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Three-step Fixed Point Iteration for Multivalued Mapping with Errors in Banach Spaces

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Abstract

In this paper, we consider the convergence of three-step fixed point iterative processes for multivalued nonexpansive mapping with errors, under some different conditions, the sequences of three-step fixed point iterates strongly or weakly converge to a fixed point of the multivalued nonexpansive mapping. Our results extend and improve some recent results.

Keywords: *Multivalued nonexpansive mapping, Fixed points, Uniform convex, Opial's condition.*

1 Introduction

Let X be a Banach space and K a nonempty subset of X. We shall denote by 2^X the family of all subsets of X, $CB(X)$ the family of all nonempty closed bounded subsets of X and denote $C(X)$ by the family of nonempty compact subsets of X. A multivalued mapping $T : K \to 2^X$ is said to be nonexpansive (resp, contractive) if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in K,$$

$$(resp, H(Tx, Ty) \leq k\|x - y\|, \; for \;\; some \;\; k \in (0, 1)).$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|\},\quad A,B\in CB(X).$$

A point x is called a fixed point of T if $x \in Tx$.

Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler in 1969 (see [7]), many authors have studied the fixed point theory for multivalued mappings (e.g. see[1, 3, 4, 5, 14]). For single-valued nonexpansive mappings, Mann [6] and Ishikawa [2] respectively introduced a new iteration procedure for approximating its fixed point in a Banach space as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \tag{1}$$

and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, y_n = (1-b_n)x_n + b_n Tx_n, \tag{2}$$

where $\{\alpha_n\}$ and $\{b_n\}$are sequences in $[0,1]$. Obviously, Mann iteration is a special case of Ishikawa iteration. Recently Song in [11] and [12] introduce the following algorithms for multivalued nonexpansive mapping,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n s_n, \tag{3}$$

where $s_n \in Tx_n$, such that $\|s_{n+1}-s_n\| \le H(Tx_{n+1}, Tx_n)$.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, y_n = (1-b_n)x_n + b_n s_n, \tag{4}$$

where $\|s_n - r_n\| \le H(Tx_n, Ty_n)$ and $\|s_{n+1} - r_n\| \le H(Tx_{n+1}, Ty_n)$ for $s_n \in Tx_n$ and $r_n \in Ty_n$. He show some strongly or weakly convergence results of the above iterates for multivalued nonexpansive mapping T under some appropriate conditions. In this paper, we introduced the following algorithm, which can be generalized as the above algorithms (3), (4):

Algorithm. For a given $x_0 \in K$ and $s_0 \in Tx_0$. Let

$$z_0 = (1-a_0-\gamma_0)x_0 + a_0 s_0 + \gamma_0 u_0.$$

There exists $t_0 \in Tz_0$ such that $\|t_0 - s_0\| \le H(Tz_0, Tx_0)$. Let

$$y_0 = (1-b_0-c_0-\mu_0)x_0 + b_0 t_0 + c_0 s_0 + \mu_0 v_0.$$

There exists $r_0 \in Ty_0$ such that $\|r_0 - t_0\| \le H(Ty_0, Tz_0)$ and $\|r_0 - s_0\| \le H(Ty_0, Tx_0)$. Let

$$x_1 = (1-\alpha_0-\beta_0-\lambda_0)x_0 + \alpha_0 r_0 + \beta_0 t_0 + \lambda_0 w_0.$$

There exists $s_1 \in Tx_1$ such that $\|s_1 - r_0\| \leq H(Tx_1, Ty_0)$ and $\|s_1 - t_0\| \leq H(Tx_1, Tz_0)$. Inductively, we can get the sequence $\{x_n\}$ as follows:

$$\begin{array}{l} z_n = (1 - a_n - \gamma_n)x_n + a_n s_n + \gamma_n u_n \\ y_n = (1 - b_n - c_n - \mu_n)x_n + b_n t_n + c_n s_n + \mu_n v_n \\ x_{n+1} = (1 - \alpha_n - \beta_n - \lambda_n)x_n + \alpha_n r_n + \beta_n t_n + \lambda_n w_n, \end{array} \tag{5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequence in $[0, 1]$ and $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are bounded sequence in K, furthermore $s_n \in Tx_n, t_n \in Tz_n, r_n \in Ty_n$ such that $\|t_n - s_n\| \leq H(Tz_n, Tx_n)$, $\|r_n - t_n\| \leq H(Ty_n, Tz_n)$, $\|r_n - s_n\| \leq H(Ty_n, Tx_n)$, $\|s_{n+1} - r_n\| \leq H(Tx_{n+1}, Ty_n)$ and $\|s_{n+1} - t_n\| \leq H(Tx_{n+1}, Tz_n)$. The iterative scheme (5) is called the three-step multivalued iterative scheme with errors. If $a_n = \gamma_n = c_n = \mu_n = \beta_n = \lambda_n \equiv 0$, then iterative scheme (5) reduces to (4). If $a_n = \gamma_n = b_n = c_n = \mu_n = \beta_n = \lambda_n \equiv 0$, then iterative scheme (5) reduces to (3). In fact let $\gamma_n = \mu_n = \lambda_n \equiv 0$ or $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, we also have the following Algorithms:

