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Injective Chromatic Sum and Injective Chromatic Polynomials of Graphs

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Abstract

The injective chromatic number $\chi_i(G)$ [5] of a graph G is the minimum number of colors needed to color the vertices of G such that two vertices with a common neighbor are assigned distinct colors. In this paper we define injective chromatic sum and injective strength of a graph and obtain the injective chromatic sum of complete graph, paths, cycles, wheel graph and complete bipartite graph. We also suggest bounds for injective chromatic sum. The injective chromatic sum of graph complements, join, union, product and corona is discussed.The concept of injective chromatic polynomial is introduced and computed for complete graphs, bipartite graphs, cycles etc. The bounds for the injective chromatic polynomial of trees is suggested.

Keywords: *injective chromatic number; chromatic sum; injective chromatic sum; injective strength; injective chromatic polynomial.*

1 Introduction

Graph theory is one of the most popular areas of research. Many research papers like [1] are available in literature. The concept of injective coloring and injective chromatic number $\chi_i(G)$ is introduced by Hahn et.al[5]. The chromatic sum $\Sigma(G)$ and strength $s(G)$ are introduced by Ewa Kubicka[7]. Several papers [8],[6],[9] are available in literature based on these concepts separately. In this paper we introduce the concepts of injective chromatic sum and injective strength of a graph. The injective chromatic sum and the injective strength of complete graphs, paths, cycles, wheels and complete bipartite graphs are studied. We also suggest bounds for injective chromatic sum of connected graphs.

In section 5,we study the injective chromatic sum of complements, join, union, product and corona of graphs . We also suggest bounds for the injective chromatic sum of operations of graphs.
In this paper we introduce the concept of injective chromatic polynomial. We also compute injective chromatic polynomial of trees.

2 Preliminaries

The chromatic sum of a graph G is defined as the smallest sum among all proper colorings of G with natural numbers and is denoted by $\Sigma(G)$ and the strength of a graph G, denoted as $s(G)$ is the minimum number of colors required to obtain the chromatic sum [7],[8].

A vertex k coloring such that two vertices having a common neighbor have distinct colors is defined as injective k coloring and the minimum number k such that G has an injective k coloring is defined as the injective chromatic number denoted as $\chi_i(G)$ [5]. The other basic concepts and notations are taken from [2] and [4].

3 Injective Chromatic Sum

The coloring of a graph such that vertices with common neighbor receive distinct colors can be done in many ways. Here we suggest injective coloring of a graph by assigning natural numbers to vertices such that 1 occurs maximum number of times, then 2, then 3 and so on which leads to the following definition.

Definition 3.1. *The injective chromatic sum of a graph G, denoted as $\Sigma_i(G)$ is the smallest sum of colors among all injective colorings with natural numbers. i.e. $\Sigma_i(G) = min \ \{\Sigma_i^k(G) : k \geq \chi_i(G)\}$, where $\Sigma_i^k(G)$ is the smallest possible sum among all proper k- injective colorings of G using natural numbers.*

Definition 3.2. *The injective strength of a graph G is the smallest number s such that $\Sigma_i(G) = \Sigma_i^s(G)$ and is denoted as $s_i(G)$ or s_i.*

Example 3.3. Consider the graph in Fig I. Injective coloring of the graph using numbers 1,2 and 3 yields $\Sigma_i(G) = 10$. Also $s_i(G) = 3$.

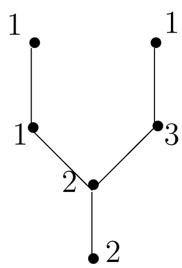

Fig I

4 Bounds for Injective Chromatic Sum

In this section we establish a few results for injective chromatic sum. We begin with an obvious result that the injective chromatic sum of a connected graph is greater than or equal to its chromatic sum. It is obvious since in injective coloring we impose the additional restriction that the vertices with common neighbor must be assigned different colors. Also injective chromatic sum of complete graphs, paths, cycles, wheels and complete bipartite graphs are obtained.

Proposition 4.1. *For a connected graph G except K_2, $\Sigma_i(G) \geq \Sigma(G)$.*

Next we give theorem for complete graphs.

Theorem 4.1. $s_i(G) = s(G) = \chi(G) = \chi_i(G) = n$ *if* $G = K_n$ *where* $n > 2$ *and*
$\Sigma_i(K_n) = \Sigma(K_n) = \frac{n(n+1)}{2}$.

