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On a New Type of Spaces Related to the Decomposition Theorem for Harmonic Functions

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Abstract

If $1 \leq p < \infty$, Ω an open subset of $\mathbb{R}^n$ and K a compact subset of Ω, we consider the space $\mathcal{A}^p(\Omega \backslash K)$ of all functions $u \in b^p(\Omega \backslash K)$ that can be decomposed as $u = v + w$ on $\Omega \backslash K$, where $v \in b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n \backslash K)$. In this paper we introduce an analogous definitions for networks and for holomorphic functions. In final, we develop a new type of regularity for distributions and obtain their useful properties.

Keywords: *Harmonic Bergman space, Decomposition theorem, Potentials on networks, Distribution.*

1 Introduction

In [2] we considered the following theorem in the framework of harmonic Bergman spaces.

Theorem 1. *1.* $(n > 2)$ *: Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. If u is harmonic on $\Omega \backslash K$, then u has a unique decomposition of the form $u = v + w$, where v is harmonic on Ω and w is harmonic function on $\mathbb{R}^n \backslash K$ satisfying* $\lim_{x \to \infty} w(x) = 0$.
2. $(n = 2)$ *: Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. If u is harmonic on $\Omega \backslash K$, then u has a unique decomposition of the form*

$u = v + w$, where v is harmonic on Ω and w is harmonic function on $\mathbb{R}^2 \backslash K$ satisfying $\lim_{x\to\infty} w(x) - b\log|x| = 0$ *for some constant b.*

For $1 \leq p < \infty$, an open set Ω in $\mathbb{R}^n$ and a compact subset K of Ω, we defined a space $\mathcal{A}^p(\Omega\backslash K)$ of all functions $u \in b^p(\Omega\backslash K)$ such that $u = v + w$ on $\Omega\backslash K$, where $v \in b^p(\Omega)$ and $w \in b^p(\mathbb{R}^n\backslash K)$. We introduced a new norm on $\mathcal{A}^p(\Omega\backslash K)$. There are also similar decomposition theorems in the cases of holomorphic functions(see [3]), harmonic functions on infinite trees (see [1]) and solutions of parabolic partial differential equations (see [6]). In the next section we consider an analogous definition for harmonic functions on infinite trees and we give the answer on an analogous problem given in Remark 2 given in [5]. In the third section we introduce an analogous results for holomorphic case and in final we introduce a new type of regularity for distributions on domains outside compact set and obtain their useful properties.

2 Network Case

Let T be an infinite tree. For any subset E of T, we write $\mathring{E}$ as a set of all $x \in T$, such that x and all its neighbours are in E. We denote $\partial E = E\backslash\mathring{E}$. Let $V(E) = E \cup \{y : y \sim x, \text{ for some } x \in E\}$. For $x \in T$, and for a function f defined on $V(x)$, define the Laplacian

$$\Delta f(x) = \sum_{x\sim x_i} t(x, x_i)[f(x_i) - f(x)].$$

We say that f is harmonic at x if $\Delta f(x) = 0$. A function f defined on an arbitrary set E is said to be harmonic on E if $\Delta f(x) = 0$ for every $x \in \mathring{E}$. Let $1 \leq p < \infty$. By $L^p(E)$ we denote the set of all functions f on E such that $\|f\|_p := \left(\sum_{x\in E} |f(x)|^p\right)^{1/p} < \infty$. Let $b^p(E)$ denote a Bergman space on E, i.e. a set of all harmonic functions f on E such that $f \in L^p(E)$.

Lemma 1. *Let $1 \leq p < \infty$ and E a subset of T. Then $b^p(E)$ is a Banach space under the $\|\cdot\|_p$ norm.*

Proof. Let f_n be a Cauchy sequence in $b^p(E)$. As $L^p(E)$ is a Banach space, there exist $f \in L^p(E)$ such that $\|f_n - f\|_p \to 0$, as $n \to \infty$. Now, for $x \in E$ we have

$$|f_n(x) - f(x)| \leq \left(\sum_{x\in E} |f_n(x) - f(x)|^p\right)^{1/p} = \|f_n - f\|_p \to 0,$$

as $n \to \infty$. So, $f_n(x) \to f(x)$, as $n \to \infty$. Now, for $x \in \mathring{E}$,

$$\begin{aligned}\Delta f(x) &= \sum_{x \sim x_i} t(x, x_i)(f(x_i) - f(x)) \\ &= \lim_{n\to\infty} \sum_{x \sim x_i} t(x, x_i)(f_n(x_i) - f_n(x)) \\ &= \lim_{n\to\infty} \Delta f_n(x) \\ &= \lim_{n\to\infty} 0 \\ &= 0,\end{aligned}$$

so f is harmonic on E and the lemma is proved. □

In [1] we can find a proof of the following theorem.

