightarrow$) Let A, B are P-$*$-septated ordered in Y. Then, $\tau_j^* cl(A) \cap B = (\tau_j^* cl(A) \cap Y) \cap B = \tau_j^* cl_Y(A) \cap B = \phi$. Similarly, $A \cap \tau_i cl(B) = A \cap (Y \cap \tau_i cl(B)) = A \cap \tau_i cl_Y(B) = \phi$.

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2, R, \mathcal{I}) \to (Y, \eta_1, \eta_2, R^*)$ be a P-continuous, surjective and increasing. If X is a P-$*$-connected ordered space, then (Y, η_1, η_2, R^*) is P-connected ordered space.*

Proof. It is known that P-connectedness ordered space is preserved by continuous, surjections and increasing (See Corollary 3.1). Also, every P-$*$-connected ordered space is P-connected ordered space (See Remark 4.2). Hence, the proof has done.

Definition 4.5. *A subset A of an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ is called P-$*s$-connected ordered if A is not the union of two P-$*$-separated ordered sets in $(X, \tau_1, \tau_2, R, \mathcal{I})$.*

Remark 4.5. *Every P-$*s$-connected set is P-$*s$-connected ordered set .*

Example 4.1 shows that the converse of Remark 4.5 is not true, as $A = \{1, 3, 4\}$ is P-$*s$-connected ordered, but not P-$*s$-connected as, $\exists\ B = \{1\}, C = \{3, 4\}$ which are P-$*$-separated sets and whose union is A.

Remark 4.6. *Every P-$*$-connected ordered set is P-$*s$-connected ordered set.*

Example 4.1 shows that the converse of Remark 4.6 is not true, as $A = \{1, 3, 4\}$ is P-$*s$-connected ordered, but not P-$*$-connected ordered set, since $(A, \tau_{1|A}, \tau_{2|A}, R_A, \mathcal{I}_A)$ is not P-$*$-connected ordered , for $\exists$ two non-empty disjoint decreasing $\tau_{1|A}$-open set $G = \{1\}$, $\exists$ an increasing $\tau^*_{2|A}$-open set $H = \{3, 4\}$ such that $A = G \cup H$. Also, $\exists$ two non-empty disjoint decreasing $\tau_{2|A}$-open set $G = \{1\}$, $\exists$ an increasing $\tau^*_{1|A}$-open set $H = \{3, 4\}$ such that $A = G \cup H$.

Theorem 4.4. *Let $Y \in \tau_1 \cap \tau_2$ and $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. Then, the following are equivalent:*

- *Y is P-$*s$-connected ordered in $(X, \tau_1, \tau_2, R, \mathcal{I})$.*
- *Y is P-$*$-connected ordered in $(X, \tau_1, \tau_2, R, \mathcal{I})$.*

Proof.

(1) $\Rightarrow$ (2) Suppose that Y is not P-$*$-connected ordered in $(X, \tau_1, \tau_2, R, \mathcal{I})$. There exist a non empty disjoint decreasing τ_i-open set A, in Y and increasing τ^*_j-open set B in Y such that $Y = A \cup B$. Since $Y \in \tau_1 \cap \tau_2$, by Lemma 2.3 A and B are τ_i-open and τ^*_j-open in X, respectively. Since A and B are disjoint, then $\tau^*_j cl(A) \cap B = \phi = A \cap \tau_i cl(B)$. This implies that A, B are P-$*$-separated ordered sets in X. Thus, Y is not P-$*s$-connected ordered in $(X, \tau_1, \tau_2, R, \mathcal{I})$.

(2) $\Rightarrow$ (1) Suppose Y is not P-$*s$-connected in $(X, \tau_1, \tau_2, R, \mathcal{I})$. There exist two P-$*$-separated sets A, B in X such that $Y = A \cup B$. By Theorem 4.1,A and B are τ_i-open and τ^*_j-open in Y, respectively $i, j = 1, 2, i \neq j$. By Lemma 2.3, A and B are τ_i-open and τ^*_j-open in X respectively. Since A and B are P-$*$-separated ordered sets in X, then A and B are nonempty and disjoint. Thus, Y is not P-$*$-connected ordered.