$$\begin{array}{l} z_n = (1 - a_n)x_n + a_n s_n \\ y_n = (1 - b_n - c_n)x_n + b_n t_n + c_n s_n \\ x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n r_n + \beta_n t_n; \end{array} \tag{6}$$

$$\begin{array}{l} z_n = (1 - a_n)x_n + a_n s_n \\ y_n = (1 - b_n)x_n + b_n t_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n r_n; \end{array} \tag{7}$$

We consider the convergence of iterative scheme (5) for multivalued nonexpansive mapping with errors, under some different conditions, we show that the sequences of iterative scheme (5) strongly or weakly converge to a fixed point of the multivalued nonexpansive mapping T. In particular, we extend some results in [12], and also give some new results are different from the [11]. The following definition was introduced in [10].

Definition 1.1 *A multivalued mapping $T : K \to CB(K)$ is said to satisfy Condition (A) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(x) > 0$ for $x \in (0, \infty)$ such that*

$$d(x, Tx) \geq f(d(x, F(T)) \quad for \quad all \ x \in K.$$

Where $F(T) \neq \emptyset$ is the fixed point set of the multivalued mapping T. From now on, $F(T)$ stands for the fixed point set of the multivalued mapping T.

2 Preliminaries

A Banach space X is said to be satisfy Opial's condition [9] if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x (n \to \infty)$ implies the following inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. We know that Hilbert spaces and $l_p (1 < p < \infty)$ have the Opial's condition.

The following Lemmas will be useful in this paper.

Lemma 2.1 *(see [13]) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \ \forall n = 0, 1, 2, \ldots$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ exists.

(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2 *(see [8]) Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$, such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|),$$

for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$.

3 Main results

Lemma 3.1 *Let X be a real Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping for which $F(T) \neq \emptyset$ and for which $T(p) = \{p\}$ for any fixed point $p \in F(T)$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n, b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \gamma_n < \infty, \sum_{n=0}^{\infty} \mu_n < \infty, \sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{u_n\}, \{v_n\}$, and $\{w_n\}$ be the bounded sequence in K. For a given $x_0 \in K$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequence defined as in (5), then we have the following conclusions:*

$$\lim_n \|x_n - p\| \ \ exists\ for\ any\ p \in F(T)$$

Proof. Let $p \in F(T)$, from iterative scheme (5), note that $T(p) = \{p\}$ for any fixed point $p \in F(T)$, we have

$$\begin{aligned}
\|z_n - p\| &\leq (1 - a_n - \gamma_n)\|x_n - p\| + a_n\|s_n - p\| + \gamma_n\|u_n - p\| \\
&= (1 - a_n - \gamma_n)\|x_n - p\| + a_n d(s_n, Tp) + \gamma_n\|u_n - p\| \\
&\leq (1 - a_n - \gamma_n)\|x_n - p\| + a_n H(Tx_n, Tp) + \gamma_n\|u_n - p\| \\
&\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\|, \\
\|y_n - p\| &\leq (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n\|t_n - p\| + c_n\|s_n - p\| \\
&\quad +\mu_n\|v_n - p\| \\
&= (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n d(t_n, Tp) + c_n d(s_n, Tp) \\
&\quad +\mu_n\|v_n - p\| \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\| + b_n H(Tz_n, Tp) + c_n H(Tx_n, Tp) \\
&\quad +\mu_n\|v_n - p\| \\
&\leq (1 - b_n - \mu_n)\|x_n - p\| + b_n\|z_n - p\| + \mu_n\|v_n - p\|,
\end{aligned}$$