Theorem 2. *Let T be an infinite tree in which every non-terminal vertex has at least two non-terminal neighbours. Let E be a finite connected subset of T and $F = V(E)$. Let A be a non-empty subset of E. Suppose u is a harmonic function on $F \backslash \mathring{A}$. Then, there exist a harmonic function s on $T \backslash \mathring{A}$ and a harmonic function t on F such that $u = s + t$ on $F \backslash \mathring{A}$. Moreover,*
i. If there are positive potentials on T, then s and t are uniquely determined if we impose restriction $|s| \leq p$ outside a finite set, where p is a positive potential on T.
ii. If there are no positive potentials on T, then s and t are uniquely determined up to an additive constant, if we impose the restriction that for some α, $[s - \alpha H]$ is bounded outside a finite set where $H(x)$ is the analogue of $\log|x|$ in T (namely $H \geq 0$ on T, unbounded, $\Delta H(x) = \delta_e(x)$ where e is a fixed vertex and $H > 0$ outside a finite set in T).

Let $1 \leq p < \infty$. Define the space $\mathcal{A}^p\left(F \backslash \mathring{A}\right)$ as a set of all functions $u \in b^p\left(F \backslash \mathring{A}\right)$ whose unique decomposition $u = s + t$ in Theorem 2 also satisfy $s \in b^p\left(T \backslash \mathring{A}\right)$ and $t \in b^p(F)$.

Lemma 2. *Let $1 \leq p < \infty$ and $u = s + t \in \mathcal{A}^p\left(F \backslash \mathring{A}\right)$ is arbitrarily chosen. Then*

$$\|u\|_{b^p\left(F \backslash \mathring{A}\right)} \leq 2^{\frac{p-1}{p}} \|u\|_{\mathcal{A}^p\left(F \backslash \mathring{A}\right)}.$$

Proof. The result follows from the following inequalities

$$\begin{aligned}
\|u\|^p_{b^p\left(F\backslash\mathring{A}\right)} = \|s+t\|^p_{b^p\left(F\backslash\mathring{A}\right)} &\leq \left(\|s\|_{b^p\left(F\backslash\mathring{A}\right)} + \|t\|_{b^p\left(F\backslash\mathring{A}\right)}\right)^p \\
&\leq 2^{p-1}\left(\|s\|^p_{b^p\left(F\backslash\mathring{A}\right)} + \|t\|^p_{b^p\left(F\backslash\mathring{A}\right)}\right) \\
&\leq 2^{p-1}\left(\|s\|^p_{b^p\left(T\backslash\mathring{A}\right)} + \|t\|^p_{b^p(F)}\right) \\
&= 2^{p-1}\|u\|_{\mathcal{A}^p\left(F\backslash\mathring{A}\right)}
\end{aligned}$$

□

Theorem 3. *Let* $1 \leq p < \infty$. *Then* $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$ *is a Banach space under the norm defined by*

$$\|u\|^p_{\mathcal{A}^p\left(F\backslash\mathring{A}\right)} = \|s\|^p_{b^p\left(T\backslash\mathring{A}\right)} + \|t\|^p_{b^p(F)},$$

where $u = s + t$ *is a decomposition of* u *in* $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$

Proof. Let (u_m) be a Cauchy sequence in $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$. From the lemma 2 follows that (u_m) is a Cauchy sequence in $b^p\left(F\backslash\mathring{A}\right)$. $b^p\left(F\backslash\mathring{A}\right)$ is a Banach space, so there exists $u \in b^p\left(F\backslash\mathring{A}\right)$ such that $u_m \to u$ in $b^p\left(F\backslash\mathring{A}\right)$. Also,

$$\|u_m - u_k\|^p_{\mathcal{A}^p\left(F\backslash\mathring{A}\right)} = \|t_m - t_k\|^p_{b^p(F)} + \|s_m - s_k\|^p_{b^p\left(T\backslash\mathring{A}\right)},$$

where $u_m = s_m + t_m$ on $F\backslash\mathring{A}$ is a decomposition mentioned in the definition of $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$. Since (u_m) is a Cauchy sequence in $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$, it follows that (t_m) is a Cauchy sequence in $b^p(F)$ and (s_m) is a Cauchy sequence in $b^p\left(T\backslash\mathring{A}\right)$. Since $b^p(F)$ and $b^p\left(T\backslash\mathring{A}\right)$ are Banach spaces, it follows that there are $t \in b^p(F)$ and $s \in b^p\left(T\backslash\mathring{A}\right)$ such that $t_m \to t$ in $b^p(F)$ and $s_m \to s$ in $b^p\left(T\backslash\mathring{A}\right)$. Now define $u' = s + t$. We have $u' \in \mathcal{A}^p\left(F\backslash\mathring{A}\right)$. Now,

$$\|u_m - u'\|^p_{\mathcal{A}^p\left(F\backslash\mathring{A}\right)} = \|t_m - t\|^p_{b^p(F)} + \|s_m - s\|^p_{b^p\left(T\backslash\mathring{A}\right)},$$

so $u_m \to u'$ in $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$. Since $u_m \to u$ in $b^p\left(F\backslash\mathring{A}\right)$ and, by lemma 2, $u_m \to u'$ in $b^p\left(F\backslash\mathring{A}\right)$, it yields $u = u' \in \mathcal{A}^p\left(F\backslash\mathring{A}\right)$. This implies that $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$ is a Banach space, so the theorem is proved. □