Proof. If $G = K_n$, each vertex v_i is joined to $n-1$ vertices and hence to color these neighbors, at least $n-1$ colors are required. Now the vertex v_i is to be assigned a different color since v_i and one of the colored vertex (say) v_j have common neighbors for each $j = 1, 2..., n-1$. Hence it follows that $s_i(G)$ $= s(G) = \chi(G) = \chi_i(G) = n$. The injective chromatic sum is obtained by assigning colors $1, 2, 3...n$ and hence the sum is $\frac{n(n+1)}{2}$.

Theorem 4.2. $s_i(P_n) = 2$ *and* $\Sigma_i(P_n) = \begin{cases} \frac{(3n-1)}{2} & , n \ odd \\ \frac{3n}{2} & , n \ even, n \equiv 0(mod4) \\ \frac{(3n-2)}{2} & , n \ even, \not\equiv 0(mod4) \end{cases}$

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of the path P_n.
Case I: If n is odd, the vertices v_j and v_{j+1} are colored 1 where $j = 1, 5, 9...$

and are colored 2, for $j = 3, 7, 11....$ Hence for any odd n, $\frac{n+1}{2}$ vertices are assigned the number 1 and $\frac{n-1}{2}$ vertices are assigned the number 2. Hence the injective chromatic sum is $\frac{n+1}{2} + 2\frac{(n-1)}{2} = \frac{(3n-1)}{2}$.
Case II: If n is even and $n \equiv 0(mod4)$, say $n = 4k$, then assigning colors as in case I, exactly $2k$ vertices are assigned color 1 and the remaining $2k$ vertices are assigned color 2. Hence $\Sigma_i(G) = 1.\frac{n}{2} + 2.\frac{n}{2} = \frac{3n}{2}$.
Case III: If n is even and $n \not\equiv 0(mod4)$, then $\frac{n}{2}$ is odd. Taking two copies of $P_{\frac{n}{2}}$ and join the last vertex of the first copy to the first vertex of the second copy and assigning the colors as in case I, we get the injective chromatic sum as $2[\frac{(n/2+1)}{2} + 2\frac{(n/2-1)}{2}] = \frac{(3n-2)}{2}$.

Now we establish the following result for cycles.

Theorem 4.3. *If n is even, then* $s_i(C_n) = \begin{cases} 3 & ,n \ odd \\ 2 & ,n \ even, n \equiv 0(mod4) \\ 3 & ,n \ even, n \not\equiv 0(mod4) \end{cases}$

and $\Sigma_i(C_n) = \begin{cases} \lfloor\frac{n}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 3 & ,n \ odd \\ \frac{3n}{2} & ,n \ even, n \equiv 0(mod4) \\ \lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n-1}{2}\rfloor + 6 & ,n \ even, n \not\equiv 0(mod4) \end{cases}$

Proof. We know, $P_n = v_1v_2...v_n$ and $C_n = v_1v_2...v_nv_1$. i.e. there exists an edge v_nv_1 in C_n.
Case I: If n is odd, by theorem 4.2, v_n is either of color 1 or 2. But since v_nv_1 is an edge in C_n, if the color of v_n is 1, v_1 becomes the common neighbor of v_n and v_2, both of color 1. Hence the vertex v_n is colored 3. If the color of v_n is 2, v_n becomes the common neighbor of v_{n-1} and v_1 which are both of color 1 and hence the color of v_{n-1} is changed to 3. Hence in both cases, $s_i(G) = 3$ and the chromatic sum $\Sigma_i(C_n) = \lfloor\frac{n}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 3$.
Case II: If n is even and $n \equiv 0(mod4)$, then the two vertices v_n and v_{n-1} of P_n are colored with 2 and hence if we add an edge v_nv_1, the same colors can be retained and hence the proof.
Case III: For n even and $n \not\equiv 0(mod4)$, the last two vertices v_n and v_{n-1} are colored with 1 and v_n is the common neighbor of v_1 and v_{n-1}, and v_1 is the common neighbor of v_2 and v_n, both v_{n-1} and v_n has to be assigned color 3 which gives $\Sigma_i(C_n) = \lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n-1}{2}\rfloor + 6$. Hence the proof.

Now we obtain the following result for wheel graph W_n.

Theorem 4.4. $s_i(W_n) = n$ *and* $\Sigma_i(W_n) = \frac{n(n+1)}{2}$.