Theorem 4.5. *Let $(X, \tau_1, \tau_2, R, \mathcal{I})$ be an ideal bitopological ordered space. If A is a P-$*s$-connected ordered set of X and H, G are P-$*$-separated ordered sets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Proof.
Let $A \subseteq H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $(A \cap G) \cap \tau^*_i cl(A \cap H) \subseteq G \cap \tau^*_i cl(H) = \phi$. By similar reasoning, we have $(A \cap H) \cap \tau_j cl(A \cap G) \subseteq H \cap \tau_j cl(G) = \phi$. Suppose that $A \cap H$ and $A \cap G$ are nonempty. Then, A is not P-$*s$-connected ordered. This is a contradiction. Thus, either $A \cap H = \phi$ or $A \cap G = \phi$. This implies that either $A \subseteq H$ or $A \subseteq G$.

Theorem 4.6. *If A is a P-$*s$-connected ordered set of an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ and $A \subseteq B \subseteq \tau_i^* cl(A)$, then B is P-$*s$-connected ordered.*

Proof.
Suppose B is not P-$*s$-connected ordered. There exist P-$*$-separated ordered sets H and G of X such that $B = H \cup G$. This implies that H and G are nonempty and $\tau_i^* cl(H) \cap G = H \cap \tau_j cl(G) = \phi$. By Theorem 4.5, we have either $A \subseteq H$ or $A \subseteq G$. Suppose that $A \subseteq H$. Then, $\tau_i^* cl(A) \subseteq \tau_i^* cl(H)$ and $G \cap \tau_i^* cl(A) = \phi$. This implies that $G \subseteq B \subseteq \tau_i^* cl(A)$ and $G = \tau_i^* cl(A) \cap G = \phi$. Thus, G is an empty set which is a contradiction. Suppose that $A \subseteq G$. By similar way, we have that H is empty, which is also a contradiction. Hence, B is P-$*s$-connected ordered.

Corollary 4.2. *If A is a P-$*s$-connected set in an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$, then $\tau_i^* cl(A)$ is P-$*s$-connected ordered.*

Theorem 4.7. *If $\{M_i : i \in I\}$ is a nonempty family of P-$*s$-connected ordered sets of an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$ with $\cap_{i \in I} M_i \neq \emptyset$. Then, $\cup_{i \in I} M_i$ is Pairwise $*s$-connected ordered.*

Proof. Suppose that $\cup_{i \in I} M_i$ is not P-$*s$-connected ordered. Then, we have $\cup_{i \in I} M_i = H \cup G$, where H and G are P-$*$-separated ordered sets in X. Since, $\cap_{i \in I} M_i \neq \phi$ we have a point x in $\cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 4.5, $M_i \subseteq H$ or $M_i \subseteq G$. Since H and G are disjoint, $\forall i \in I$ $M_i \subseteq H$ and hence $\cup_{i \in I} M_i \subseteq H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By similar way, we have that H is empty. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is P-$*s$-connected ordered.

On account of Remarks 3.2,3.3,4.1,4.2,4.5 and 4.6 we have the following proposition which studies the relationship between the current definitions and the previous definitions.

Proposition 4.1. *For an ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathcal{I})$, we have the following implications*

1. *P-$*$-connected spaces $\Rightarrow$ P-$*$-connected ordered spaces.*
   $\Downarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\Downarrow$
   P-connected spaces $\Rightarrow$ P-connected ordered spaces.

2. *P-$*s$-connected sets $\Rightarrow$ P-$*s$-connected ordered sets $\Leftarrow$ P-$*$-connected ordered set.*

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References

[1] F.S.A. Hamied, On bitopological ordered spaces, *M.Sc. Thesis*, South Valley University, (1994).