Theorem 4. *Let* $1 \leq p < \infty$. *Then*

$$\mathcal{A}^p\left(F\backslash\mathring{A}\right) = b^p(F)\,|_{F\backslash\mathring{A}} \oplus b^p\left(T\backslash\mathring{A}\right)|_{F\backslash\mathring{A}}.$$

Proof. From the definition of the space $\mathcal{A}^p\left(F\backslash\mathring{A}\right)$ it is obvious that $\mathcal{A}^p\left(F\backslash\mathring{A}\right) = b^p(F)\,|_{F\backslash\mathring{A}} + b^p\left(T\backslash\mathring{A}\right)|_{F\backslash\mathring{A}}$. So, we only need to prove that this sum is direct. Let $u \in b^p(F)\,|_{F\backslash\mathring{A}} \cap b^p\left(T\backslash\mathring{A}\right)|_{F\backslash\mathring{A}}$. So, there exists $s \in b^p\left(T\backslash\mathring{A}\right)$, $t \in b^p(F)$, such that $u = s = t$ on $F\backslash\mathring{A}$. Then a function defined as $\tilde{u}(x) = t(x)$ for $x \in F$ and $\tilde{u}(x) = s(x)$ for $x \in T\backslash\mathring{A}$. We obtain that $\tilde{u} \in b^p(T) = \{0\}$, so we get $b^p(F)\,|_{F\backslash\mathring{A}} \cap b^p\left(T\backslash\mathring{A}\right)|_{F\backslash\mathring{A}} = \{0\}$, so the sum is direct and the proof follows. $\square$

In [5] we remarked that it would be interesting to see when $\mathcal{A}^p(\Omega\backslash K) = b^p(\Omega\backslash K)$. We also proved that if K is one-element set, then the equality holds. That was the case for harmonic functions on Euclidean domains. If we consider the same problem in the context of infinite trees, equality never holds, as we prove in the the following theorem.

Theorem 5. *Let* $1 \leq p < \infty$. *Then* $\mathcal{A}^p\left(F\backslash\mathring{A}\right) \neq b^p\left(F\backslash\mathring{A}\right)$.

Proof. As harmonic functions on finite graphs are also integrable there, we obtain that $\mathcal{A}^p\left(F\backslash\mathring{A}\right) = b^p\left(F\backslash\mathring{A}\right)$ holds if and only if every harmonic function on $T\backslash\mathring{A}$ is also in $b^p\left(T\backslash\mathring{A}\right)$. This is not true because a harmonic function $u(x) = 1$ on $T\backslash\mathring{A}$ is not in $b^p\left(T\backslash\mathring{A}\right)$. $\square$

By this theorem we solved an analogous problem of Remark 2 in [5] in the case of harmonic functions on trees. A decomposition theorem also holds for polyharmonic functions, as we see from the following theorem in [1].

Theorem 6. *Let T be an infinite tree in which every non-terminal vertex has at least two non-terminal neighbhours. Let E be a finite connected set of T and $F = V(E)$. Let A be a non-empty subset of E. Suppose u is an m-harmonic function on $F\backslash\mathring{A}$. Then, there exist an m-harmonic function s on $T\backslash\mathring{A}$ and an m-harmonic function t on F such that $u = s + t$ on $F\backslash\mathring{A}$. This representation can be chosen to be unique up to an additive $(m-1)$-harmonic function on T if T has positive potentials; otherwise, the representation is unique up to an m-harmonic function generated by a constant.*

Remark 1. *The results we have obtained for harmonic functions on infinite trees we can also obtain for m-harmonic functions, so it would be interesting to consider an analogous problems for m-harmonic functions in this direction.*

3 Holomorphic Case

In this section we consider the following theorem (see [3]).

Theorem 7. *In the complex plane, let K be a compact set and ω be an open set such that $K \subset \omega$. Assume that there is an open disk D such that $K \subset D \subset \overline{D} \subset \omega$. Suppose f is analytic on $\omega \backslash K$. Then $f = f_1 + f_2$ on $\omega \backslash K$, where f_1 is analytic on ω and f_2 is analytic on $\mathbb{C} \backslash K$ tending to 0 at the point at infinity. This decomposition is unique.*

Let $1 \leq p < \infty$ and $A^p(\omega \backslash K)$ be the set of all $f \in B^p(\omega \backslash K)$ such that $f = f_1 + f_2$ on $\omega \backslash K$, where $f_1 \in B^p(\omega)$, $f_2 \in B^p(\mathbb{C} \backslash K)$.