Proof. In the wheel graph since all the outer vertices v_j for $j = 1, 2..., n-1$ are neighbors of the central vertex v_n, they must be assigned different colors in injective coloring. Hence we use $1, 2, 3...n-1$ to color the vertices v_j for $j = 1, 2..., n-1$. Now the central vertex v_n must be assigned the color n since one of the outer vertices v_j and the central vertex v_n have common neighbors.

Hence the minimum sum is obtained by using $1, 2...n$ to color all the vertices of the wheel graph and hence the proof.

Theorem 4.5. *$s_i(K_{m,n}) = max\ \{m, n\}$ and $\Sigma_i(K_{m,n}) = \frac{n(n+1)}{2} + \frac{m(m+1)}{2}$.*

Proof. Since in complete bipartite graph with vertex set $V = V_1 \bigcup V_2$ where $|V_1| = m$ and $|V_2| = n$, every vertex in V_1 is adjacent to all vertices in V_2 and vice versa, and by coloring all the vertices of V_1 with $1, 2, ..., m$ and the vertices of V_2 with $1, 2, ..., n$, the minimum number required for injective coloring is the maximum of m and n and hence the result. The injective chromatic sum is obvious since the graph is complete bipartite.

The result for star graph follows from the above theorem.

Corollary 4.6. *$s_i(K_{1,n}) = n$ and $\Sigma_i(K_{1,n}) = 1+ \frac{n(n+1)}{2}$.*

For any graph G with n vertices and e edges, the chromatic sum is bounded by $\Sigma(G) \leq n + e$ [9]. The following bounds can be obtained for injective chromatic sum.

Theorem 4.7. *For every connected graph G , the injective chromatic sum is bounded by $1+ \frac{\Delta(\Delta+1)}{2} \leq \Sigma_i(G) \leq \frac{n(n+1)}{2}$. Also $\Sigma_i(G) = 1+ \frac{\Delta(\Delta+1)}{2}$ if G is a star graph and $\Sigma_i(G) = \frac{n(n+1)}{2}$ if and only if (i) $d(G) \leq 2$ where $d(G)$ is the diameter of G and (ii) every edge of G lies in a triangle.*

Proof. Consider a connected graph G with maximum degree Δ. Hence for injective coloring, minimum Δ colors are required. Since maximum degree is Δ, the graph has at least $1+\Delta$ vertices and hence we use the numbers 1,2,3...Δ for injective coloring and 1 is used again to color the vertex with degree Δ. Hence the injective chromatic sum is at least $1+ \frac{\Delta(\Delta+1)}{2}$.
Now consider the colors 1,2,3...n used for injective coloring of a graph of order n. This is the maximum possible number of colors which can be used in injective coloring of a graph G of order n and hence the upper bound.
Next assume that $G = K_{1,n}$. Then for injective coloring of the vertices such that the sum is minimized, the numbers 1,2,3...Δ is used and the common vertex is colored 1. Hence $\Sigma_i(G) = 1+ \frac{\Delta(\Delta+1)}{2}$.
Now to prove $\Sigma_i(G) = \frac{n(n+1)}{2}$ if and only if (i) $d(G) \leq 2$ where $d(G)$ is the diameter of G and (ii) every edge of G lies in a triangle.
Assume $\Sigma_i(G) = \frac{n(n+1)}{2}$.
To prove (i), assume $d(G) > 2$. Then there exists at least two vertices u and v in G having color 1(since we aim for minimum sum). Hence at the most $n-1$ colors 1,2,...$n-1$ are required for injective coloring. Hence $\Sigma_i(G) < \frac{n(n+1)}{2}$ which is a contradiction. Hence (i) is satisfied.
Now assume $\Sigma_i(G) = \frac{n(n+1)}{2}$ and $d(G) \leq 2$ by (i). To prove (ii), assume at least one edge is not in a triangle.

Claim: $\exists$ at least two vertices of same color.
proof: Since one edge say uv is not in a triangle, both u and v can have the same color since they don't have any common neighbor.
Hence $\exists$ at least two vertices in G having color 1 and $\Sigma_i(G) < \frac{n(n+1)}{2}$, a contradiction. Hence both (i) and (ii) holds.
Conversely, assume (i) and (ii) holds. Also assume $\Sigma_i(G) \neq \frac{n(n+1)}{2}$. Then $\exists$ vertices x, y and z in G such that xyz is a triangle (by (ii)) and $c(x) = c(z)$ (where $c : V \to \mathbf{N}$ is the coloring function to the set of natural numbers) since $\Sigma_i(G) < \frac{n(n+1)}{2} \Rightarrow x$ and z have same color. But x and z have a common neighbor y which contradicts the concept of injective coloring. Also by (i) no two vertices are at a distance more than 2 and this also restricts the repetition of colors. Hence $\Sigma_i(G) = \frac{n(n+1)}{2}$.

