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Some Symmetry Properties of Solutions to General Heun's Differential Equation

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Abstract

Based on properties of solutions of Heuns differential equations, the derivative of a solution of general form of Heun's differential equation, in some particular cases, can be expressed from the solution of another form of Heun's equation. Relevant symmetry properties for general Heun's differential equations are investigated.

Keywords: *Heun's, Symmetry, Fuchsian.*

1 Introduction

Heun's differential equation and its confluent forms have been subject to recent investigations in previous years due to a large number of their applications in mathematical physics and quantum mechanics [1, 2]. They indeed play a central role in a number of physical problems, like quasi-exactly solvable systems [5], higher dimensional correlated systems [11], Kerr-de Sitter black holes [12], Calogero-Moser-Sutherland systems [14], finite lattice Bethe ansatz systems [15], etc. Besides, this equation appears as a natural generalization of the hypergeometric equation and its special cases including the Gauss hypergeometric, confluent hypergeometric, Mathieu, Ince, Lame, Bessel, Legendre, Laguerre equations, etc.

The general second order Heun's differential equation (GHE) can be written, in canonical form, as follows [1]

$$\mathcal{D}^2 y + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a})\mathcal{D} y + \frac{(\alpha\beta x - q)}{x(x-1)(x-a)} y = 0, \tag{1.1}$$

where $\mathcal{D} = \frac{d}{dx}$, $\{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}$ $(a \neq 0, 1)$ are parameters, generally complex and arbitrary, linked by the Fuchsian constraint $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. This equation has four regular singular points at $\{0, 1, a, \infty\}$, with the exponents of these singularities being respectively, $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$, and $\{\alpha, \beta\}$.

The Heun differential equation (1.1) can be rewritten in the general form

$$P\mathcal{D}^2 y + Q\mathcal{D} y + R y = 0, \tag{1.2}$$

where P, Q,R, are polynomial functions. Setting $\mathcal{D}y = z$, taking the derivative of (1.2), subtracting the equation (1.2) multiplied by some function $\psi(x)$, we obtain the differential equation of the form

$$P\mathcal{D}^2 z + (P' + Q - \psi(x)P)\mathcal{D}z + (Q' + R - \psi(x)Q)z + (R' - \psi(x)R)y = 0, \tag{1.3}$$

where $V' := \mathcal{D}V$. The condition which render (1.2) invariant under derivative operation, i.e, allowing to write (1.3) in the form

$$\bar{P}\mathcal{D}^2 z + \bar{Q}\mathcal{D}z + \bar{R}z = 0, \tag{1.4}$$

where $\bar{P}, \bar{Q}, \bar{R}$ being appropriate polynomials, are of great interest in the investigation of solutions of the Heun equation [6]. The suitable ansatz reads $\psi = \frac{R'}{R}$. In this case, (1.2) reads

$$P\mathcal{D}^2 z + (P' + Q - \frac{R'}{R}P)\mathcal{D}z + (Q' + R - \frac{R'}{R}Q)z = 0, \tag{1.5}$$

where $R(x) = xR' + R(0)$ is a polynomial of degree 1 which can be written as

$$R(x) = R'(x - c), \qquad \text{with} \qquad c = -\frac{R(0)}{R'}. \tag{1.6}$$

Therefore, P/R and Q/R have to be polynomials defining the singularities of the equation (1.5). In general, such a transformation leads to more singular points than in the initial equations. When the singular points coincide with already existing ones, the number of singularities to that of the initial Heun equation increases by one . The derived equation can be transformed to Heun equation (1.1) and the derivative of the solution to the initial Heun equation can be expressed in terms of a solution to another Heun equation . This property may lead to interesting series solutions to Heun equations in term of hypergeometric functions. In [6], Ishkhanyan and Suominem investigate new solutions of Heun general equation, using such properties. Unfortunately, their result contain some misprints which have been corrected in the work [7].

This letter reports on relevant properties of solutions for general Heun differential equation. After appropriate manipulation, the Heun equation (1.1) can be cast in form (1.5) and the polynomial $\bar{P}, \bar{Q}, \bar{R}$ identified. For any such transformed equation, the pole can be eliminated setting either $R' = 0$ or $R(0) = 0$. Then, the compatibility condition for (1.5) and (1.2) can yield symmetry properties between original Heun equation and its derivative. To explicate such properties, let us denote $\mathcal{G}n(a, q; \alpha, \beta, \gamma, \delta, \epsilon; x)$, the corresponding solutions to GHE.