Lemma 3. *Let $1 \leq p < \infty$ and $f \in A^p(\omega \backslash K)$ is arbitrarily chosen. Then*

$$\|f\|_{B^p(\omega \backslash K)} \leq 2^{\frac{p-1}{p}} \|f\|_{A^p(\omega \backslash K)}$$

Proof. The result follows from the following inequalities

$$\begin{aligned} \|f\|^p_{B^p(\omega \backslash K)} = \|f_1 + f_2\|^p_{B^p(\omega \backslash K)} &\leq \left(\|f_1\|_{B^p(\omega \backslash K)} + \|f_2\|_{B^p(\omega \backslash K)}\right)^p \\ &\leq 2^{p-1} \left(\|f_1\|^p_{B^p(\omega \backslash K)} + \|f_2\|^p_{B^p(\omega \backslash K)}\right) \\ &\leq 2^{p-1} \left(\|f_1\|^p_{B^p(\omega)} + \|f_2\|^p_{B^p(\mathbb{C} \backslash K)}\right) \\ &= 2^{p-1} \|f\|^p_{A^p(\omega \backslash K)}. \end{aligned}$$

□

Lemma 4. *Let $1 \leq p < \infty$, $n \geq 2$. If $f \in B^p(\mathbb{C} \backslash K)$, then $\lim_{z \to \infty} f(z) = 0$.*

Proof. If $f \in B^p(\mathbb{C} \backslash K)$, then $f = u + iv$, where $u, v \in b^p(\mathbb{R}^2 \backslash K)$. This implies $u(x, y) \to 0$, $v(x, y) \to 0$ as $(x, y) \to \infty$. Now, $|f(z)|^2 = |u(x, y)|^2 + |v(x, y)|^2 \to 0$, as $(x, y) \to \infty$ (see Lemma 2 in [5]), from where the result follows. □

Theorem 8. *$A^p(\omega \backslash K)$ is a Banach space under the norm defined by*

$$\|f\|^p_{A^p(\omega \backslash K)} = \|f_1\|^p_{B^p(\omega)} + \|f_2\|^p_{B^p(\mathbb{C} \backslash K)}$$

where $f = f_1 + f_2$ is a decomposition of f in $A^p(\omega \backslash K)$.

Proof. Let (u_m) be a Cauchy sequence in $A^p(\omega \backslash K)$. From the lemma 3 follows that (u_m) is a Cauchy sequence in $B^p(\omega \backslash K)$. $B^p(\omega \backslash K)$ is a Banach space, so there exists $u \in B^p(\omega \backslash K)$ such that $u_m \to u$ in $B^p(\omega \backslash K)$. Also,

$$\|u_m - u_k\|^p_{A^p(\omega \backslash K)} = \|v_m - v_k\|^p_{b^p(\omega)} + \|w_m - w_k\|^p_{b^p(\mathbb{C} \backslash K)},$$

where $u_m = v_m + w_m$ on $\omega \backslash K$ is a decomposition of u_m in $A^p(\omega \backslash K)$. Since (u_m) is a Cauchy sequence in $A^p(\omega \backslash K)$, it follows that (v_m) is a Cauchy sequence in $B^p(\omega)$ and (w_m) is a Cauchy sequence in $B^p(\mathbb{C} \backslash K)$. Since $B^p(\omega)$ and $B^p(\mathbb{C} \backslash K)$ are Banach spaces, it follows that there are $v \in B^p(\omega)$ and $w \in B^p(\mathbb{C} \backslash K)$ such that $v_m \to v$ in $B^p(\omega)$ and $w_m \to w$ in $B^p(\mathbb{C} \backslash K)$. Now define $u' = v + w$. By lemma 4, $u' \in A^p(\omega \backslash K)$. Now,

$$\|u_m - u'\|^p_{A^p(\omega \backslash K)} = \|v_m - v\|^p_{b^p(\omega)} + \|w_m - w\|^p_{b^p(\mathbb{C} \backslash K)},$$

so $u_m \to u'$ in $A^p(\omega \backslash K)$. Since $u_m \to u$ in $B^p(\omega \backslash K)$ and, by lemma 3, $u_m \to u'$ in $B^p(\omega \backslash K)$, it yields $u = u' \in A^p(\omega \backslash K)$. This implies that $A^p(\omega \backslash K)$ is a Banach space, so the theorem is proved. □

Theorem 9. *Let $1 \leq p < \infty$. Then $B^p(\omega)$ and $B^p(\mathbb{C} \backslash K)$ are closed subspaces of $A^p(\omega \backslash K)$.*

Proof. The topology on $B^p(\omega)$ is the same as the subspace topology from $A^p(\omega \backslash K)$. Because a subspace of a complete metric space is complete if and only if it is closed in subspace topology, it follows that $B^p(\omega)$ is closed in $A^p(\omega \backslash K)$. In the same way we prove that $B^p(\mathbb{C} \backslash K)$ is closed in $A^p(\omega \backslash K)$. □