Remark 4.8. The upper bound in Theorem 4.7 is attained by K_n and W_n (Theorems 4.1 and 4.4)

5 Operations on Graphs

In this section we find the relationship between the injective chromatic sum of the constituent graphs and their resultant graph after performing operations like complement, join, union, product and corona.

First we establish the results for complement of graphs. As we know, the complement $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ is the graph with vertex set V such that $uv \in \overline{E}$ if and only if $uv \notin E$.

Theorem 5.1. *For any graph G on n vertices,*
$\Sigma_i(G) + \Sigma_i(\overline{G}) \geq \frac{n(n+1)}{2}$.
The lower bound holds for C_4 and P_3 .

Proof. Let $1, 2, 3, ..., k$ are used for injective coloring of G . Then $1, 2, 3, ..., n$ are the integers required for injective coloring of G and its complement together since for all other graphs except C_4 and P_3, the integers are repeated and hence the sum becomes larger. Hence the proof.

Theorem 5.2. $\Sigma_i(G) = \Sigma_i(\overline{G})$ *if and only if G is self complementary or C_6.*

Proof. The result is obvious for self complementary graphs. Now consider C_6. For C_6 and $\overline{C_6}$, the colors 1,2 and 3 are used twice for injective coloring and hence $\Sigma_i(G) = \Sigma_i(\overline{G})$.
Now assume $\Sigma_i(G) = \Sigma_i(\overline{G})$. This implies that the vertices of G and $\overline{G}$ are assigned injective coloring with the same integers each integer being used the same number of times. This is possible only if G and $\overline{G}$ are isomorphic or if G is C_6. For no other graph, the same colors repeats the same number of times. Hence the proof.

The join of G_1 and G_2, denoted as $G_1 + G_2$ consists of vertex set $V_1 \cup V_2$, edge set $E_1 \cup E_2 \cup \{xy : x \in V_1, y \in V_2\}$.

Lemma 5.3. *If G_1 and G_2 are connected, then the join G_1+G_2 is triangulated and has diameter 2.*

Proof. Note that there exists a path of length atmost 2 between every pair of vertices of $G_1 + G_2$. A vertex of V_2 is a common vertex of two vertices of V_1 and vice versa and hence every edge of $G_1 + G_2$ lies in a triangle.

Theorem 5.4. $\chi_i(G_1 + G_2) = |V(G_1)| + |V(G_2)|$ *and* $\Sigma_i(G_1 + G_2) = \frac{m(m+1)}{2}$, *where* $m = |V(G_1)| + |V(G_2)|$.

Proof. The theorem follows from Lemma 4 of [5]. Since every edge lies in a triangle, the number of colors required for injective coloring is equal to $|V(G_1+G_2)| = |V(G_1)| + |V(G_2)|$. The injective chromatic sum follows from Theorem 4.7. Hence the proof.

The following are the results obtained for product of graphs. The product of G_1 and G_2, denoted by $G_1 \times G_2$ has vertex set $V_1 \times V_2$ with $(u, x)(v, y) \in E(G_1 \times G_2)$ if either $x = y$ and $uv \in E_1$, or if $u = v$ and $xy \in E_2$.

By Lemma 8 [5], if G_1 and G_2 are connected graphs both distinct from K_2, $\chi_i(G_1 \times G_2) \leq \chi_i(G_1)\chi_i(G_2)$. But the inequality is not the same for injective chromatic sum. We have also obtained a different bound for $\chi_i(G_1 \times G_2)$.