2 Symmetry Properties of Solutions to General Heun's Differential Equation (GHE)

In this section, we examine the symmetry form of solutions of GHE and its transform using the symmetry relations. To do this, we state the following:

Proposition 2.1 *The following symmetry are verified for GHE:*

(1) $D[x^s\mathcal{G}n(a, 0; \alpha, \beta, 0, \delta, \epsilon; x)] = [x^{s'}\mathcal{G}n(a, 0; \alpha + \frac{\delta+\epsilon}{\beta}, \beta, \delta + 1, \epsilon + 1; x)]$,
$s = s' = 0$ or $s = s' = 1$ *at* $x = 0,\ q = 0$;

(2) $D[(x-1)^s\mathcal{G}n(a, \alpha\beta; \alpha, \beta, 0, \delta, \epsilon;\ x-1)] = [(x-1)^{s'}\mathcal{G}n(a, \alpha\beta; \alpha+\frac{\gamma+\epsilon}{\beta}, \beta, \gamma, 0, \epsilon+1; x-1)]$,
$s = s' = 0$ or $s = s' = 1$ *at* $x = 1,\ q = \alpha\beta$;

(3) $D[(x-a)^s\mathcal{G}n(a, a\alpha\beta; \alpha, \beta, \gamma, \delta, 0;\ x-1)] = [(x-a)^{s'}\mathcal{G}n(a, a\alpha\beta; \alpha+\frac{\gamma+\delta}{\beta}, \beta, \gamma+1, \delta+1, 0; x-a)]$,
$s = s' = 0$ or $s = s' = 1$ *at* $x = a,\ q = a\alpha\beta$;

(4) At ∞ *and* $\alpha\beta$*, no symmetry solution exist.*

where s *and* s' *are exponents of the old and new equations respectively.*

Proof. From (1.1) and (1.4), we observe the followings:

$$\begin{aligned}
&P(x) = x(x-1)(x-a), \;\; Q(x) = \gamma(x-1)(x-a) + \delta x(x-a) + \epsilon x(x-1),\\
&R(x) = \alpha\beta x - q,\\
&\bar{Q}(x) = (x-a)(x-1)(\gamma+1) + (\delta+1)x(x-a) + (1+\epsilon)x(x-1)\\
&-\frac{\alpha\beta x(x-1)(x-a)}{\alpha\beta x - q}\\
&\bar{R}(x) = (\epsilon+\gamma)(x-1) + (\gamma+\delta)(x-a) + (\delta+\epsilon)x + \alpha\beta x - q\\
&-\frac{\alpha\beta((x-1)(x-a)\gamma) + \delta x(x-a) + \epsilon x(x-1)}{\alpha\beta x - q}. \qquad (2.1)
\end{aligned}$$

Equation (1.1) has regular singularities $x = 0,\ 1,\ a,\ \infty$. Using (1.4), we obtain a new equation having extra singularity given by $x = \frac{q}{\alpha\beta}$. The extra singularity coincides with the previous singularities of (1.1) when

(i) $q = 0,\ x = 0,$

(ii) $q = \alpha\beta,\ x = 1,$

(iii) $q = a\alpha\beta,\ x = a,$

(iv) $\alpha\beta = 0,\ x = \infty$

leading to four cases to be examined.

Case (i): $q = 0$.

Equations (1.1) and (1.4) for GHE reduce to

$$\mathcal{D}^2 y + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a})\mathcal{D} y + \frac{(\alpha\beta x)}{x(x-1)(x-a)} y = 0, \qquad (2.2)$$

and

$$\begin{aligned}
&x(x-1)(x-a)\mathcal{D}^2 z + ((x-1)(x-a)\gamma + (\delta+1)x(x-a) + (1+\epsilon))x(x-1)\mathcal{D} z\\
&+(\gamma(2x-a-1) + (\delta+\epsilon)x - \alpha\beta x - \frac{(x-1)(x-a)\gamma}{x})\mathcal{D} z = 0. \qquad (2.3)
\end{aligned}$$