[2] B. Dvalishvili, Bitopological spaces theory, *Relations with Generalized Algebraic Structures and Applications*, North Holland, (2005).

[3] E. Ekici and T. Noiri, Connectedness in ideal topological spaces, *Novi Sad J. Math.*, 38(2) (2008), 65-70.

[4] S.A. El-Sheikh and M. Hosny, $\mathcal{I}$-increasing (decreasing) sets and $\mathcal{I}P^*$-separation axioms in bitopological ordered spaces, *Pensee Journal*, 76(3) (2014), 429-443.

[5] S.A. El-Sheikh and M. Hosny, P^*-$*$Connectedness in ideal bitopological spaces, *To appear in Mathematical Sciences Letters.*

[6] R. Engelking, *General Topology*, Warszwawa, (1977).

[7] M.H. Ghanim, On some types of topologies, *M.Sc. Thesis*, Al-Azher University, (1976).

[8] E. Hatir, A. Keskin and T. Noiri, A note on strong β-$\mathcal{I}$-sets and strongly β-$\mathcal{I}$-continuous functions, *Acta Math. Hungar.*, 108(1-2) (2005), 87-94.

[9] S. Jafari and N. Rajesh, On qI-open sets in ideal bitopological spaces, University of Bacau, Faculty of Sciences, Scientific Studies and Research, *Series Mathematics and Informatics*, 20(2) (2010), 29-38.

[10] D. Jankovic and T.R. Hamlet, New topologies from old via ideals, *The American Mathematical Monthly*, 97(1990), 295-310.

[11] D. Jankovic and T.R. Hamlet, Compatible extensions of ideals, *Boll. Un. Mat. tal.*, 7(6-B) (1992), 453-465.

[12] A. Kandil, O. Tantawy, S.A. El-Sheikh and M. Hosny, New approach to bitopological ordered spaces, *Jökull Journal*, 63(6) (2013), 10-25.

[13] A. Kandil, O. Tantawy, S.A. El-Sheikh and M. Hosny, $\mathcal{I}P$-separation axioms in bitopological ordered spaces I, *To appear in Sohag Journal of Mathematics.*

[14] J.C. Kelly, Bitopological spaces, *Proc. London Math. Soc.*, 13(1963), 71-89.

[15] K. Kuratowski, *Topology (Vol. I)*, Academic Press, New York, (1966).

[16] M. Kar and S.S. Thakur, Connectedness in ideal bitopological spaces, *Int. Journal of Engineering Research and Applications*, 4(1) (2014), 165-167.

[17] D.N. Misra and K.K. Dube, Pairwise R_0-space, *Ann. Soc. Sci. Bruxelles*, 87(1976), 3-15.

[18] G.M. Murdeshwar, *General Topology*, New Age International (P) Ltd. Publishers, (1990).

[19] L. Nachbin, *Topology and Order*, Van Nostrand Mathematical Studies, Princeton, New Jersey, (1965).

[20] W.J. Pervin, Connectedness in bitopological spaces, *Indog Math. J.*, 29(1967), 369-372.

[21] I.L. Reilly, On pairwise connected bitopological spaces, *Kyungpook Math. J.*, 11(1) (1971), 25-28.

[22] M.K. Singal and A.R. Singal, Some more separation axioms in bitopological spaces, *Ann. Soc. Sci. Bruelles*, 84(1970), 207-230.

[23] M.K. Singal and A.R. Singal, Bitopological ordered spaces, *Math. Student*, 5(1971), 440-447.

[24] J. Swart, Total disconectedness in bitopological spaces and product bitopological spaces, *Indagationes Mathematicae (Proceedings)*, 74(1971), 135-145.

[25] R. Vaidynathaswamy, The localization theory in set topology, *Proc. Ind. Acad. of Sci.*, 20(1945), 515-61