Lemma 5.5. $\Delta(G_1 \times G_2) = \Delta_1 + \Delta_2$ *where $\Delta_1$is the maximum degree of G_1 and $\Delta_2$is the maximum degree of G_2.*

Proof. By the construction of $G_1 \times G_2$, each vertex say u of G_1 is paired with every vertex of G_2 and the corresponding edges are drawn such that $u_2 = v_2$ and u_1 is adjacent to v_1 for the vertices (u_1, u_2),(v_1, v_2). Let u be the vertex of maximum degreeΔ_1 in G_1. u is paired with every vertex of G_2. Let v be the vertex of maximum degree Δ_2 in G_2. u is paired with every vertex of $G_2 \Rightarrow (u, v)$ is the vertex of degree Δ_2. Since u is of maximum degree Δ_1 in G_1, u is adjacent with Δ_1 other vertices of G_1 and hence all the nodes (u, v_i) are joined to the corresponding node (u, v) in $G_1 \times G_2$. Hence there are Δ_1 more edges from (u, v). Hence the total degree of (u, v) is $\Delta_1 + \Delta_2$. Hence the proof.

Theorem 5.6. *If G and H are connected graphs with $G = (p_1, q_1)$ and $H = (p_2, q_2)$ such that Δ_1 and Δ_2 are the maximum degrees of G and H respectively, then $\Delta_1 + \Delta_2 \leq \chi_i(G \times H) \leq p_1p_2$. The upper bound is attained if(i) $G$and H are complete graphs, or (ii)$G \times H$ is triangulated with diameter 2.The lower bound is attained by $G = K_2$ and $H = P_n$ for $n \geq 3$.*

Proof. We know $\chi_i(G) \geq \Delta(G)$ [5]. Hence $\chi_i(G \times H) \geq \Delta(G \times H)$. i.e $\chi_i(G \times H) \geq \Delta_1 + \Delta_2$. The maximum number of colors required for injective coloring of G is p_1 and for H is p_2. By the construction of $G \times H$, the maximum number of colors required for injective coloring is p_1p_2. Hence the upper bound.

The cube graphs are bipartite graphs which have a significant role in coding theory. The first cube graph Q_1 is same as K_2, second cube graph Q_2 is $Q_1 \times K_2$, third Q_3 is $Q_2 \times K_2$ and so on.

Theorem 5.7. $\Sigma_i(G_1 \times G_2) \leq \Sigma_i(G_1)\Sigma_i(G_2)$ *, if either* G_1 *or* G_2 *is a triangle.* $\Sigma_i(G_1 \times G_2) > \Sigma_i(G_1)\Sigma_i(G_2)$*, for cubes.*

Proof. If either G_1 or G_2 is a triangle, then the maximum degree of the product will be one more then the largest of $\Delta(G_1)$ and $\Delta(G_2)$. Without loss of generality assume that $\Delta(G_1) > \Delta(G_2)$. Also assume G_1 is triangle. Hence we require $1, 2, 3, ..., \Delta(G_1)$ for the injective coloring of the product graph with each color repeating k times where k is the number of vertices of the graph G_2. Hence definitely $k + 2k + 3k... + \Delta(G_1)k \leq \Sigma_i(G_1)\Sigma_i(G_2)$.
But in the case of cubes, the integers used for injective coloring are repeated more number of times and hence the result.

The upper bound is sharp. For example it holds for $K_m \times K_2$, $m \geq 3$.

We establish the following result analogous to Theorem 9 [5].

Theorem 5.8. $\Sigma_i(Q_n) = 1.n + 2.n + ...n.n = \frac{n^2(n+1)}{2}$ *if and only if* $n = 2^r$ *for some* $r \geq 0$*.*

Proof. In order to obtain the injective chromatic sum, each of the colors $1, 2, ..., n$ used for injective coloring of Q_n is used maximum number of times. Let u be the vertex in Q_n which is colored i, $i = 1, 2, ..., n$. We know $deg(u)$ $= n$ and let $v \in N(u)$(neighborhood of u), which is also colored i. By the construction of Q_n, there are $n - 2$ other vertices not in $N(u)$ which can also be colored with same color i. Hence the total number of vertices which can be colored i is n. Now the result follows from Theorem 9 of [5].

The corona of G_1 and G_2 denoted as $G_1 \circ G_2$ is obtained by taking one copy of G_1 and m copies of G_2, where $m = |V(G_1)|$and joining each of the m vertices of G_1 with the corresponding copy of G_2.

The following result is established for injective chromatic sum of corona of graphs.

Theorem 5.9. $\Delta_1 + |V(G_2)| \leq \chi_i(G_1 \circ G_2) \leq |V(G_1)| + |V(G_2)|$.