For symmetry to exist between the two equations we must have $\gamma = 0$, leading to the reduced equations

$$\mathcal{D}^2 y + (\frac{\delta}{x-1} + \frac{\epsilon}{x-a})\mathcal{D} y + \frac{\alpha\beta x}{x(x-1)(x-a)} y = 0, \qquad (2.4)$$

and

$$\begin{aligned}
&x(x-1)(x-a)\mathcal{D}^2 z + ((\delta+1)x(x-a) + (1+\epsilon))x(x-1)\mathcal{D} z\\
&+((\delta+\epsilon)x - \alpha\beta x)z = 0, \qquad (2.5)
\end{aligned}$$

giving rise to the symmetry transformations
$q = 0,\ \gamma = 0,\ \delta \longrightarrow \delta + 1,\ \epsilon \longrightarrow \epsilon + 1,\ \ \alpha\beta \longrightarrow \delta + \epsilon + \alpha\beta$ and $\alpha \longrightarrow \alpha + \frac{\delta+\epsilon}{\beta}$ or $\beta \longrightarrow \beta + \frac{\delta+\epsilon}{\alpha}$.

By application of Frobenius method to equations (2.4) and (2.5), we obtain the first statement of the proposition.

Case(ii): $q = \alpha\beta$.
Similarly at $x = 1$, we have $q = \alpha\beta$ reducing equations (1.1) and (1.4)

$$x(x-1)(x-a)\mathcal{D}^2y+(\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon(x-1)x)\mathcal{D}y+\alpha\beta(x-1)y = 0, \tag{2.6}$$

and

$$x(x-1)(x-a)\mathcal{D}^2z + ((1+\gamma)(x-1)(x-a) + \delta x(x-a) + (1+\epsilon)(x-1)x)\mathcal{D}z$$
$$+(\gamma(2x-a-1) + \delta(x-a) + \alpha\beta x - \frac{\delta(x-a)}{x-1})z = 0. \tag{2.7}$$

For symmetry to exist we must have $\delta = 0$, leading to the reduce equations

$$x(x-1)(x-a)\mathcal{D}^2y + (\gamma(x-1)(x-a) + \epsilon(x-1)x)\mathcal{D}y + \alpha\beta(x-1)y = 0, \tag{2.8}$$

and

$$x(x-1)(x-a)\mathcal{D}^2z + ((1+\gamma)(x-1)(x-a) + (1+\epsilon)(x-1)x)\mathcal{D}z$$
$$+(\gamma(2x-a-1) + \alpha\beta(x-1))z = 0, \tag{2.9}$$

giving rise to the following transformation
$q = \alpha\beta,\ \delta = 0,\ \gamma \longrightarrow \gamma+1,\ \epsilon \longrightarrow \epsilon+1,\ \ \alpha\beta \longrightarrow \gamma+\epsilon+\alpha\beta$ and $\alpha \longrightarrow \alpha+\frac{\gamma+\epsilon}{\beta}$ or $\beta \longrightarrow \beta + \frac{\gamma+\epsilon}{\alpha}$.
By application of Frobenius method to equations (2.8) and (2.9), we obtain the second statement of the proposition.

Case(iii); $q = a\alpha\beta$.
In this case equations (1.1) and (1.4) reduces to

$$x(x-1)(x-a)\mathcal{D}^2y+(\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon(x-1)x)\mathcal{D}y+\alpha\beta(x-a)y = 0, \tag{2.10}$$

and

$$x(x-1)(x-a)\mathcal{D}^2z + ((1+\gamma)(x-1)(x-a) + (1+\delta)x(x-a) + \epsilon(x-1)x)\mathcal{D}z$$
$$+(\epsilon x + \epsilon(x-1) + \alpha\beta x - \frac{\epsilon(x-1)x}{x-a})z = 0. \tag{2.11}$$