Theorem 10. *Let $1 \leq p < \infty$. Then $A^p(\omega \backslash K)$ is a set of all holomorphic functions $f = u + iv$ on $\omega \backslash K$, where u and v are real valued and in $\mathcal{A}^p(\omega \backslash K)$.*

Proof. Let $f \in A^p(\omega \backslash K)$ be arbitrary. Then $f = f_1 + f_2$ on $\omega \backslash K$, where $f_1 \in B^p(\omega)$ and $f_2 \in B^p(\mathbb{C} \backslash K)$. Now, $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$, where $u_1, u_2 \in b^p(\omega)$, $v_1, v_2 \in b^p(\mathbb{R}^2 \backslash K)$. Now, $f = u + iv$, where $u = u_1 + u_2 \in A^p(\omega \backslash K)$, $v = v_1 + v_2 \in \mathcal{A}^p(\omega \backslash K)$. Another implication follows easily, so the theorem is proved. □

Theorem 11. *Let $1 \leq p < \infty$. Then*

$$A^p(\omega \backslash K) = B^p(\omega)|_{\omega \backslash K} \oplus B^p(\mathbb{C} \backslash K)|_{\omega \backslash K}$$

Proof. It follows immidiately from the definition of $A^p(\omega \backslash K)$ and from the fact that there is no non-zero function in $B^p(\mathbb{C})$. □

$A^2(\omega \backslash K)$ is a Hilbert space under the inner product defined by

$$\langle f_1, f_2 \rangle_{A^2(\omega \backslash K)} = \langle g_1, g_2 \rangle_{B^2(\omega)} + \langle h_1, h_2 \rangle_{B^2(\mathbb{C} \backslash K)},$$

where $f_1 = g_1 + h_1$ and $f_2 = g_2 + h_2$ are decompositions in $A^2(\omega \backslash K)$.

Theorem 12. *Let $f \in A^2(\omega \backslash K)$ and $z \in \omega \backslash K$ be arbitrarily chosen. Then*

$$|f(z)| \leq \frac{\sqrt{2}\|f\|_{A^2(\omega \backslash K)}}{\sqrt{V(B)}d(z, \partial(\omega \backslash K))^{n/2}}$$

Proof. The assertion follows directly from Theorem 2 in [5] when applied on real and imaginary part of f. $\square$

Let $z \in \omega \backslash K$ be arbitrarily chosen. Then a mapping $f \mapsto f(z)$ is a bounded linear functional on $A^2(\omega \backslash K)$. It follows that there exist $H_{\omega \backslash K}(z, \cdot) \in A^2(\omega \backslash K)$ such that $f(z) = \langle f, H_{\omega \backslash K}(z, \cdot) \rangle_{A^2(\omega \backslash K)}$.

Theorem 13. *It holds that*

$$H_{\omega \backslash K}(z, \cdot) = K_\omega(z, \cdot) + K_{\mathbb{C} \backslash K}(z, \cdot)$$

on $\omega \backslash K$, where K_ω and $K_{\mathbb{C} \backslash K}$ are Bergman kernels on ω and $\mathbb{C} \backslash K$, respectively.

Proof. Let $z \in \omega \backslash K$ be arbitrarily chosen. As $H_{\omega \backslash K}(z, \cdot) \in A^2(\omega \backslash K)$, there exists unique $V_\omega(z, \cdot) \in B^2(\omega)$ and $W_{\mathbb{C} \backslash K}(z, \cdot) \in B^2(\mathbb{C} \backslash K)$ such that $H_{\omega \backslash K}(z, \cdot) = V_\omega(z, \cdot) + W_{\mathbb{C} \backslash K}(z, \cdot)$ on $\omega \backslash K$. It follows that

$$f(z) = \int_\omega g(w) \overline{V_\omega(z, w)} dw + \int_{\mathbb{C} \backslash K} h(w) \overline{W_{\mathbb{C} \backslash K}(z, w)} dw,$$

where $f = g + h$ is a decomposition of $f \in A^2(\omega \backslash K)$. By the fact that $b^2(\omega) \subseteq A^2(\omega \backslash K)$ and $b^2(\mathbb{C} \backslash K) \subseteq A^2(\omega \backslash K)$ it follows that $V_\omega(z, \cdot) = K_\omega(z, \cdot)$ on ω and $W_{\mathbb{C} \backslash K}(z, \cdot) = K_{\mathbb{C} \backslash K}(z, \cdot)$ on $\mathbb{C} \backslash K$, from where the assertion follows. $\square$

Now we will mention some open problems analogous to those given in [5].

1. Find all triples (p, ω, K) such that $A^2(\omega \backslash K) = B^2(\omega \backslash K)$.
2. If $A^2(\omega \backslash K) = B^2(\omega \backslash K)$, then $K_{\omega \backslash K}(z, \cdot) \in A^2(\omega \backslash K)$, so it would be interesting to find a relation between $H_{\omega \backslash K}(z, \cdot)$ and $K_{\omega \backslash K}(z, \cdot)$ for a given $z \in \omega \backslash K$.