Proof. Proof: Let $p_1 = |V(G_1)|$ and $p_2 = |V(G_2)|$. We assume $p_2 < p_1$. Let Δ_1 be the maximum degree of $G_1$and let $deg(u) = \Delta_1$. By the definition of corona of two graphs, u is joined to each vertex of the corresponding copy of G_2. Hence in $(G_1 \circ G_2)$,$deg(u) = \Delta_1 + p_2 = \Delta(G_1 \circ G_2)$.
Since $\chi_i(G) \geq \Delta(G)$[5], $\chi_i(G_1 \circ G_2) \geq \Delta(G_1 \circ G_2)$. Hence we have
$\Delta_1 + |V(G_2)| \leq \chi_i(G_1 \circ G_2)$. Now to prove the upper bound. Each vertex u of G_1 is joined with every vertex v of the corresponding copy of G_2. If G_2 is a complete graph, $\chi_i(G_2) = p_2$ and u is assigned a new color say $p_2 + 1$. This

same set of p_2 colors can be used for injective coloring of all copies of G_2. If all the vertices of G_1 are assigned different colors, then a maximum of p_l colors are required. Thus the total number of colors required for injective coloring of $G_1 \circ G_2$ is at the most $|V(G_1)| + |V(G_2)|$. Hence the proof.

Observation 5.10. $\Sigma_i(G_1 \circ G_2) > \Sigma_i(G_1)\Sigma_i(G_2)$, for any two connected graphs G_1 and G_2 other than complete graphs K_n, $n > 2$
$\Sigma_i(G_1 \circ G_2) < \Sigma_i(G_1)\Sigma_i(G_2)$ for complete graphs with $n > 2$.

6 Injective Chromatic Polynomials

The injective chromatic number $\chi_i(G)$ [5] of a graph G is the minimum number of colors needed to color the vertices of G such that two vertices with a common neighbor are assigned distinct colors. A large amount of work has been done on injective chromatic number like [6].
The chromatic polynomial is the bridge between algebra and graph theory. The chromatic polynomial was introduced by George David Birkhoff in 1912, defining it only for planar graphs, in an attempt to prove the four color theorem. In 1932, Hassler Whitney generalized Birkhoffs polynomial from the planar case to general graphs . Reed introduced and studied the concept of chromatically equivalent graphs in 1968.
A vertex k coloring such that two vertices having a common neighbor have distinct colors is defined as injective k coloring and the minimum number k such that G has an injective k coloring is defined as the injective chromatic number denoted as $\chi_i(G)$.

The chromatic polynomial counts the number of colorings of a graph G as a function of the number of colors used.[3] i.e $P(G, k)$ denotes the number of ways of coloring the vertices of graph G with k colors. The smallest k for which $P(G, k) > 0$ is the chromatic number of G [4], [2].

A graph G can be assigned injective coloring using k colors in different ways. We discuss the number of ways of injective coloring of the vertices of some class of graphs using k colors.

Definition 6.1. *The number of different injective colorings of a graph G using k colors is denoted by $I(G, k)$. The function $I(G, k)$ is a polynomial in k of degree $n(G)$ and hence $I(G, k)$ is called injective chromatic polynomial(ICP).*
$I(G, k) = 0$, if $k < \chi_i(G)$.
The smallest number k for which $I(G, k) > 0$ is the injective chromatic number $\chi_i(G)$.

Now we establish a few results on i-chromatic polynomial of complete graphs, wheels, cycles and complete bipartite graphs. The following theorem is that of complete graph K_n.

Theorem 6.2. *For a complete graph K_n, $I(K_n, k) = k(k-1)...(k-n+1)$.*

Proof. The complete graph K_n can be injectively colored using k colors as follows. Let $v_1, v_2...v_n$ be the vertices of K_n.
Color some vertex say v_1 with color i. Here i can be any value from 1 to k. The remaining vertices $v_2...v_n$ can be colored using $k-1$, $k-2$ ...$k-n+1$ different colors. Hence the proof.

The wheel graph W_n also has ICP as that of K_n as shown in the following theorem.

Theorem 6.3. $I(W_n, k) = k(k-1)...(k-n+1)$.

Proof. For the injective coloring of the wheel graph using k colors, the middle vertex say v_1 can be colored i where $i = 1, 2, ...k$. The outer vertices $v_2...v_n$ can be colored in $k-1$, $k-2$, ...$k-n+1$ different ways since the middle vertex is the common neighbor of any two outer vertices. Hence $I(W_n, k) = k(k-1)...(k-n+1)$.

For cycles the following theorem is established.