For symmetry to exist we must have $\epsilon = 0$, leading to the reduce equations

$$x(x-1)(x-a)\mathcal{D}^2y + (\gamma(x-1)(x-a) + \delta x(x-a))\mathcal{D}y + \alpha\beta(x-a)y = 0, \tag{2.12}$$

and

$$x(x-1)(x-a)\mathcal{D}^2z+((1+\gamma)(x-1)(x-a)+(1+\delta)x(x-a))\mathcal{D}z$$
$$+(\alpha\beta x-\frac{\epsilon(x-1)x}{x-a})z=0, \tag{2.13}$$

giving rise to the following symmetry transformation
$q=a\alpha\beta,\ \epsilon=0,\ \gamma\longrightarrow\gamma+1,\ \delta\longrightarrow\delta+1,\ \ \alpha\beta\longrightarrow\gamma+\delta+\alpha\beta$ and $\alpha\longrightarrow\alpha+\frac{\gamma+\delta}{\beta}$ or $\beta\longrightarrow\beta+\frac{\gamma+\delta}{\alpha}$.
By application of Frobenius method to equations (2.12) and (2.13), we obtain the second statement of the proposition.

Case(iv); $\alpha\beta=0$. Following the same process as above no symmetry relations exists at all.

3 Conclusion and Remarks

Symmetry properties were established for the general Heun's equation. The properties were established in terms of symmetry form solution derived by the introduction of an arbitrary function $\psi(x)$, into the original equation leading to perturbed equations.The resultant equation leads to the closed form solutions. However, The symmetry closed forms discussed above are particular cases of closed forms solutions. These cases arise from the symmetries of the equations. The symmetry properties here does not give the general cases at all. The genaral cases are currently being investigated.

References

[1] A. Ronveaux, *Heun's Differential Equation*, Oxford University Press, Oxford, (1995).

[2] A.O. Smirnov, Elliptic solutions and Heun's equations, *C.R.M. Proceedings and Lecture notes*, 32(2002), 287-305.

[3] P.A. Clarkson and P.J. Oliver, Symmetry and the Chazy equation, *J. Differential Equations*, 124(1996), 225-246.

[4] R.S. Maier, Heun-to-hypergeometric transformations, *Contribution to the Conference of Foundations of Computational Mathematics*, 02(2002), downloadable from http://www.math.umn.edu/~focm/c_/Maier.pdf.13.

[5] N.H. Christ and T.D. Lee, Field theory and symmetry principle, *Phys. Rev. D*, 12(1975), 1606.

[6] A. Ishkhanyan and K.A. Souminen, New solution of Heun's general equation, *J. Phys. A: Math. Gen.*, 36(2003), L81-L85.

[7] G. Valent, Heun functions versus elliptic functions, *International Conference on Differential Equations, Special Functions and Applications*, Munich, (2005), (e-print math-ph/0512006).

[8] V.S. Grossman, *Multivariate Calculus, Linear Algebra, and Differential Equation*, Saunder College Publishing, New York, (1995).

[9] S.P. Tsarev, An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator, *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, Switzerland, (1996), 226-231.

[10] M. Van Hoeij, Linear differential opreator for polynomial equation, *Journal of Symbolic Computation*, 24(5) (1997), 537-561.

[11] R.K. Bhadari, A. Khare, J. Law, M.V.N. Murthy and D. Sen, On polynomial solution of Heun's equation, *J. Phys. A: Math. Gen.*, 30(1997), 2557-2260.

[12] M. Suzuki, E. Takasugi and H. Umetsu, Symmetry operator and separability of massive Klein, *Prog. Theor. Phys.*, 100(1998), 491-505.

[13] K. Takemura, On polynomial solutions of the Heun's equation, *Commun. Math. Phys.*, 235(2003), 467-494.

[14] K. Takemura, Integral transformation of Heun's equation and some application, *J. Nonlinear Math. Phys.*, 11(2004), 480-498.

[15] P. Dorey, J. Suzuki and R. Tateo, An application of Berthe ansatz method to certain classes of ODE, *J. Phys. A: Math. Gen.*, 37(2004), 2047-2061.

[16] M.N. Hounkonnou, A. Ronveaux and A. Anjorin, Derivatives of Heun's function from some properties of hypergeometric equation, *Proceeding of International Workshop on Special Functions*, Marseille, (2007), (In Press), Preprint: ICMPA-MPA/2007/21.

[17] K. Kuiken, Heun's equations and the hypergeometric equations, *S.I.A.M. J. Math. Anal.*, 10(3) (1979), 655-657.