4 A New Type of Regularity for Distributions

Definition 1. *For $u \in h(\Omega \backslash K)$, we define $T^u_{\Omega,K} : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ as*

$$T^u_{\Omega,K}(\phi) = \int_\Omega v(x) \phi(x) dx + \int_{\mathbb{R}^n \backslash K} w(x) \phi(x) dx$$

for all $\phi \in \mathcal{D}(\mathbb{R}^n)$, where $u = v + w$ is a decomposition of u.

Theorem 14. *$T^u_{\Omega,K}$ is a distribution of order zero on $\mathbb{R}^n$.*

Proof. Linearity is obvious. Let $K' \subset\subset \mathbb{R}^n$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $supp\,(\phi) \subset K'$ be arbitrarily chosen. We now have

$$\begin{aligned}
&|T^u_{\Omega,K}(\phi)| = |\int_{\Omega} v(x)\,\phi(x)\,dx + \int_{\mathbb{R}^n\setminus K} w(x)\,\phi(x)\,dx| \\
&\leq \int_{\Omega} |v(x)||\phi(x)|dx + \int_{\mathbb{R}^n\setminus K} |w(x)||\phi(x)|dx \\
&= \int_{K'\cap\Omega} |v(x)||\phi(x)|dx + \int_{K'\cap(\mathbb{R}^n\setminus K)} |w(x)||\phi(x)|dx \\
&\leq \max_{x\in K'\cap\Omega} |\phi(x)| \int_{K'\cap\Omega} |v(x)|dx + \max_{x\in K'\cap(\mathbb{R}^n\setminus K)} |\phi(x)| \int_{K'\cap(\mathbb{R}^n\setminus K)} |w(x)|dx \\
&\leq \max\left\{C^1_{K'}, C^2_{K'}\right\}\left(\max_{x\in K'\cap\Omega} |\phi(x)| + \max_{x\in K'\cap(\mathbb{R}^n\setminus K)} |\phi(x)|\right) \\
&\leq 2\max\left\{C^1_{K'}, C^2_{K'}\right\} \max_{x\in K'} |\phi(x)| \\
&= C_{K'} \max_{x\in K'} |\phi(x)|.
\end{aligned}$$

Here, $C_{K'} = 2\max\{C^1_{K'}, C^2_{K'}\}$ and
$C^1_{K'} = \int_{K'\cap\Omega} |v(x)|dx$, $C^2_{K'} = \int_{K'\cap(\mathbb{R}^n\setminus K)} |w(x)|dx$. By using Proposition 1.3.1 from [4] we finish the proof. □

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset\subset \Omega$. We say a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is (Ω, K)-regular if there exists $u \in h(\Omega\setminus K)$ such that $T = T^u_{\Omega,K}$ in $\mathcal{D}'(\mathbb{R}^n)$. Otherwise we say it is (Ω, K)-singular.*

Definition 3. *For a distribution $T \in \mathcal{D}(\mathbb{R}^n)$ we say it is h-regular if it is (Ω, K)-regular for some open set $\Omega \subset \mathbb{R}^n$ and a compact set $K \subset\subset \Omega$. Otherwise we say it is h-singular.*

If $u \in h(\mathbb{R}^n)$, then for every open set $\Omega \subset \mathbb{R}^n$ and every $K \subset\subset \Omega$, $u = v + w$ on $\Omega\setminus K$, where $v = u$ on Ω and $w = 0$ on $\mathbb{R}^n\setminus K$. So, in this case we obtain

$$T^u_{\Omega,K}(\phi) = \int_{\Omega} u(x)\,\phi(x)\,dx.$$

In this case $T^u_{\Omega,K}$ doesn't depend on K, so we denote it by T^u_{Ω}.

Definition 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is Ω-regular if there exists $u \in h(\mathbb{R}^n)$ such that $T = T^u_{\Omega}$ in $\mathcal{D}'(\mathbb{R}^n)$. Otherwise we say it is Ω-singular.*

Definition 5. *We say that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is m-regular if it is Ω-regular for some open set $\Omega \subset \mathbb{R}^n$. Otherwise we say it is m-singular.*

In the theory of distributions we know that a distribution T is called regular if there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that $T = T_f$ in $\mathcal{D}'(\mathbb{R}^n)$. Here T_f is defined by

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx$$

for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. In the next theorem we will see that every h-regular distribution is regular.