Theorem 6.4. *For cycle C_n,*

$$I(C_n, k) = \begin{cases} k^2(k-1)^{n-3}(k-2) & , n \ odd \\ k^2(k-1)^{n-2} & , n \ even, n \equiv 0(mod4) \\ k^2(k-1)^{n-2}(k-2)^2 & , n \ even, n \not\equiv 0(mod4) \end{cases}$$

Proof. For the cycle $C_n = v_1v_2...v_nv_1$, where n is odd, the first vertex v_1 and the second vertex v_2 can be injectively colored by assigning a color i where $i = 1, 2, ..k$. The remaining $n-3$ vertices have one choice lesser than these two vertices and hence can be colored only in $k-1$ different ways and the last vertex has only $k-2$ different colors left since two colors are already allotted to the vertices at a path length 2 . If n is even and multiple of 4, the first vertex v_1 and the second vertex v_2 can be injectively colored by assigning a color i where $i = 1, 2, ..k$ and the remaining vertices can be colored in $k-1$ different ways. If n is even and not a multiple of 4, coloring is done injectively as in the case of odd n but the last two vertices have $k-2$ different colors left for injective coloring. Hence the proof.

The following theorem is established for complete bipartite graphs.

Theorem 6.5. $I(K_{m,n}, k) = k^2(k-1)^2...(k-m+1)^2(k-m)...(k-n+1)$ *if* $m < n$.

Proof. Consider a complete bipartite graph with vertex set $V = V_1 \bigcup V_2$ where $|V_1| = m$ and $|V_2| = n$ where $m < n$. The injective coloring of vertices in V_1 can be done in $k(k-1)(k-2)...(k-m+1)$ ways since any two of them have a common neighbor in V_2 and the vertices in V_2 in $k(k-1)(k-2)...(k-n+1)$ different ways. Hence if $m < n$, one vertex of V_1 and one vertex of V_2 can be assigned colors up to $k-m+1$ and hence the proof.

7 Injective Chromatic Polynomial of Trees

In this section we obtain the injective chromatic polynomial of star and path using which Injective chromatic polynomial of tree is obtained.

Theorem 7.1. *For star graph,* $I(T,k) = k^2(k-1)(k-2)...(k-n+1)$.

Proof. The proof follows from theorem of bipartite graphs as star graph is a special case of bipartite graph where $m = 1$.

Theorem 7.2. $I(T,k) = k^2(k-1)^{n-2}$ *if and only if* T *is a path.*

Proof. If T is a path, then the first and second vertex can be colored injectively in k ways while all the remaining vertices can be assigned $k-1$. Conversely, for any tree other than a path, there is at least one vertex of degree ≥ 3 say v. Hence for that component having the vertex v, one of the neighbors of v can be colored injectively in $k-2$ ways only and hence $I(T,k) < k^2(k-1)^{n-2}$. Hence if T is a tree with$I(T,k) = k^2(k-1)^{n-2}$ and T is not a path, then $I(T,k) < k^2(k-1)^{n-2}$ which is a contradiction. Hence the proof.

A tree T on n vertices can be considered as composed of say r branches or r star graphs connected together to form a single component either by edges or clinged together.

Theorem 7.3. *For any tree* T *on* n *vertices,* $I(T,k) = k^2 \prod_r (k-1)(k-2)...(k-\Delta_r+1)$ *where* Δ_r *is the maximum degree of the* r^{th} *branch.*

Proof. Let there be r connected star graphs for T. Choose the vertex with maximum degree Δ. If there are more than one vertex, choose the one having maximum number of edges in its branches. Let the vertex be u. Now its neighbors can be colored injectively in $k(k-1)...(k-\Delta+1)$ ways such that the neighbor with largest degree receives color $k-1$. Repeating this till all the vertices in the r branches are colored. Then $I(T,k) = k^2 \prod_r (k-1)(k-2)...(k-\Delta_r+1)$.

8 Conclusion

The injective chromatic sum and injective strength of graph are defined and studied for various graphs and bounds are suggested. The injective coloring of graphs with minimum sum finds applications in star topology in optimal routing problems in communication networks where the objective is to optimize delay. Also the bounds of injective chromatic sum of operations of graphs are suggested which helps to characterize the combinations of graphs based on injective chromatic number of individual graphs.The concept of injective chromatic polynomial is introduced and computed for complete graphs, bipartite graphs, cycles etc. The bounds for the injective chromatic polynomial of trees is suggested.

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