Theorem 15. *Every h-regular distribution is regular.*

Proof. Let T be an arbitrarily chosen h-regular distribution. There exists an open set $\Omega \subset \mathbb{R}^n$, $K \subset\subset \Omega$ and $u \in h(\Omega \backslash K)$ such that $T = T^u_{\Omega,K}$. We have

$$T(\phi) = \int_{\Omega} v(x)\,\phi(x)\,dx + \int_{\mathbb{R}\backslash K} w(x)\,\phi(x)\,dx = T_g(\phi) + T_h(\phi),$$

where $g(x) = v(x)$ on Ω and zero outside Ω, and $h(x) = w(x)$ on $\mathbb{R}^n \backslash K$ and zero on K. This implies $T = T_{g+h}$. Since g and h are in $L^1_{loc}(\mathbb{R}^n)$, so is $g + h$. This implies that T is a regular distribution in $\mathcal{D}'(\mathbb{R}^n)$. □

From the proof of the theorem we see that $T^u_{\Omega,K} = T_f$, where $f = u$ on $\Omega \backslash K$, $f = v$ on K and $f = w$ on $\mathbb{R}^n \backslash \Omega$. This is the motivation for introducing the following definition.

Definition 6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset\subset \Omega$. We say a function $f \in L^1_{loc}(\mathbb{R}^n)$ is (Ω, K)-harmonic if there exists $u \in h(\Omega \backslash K)$ such that $f = u$ on $\Omega \backslash K$, $f = v$ on K and $f = w$ on $\mathbb{R}^n \backslash \Omega$. We use notation $f = u\chi_{\Omega\backslash K} + v\chi_K + w\chi_{\mathbb{R}^n\backslash\Omega}$ for the function of this form.*

The set of all (Ω, K)-harmonic functions on $\mathbb{R}^n$ is a vector subspace of $L^1_{loc}(\mathbb{R}^n)$. We denote it by $H(\Omega, K)$.

Corollary 1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then T_f is (Ω, K)-regular if and only if $f \in H(\Omega, K)$.*

Theorem 16. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset\subset \Omega$. If $u_1, u_2 \in h(\Omega \backslash K)$ are such that $T^{u_1}_{\Omega,K} = T^{u_2}_{\Omega,K}$ in $\mathcal{D}'(\mathbb{R}^n)$, then $u_1 = u_2$ on $\Omega \backslash K$.*

Proof. Let $u_1 = v_1 + w_1$ and $u_2 = v_2 + w_2$ be decompositions of u_1 and u_2. Then $T^{u_1}_{\Omega,K} = T_{f_1}$, $T^{u_2}_{\Omega,K} = T_{f_2}$ implies $T_{f_1} = T_{f_2}$ from where we get $f_1 = f_2$ in $L^1_{loc}(\mathbb{R}^n)$. Here, $f_1 = u_1\chi_{\Omega\backslash K} + v_1\chi_K + w_1\chi_{\mathbb{R}^n\backslash\Omega}$ and $f_2 = u_2\chi_{\Omega\backslash K} + v_2\chi_K + w_2\chi_{\mathbb{R}^n\backslash\Omega}$. From this we get $u_1 = u_2$ a.e. on $\Omega \backslash K$ and since both are harmonic on $\Omega \backslash K$, it holds that $u_1 = u_2$ on $\Omega \backslash K$ so the proof is finished. □

Theorem 17. *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be an open set and $K \subset\subset \Omega$. Let $u \in \mathcal{A}^p(\Omega \backslash K)$ be arbitrarily chosen. If $f = u\chi_{\Omega\backslash K} + v\chi_K + w\chi_{\mathbb{R}^n\backslash\Omega} \in H(\Omega, K)$ is harmonic on $\mathbb{R}^n$, then u must be zero function on $\Omega \backslash K$.*

Proof. If $u \in \mathcal{A}^p(\Omega \backslash K)$ then $u \in L^p(\Omega \backslash K)$, $v \in L^p(\Omega)$ and $w \in L^p(\mathbb{R}^n \backslash K)$. From this we conclude that $f \in L^p(\mathbb{R}^n)$. If we suppose that f is harmonic on $\mathbb{R}^n$, then we now have $f \in b^p(\mathbb{R}^n) = \{0\}$, so $f = 0$ on $\mathbb{R}^n$. This implies that $u = 0$ on $\Omega \backslash K$, so the proof is finished. $\square$

Remark 2. *We see that functions $f \in H(\Omega, K)$ are not harmonic in general but they are in some sense piecewise harmonic. This is the reason why we take the name (Ω, K)-harmonic for the definition of this functions.*

Definition 7. *Let $1 \leq p < \infty$. We define a set $H^p(\Omega, K)$ to be the set of all functions $f \in H(\Omega, K)$, $f = u\chi_{\Omega \backslash K} + v\chi_K + w\chi_{\mathbb{R}^n \backslash \Omega}$, such that $u \in \mathcal{A}^p(\Omega \backslash K)$.*

Theorem 18. *Let $1 \leq p < \infty$. It holds that $H^p(\Omega, K) \subseteq H(\Omega, K) \cap L^p(\mathbb{R}^n)$.*

Proof. The proof immidiately follows by using definitions of $H^p(\Omega, K)$ and $\mathcal{A}^p(\Omega, K)$. $\square$

We would like to know when equality holds, i.e. when is $L^p(\mathbb{R}^n) \cap H(\Omega, K) = H^p(\Omega, K)$? For that purpose we introduce the following definition.

Definition 8. *Let $1 \leq p < \infty$. We define $\mathcal{B}^p(\Omega \backslash K)$ to be the set of all $u \in b^p(\Omega \backslash K)$ such that $w \in L^p(\mathbb{R}^n \backslash \Omega)$ in the decomposition $u = v + w$ of u.*

Definition 9. *Let $1 \leq p < \infty$. We define $\mathcal{M}^p(\Omega \backslash K)$ to be the set of all functions $u \in b^p(\Omega \backslash K)$ such that $v \in L^p(\Omega \backslash K)$ and $w \in L^p(\Omega \backslash K)$ in the decomposition $u = v + w$ of u.*

Theorem 19. *Let $1 \leq p < \infty$. Then the following assertions are equivalent*
(i) $H^p(\Omega, K) = H(\Omega, K) \cap L^p(\mathbb{R}^n)$
(ii) $\mathcal{A}^p(\Omega \backslash K) = \mathcal{B}^p(\Omega \backslash K)$
(iii) $\mathcal{B}^p(\Omega \backslash K) \subseteq \mathcal{M}^p(\Omega \backslash K)$.

Proof. $(i) \Rightarrow (ii)$. Suppose (i) holds. It is obvious that $\mathcal{A}^p(\Omega \backslash K) \subseteq \mathcal{B}^p(\Omega \backslash K)$. Let $u \in \mathcal{B}^p(\Omega \backslash K)$ be arbitrarily chosen. Then $u = v + w$, where $u \in b^p(\Omega \backslash K)$ and $w \in L^p(\mathbb{R}^n \backslash \Omega)$. This implies that $f = u\chi_{\Omega \backslash K} + v\chi_K + w\chi_{\mathbb{R}^n \backslash \Omega}$ belongs to $L^p(\mathbb{R}^n) \cap H(\Omega, K)$, which is by (i) equal to $H^p(\Omega, K)$. So, $u \in \mathcal{A}^p(\Omega \backslash K)$ and we proved that $\mathcal{B}^p(\Omega \backslash K) \subseteq \mathcal{A}^p(\Omega \backslash K)$, so (ii) holds.
$(ii) \Rightarrow (iii)$. Suppose $u \in \mathcal{B}^p(\Omega \backslash K)$ is arbirarily chosen. Then by (ii) we have $u \in \mathcal{A}^p(\Omega \backslash K)$ and this obviously implies $u \in \mathcal{M}^p(\Omega \backslash K)$, so (iii) holds.
$(iii) \Rightarrow (i)$. We only need to prove that $L^p(\mathbb{R}^n) \cap H(\Omega, K) \subseteq H^p(\Omega, K)$. Let $f = u\chi_{\Omega \backslash K} + v\chi_K + w\chi_{\mathbb{R}^n \backslash \Omega}$ in $L^p(\mathbb{R}^n)$ be arbitrarily chosen. From this we have $u \in b^p(\Omega \backslash K)$ and $w \in Ł^p(\mathbb{R}^n \backslash \Omega)$, i.e. $u \in \mathcal{B}^p(\Omega \backslash K)$ and by (iii) it holds that $u \in \mathcal{M}^p(\Omega \backslash K)$. This implies that $v \in L^p(\Omega \backslash K)$ and $w \in L^p(\Omega \backslash K)$. But, we know that $w \in L^p(\mathbb{R}^n \backslash \Omega)$ which implies $v \in L^p(\Omega)$ and $w \in L^p(\mathbb{R}^n \backslash K)$ i.e. $u \in \mathcal{A}^p(\Omega \backslash K)$. This implies $f \in H^p(\Omega, K)$, so (i) holds. So, the theorem is proved. $\square$

Definition 10. *Let $1 \leq p < \infty$. Let $u \in \mathcal{A}^p(\Omega \backslash K)$. We can extend $T^u_{\Omega,K}$ on $L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$, by*

$$T^u_{\Omega,K}(f) = \int_{\Omega} v(x) f(x)\, dx + \int_{\mathbb{R}^n \backslash K} w(x) f(x)\, dx$$

for all $f \in L^q(\mathbb{R}^n)$.

Theorem 20. *Let $1 < p < \infty$. Then $T^u_{\Omega,K}$ is a bounded linear functional on $L^q(\mathbb{R}^n)$ whose norm is equal to $\|f\|_p$, where $f = u\chi_{\Omega\backslash K} + v\chi_K + w\chi_{\mathbb{R}^n\backslash\Omega}$.*

Proof. We easily get that

$$T^u_{\Omega,K}(g) = \int_{\mathbb{R}^n} f(x)\, g(x)\, dx,$$

so $T^u_{\Omega,K} = T_f$, from where the assertion follows. $\square$

Remark 3. *This new type of regularity could be considered in the case of parabolic PDE's because the decomposition theorem holds in that case also (see [6]). This is a new tool that gives a new approach to the problems related to the boundary value problems for parabolic PDE's on domains of the form $\Omega \backslash K$.*

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