and so we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n\|r_n - p\| \\
&\quad +\beta_n\|t_n - p\| + \lambda_n\|w_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n H(Ty_n, Tp) \\
&\quad +\beta_n H(Tz_n, Tp) + \lambda_n\|w_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\
&\quad +\beta_n\|z_n - p\| + \lambda_n\|w_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n[(1 - b_n - \mu_n)\|x_n - p\| \\
&\quad +b_n\|z_n - p\| + \mu_n\|v_n - p\|] + \beta_n\|z_n - p\| + \lambda_n\|w_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - p\| + \alpha_n(1 - b_n - \mu_n)\|x_n - p\| \\
&\quad +\alpha_n b_n(1 - \gamma_n)\|x_n - p\| + \alpha_n b_n\gamma_n\|u_n - p\| + \alpha_n\mu_n\|v_n - p\| \\
&\quad +\beta_n(1 - \gamma_n)\|x_n - p\| + \beta_n\gamma_n\|u_n - p\| + \lambda_n\|w_n - p\| \\
&= \|x_n - p\| - (\beta_n\gamma_n + \alpha_n\mu_n + \alpha_n b_n\gamma_n + \lambda_n)\|x_n - p\| \\
&\quad +(\beta_n\gamma_n + \alpha_n b_n\gamma_n)\|u_n - p\| + \alpha_n\mu_n\|v_n - p\| + \lambda_n\|w_n - p\| \\
&\leq \|x_n - p\| + \gamma_n(\beta_n + \alpha_n b_n)\|u_n - p\| + \alpha_n\mu_n\|v_n - p\| + \lambda_n\|w_n - p\| \\
&\leq \|x_n - p\| + 2\gamma_n\|u_n - p\| + \mu_n\|v_n - p\| + \lambda_n\|w_n - p\|.
\end{aligned}$$

From the assumption we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + L\gamma_n + M\mu_n + N\lambda_n,$$

where $L = \sup\{2\|u_n - p\|, n \geq 0\}$, $M = \sup\{\|v_n - p\|, n \geq 0\}$ and $N = \sup\{\|w_n - p\|, n \geq 0\}$. If we let $K = \max\{L, M, N\}$ then we get that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + K(\gamma_n + \mu_n + \lambda_n). \tag{8}$$

It follows from Lemma 2.1 that $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$.

Lemma 3.2 *Let X be a uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping for which $F(T) \neq \emptyset$ and for which $T(p) = \{p\}$ for any fixed point $p \in F(T)$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n, b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \gamma_n < \infty, \sum_{n=0}^{\infty} \mu_n < \infty, \sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{u_n\}, \{v_n\}, \{w_n\}$ be the bounded sequence in K. For a given $x_0 \in K$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequence defined as in (5).*

(i) If $\liminf_n \alpha_n > 0$ and $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n + \mu_n) < 1$, then $\lim_n d(x_n, Tz_n) = 0$.

(ii) If $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_n d(x_n, Ty_n) = 0$.

(iii) If $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n + \mu_n) < 1$ and $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_n d(x_n, Tx_n) = 0$.

Proof. (i) It is well known that T has a fixed point $p \in K$ (see [5]). By lemma 3.1, we know that $\lim_n \|x_n - p\|$ exists for any $p \in F(T)$, then it follows that $\{s_n - p\}, \{t_n - p\}, and\{r_n - p\}$ are all bounded and from the assumption we known that $\{u_n - p\}, \{v_n - p\}, \{w_n - p\}$ are all bounded. We may assume that these sequences belong to B_r where $r > 0$. Note that $T(p) = \{p\}$ for any fixed point $p \in F(T)$. By Lemma 2.2, we get

$$\begin{aligned}
\|z_n - p\|^2 &\leq (1 - a_n - \gamma_n)\|x_n - p\|^2 + a_n\|s_n - p\|^2 + \gamma_n\|u_n - p\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g(\|x_n - s_n\|) \\
&= (1 - a_n - \gamma_n)\|x_n - p\|^2 + a_n d(s_n, Tp)^2 + \gamma_n\|u_n - p\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g(\|x_n - s_n\|) \\
&\leq (1 - a_n - \gamma_n)\|x_n - p\|^2 + a_n H(Tx_n, Tp)^2 + \gamma_n\|u_n - p\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g(\|x_n - s_n\|) \\
&\leq (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 - a_n(1 - a_n - \gamma_n)g(\|x_n - s_n\|), \\
&\leq (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2, \\
\|y_n - p\|^2 &\leq (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n\|t_n - p\|^2 + c_n\|s_n - p\|^2 \\
&\quad + \mu_n\|v_n - p\|^2 - b_n(1 - b_n - c_n - \mu_n)g(\|x_n - t_n\|) \\
&\leq (1 - b_n - c_n - \mu_n)\|x_n - p\|^2 + b_n H(Tz_n, Tp)^2 + c_n H(Tx_n, Tp)^2
\end{aligned}$$

$$
\begin{aligned}
&+\mu_n\|v_n-p\|^2-b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\\
\leq\ &(1-b_n-\mu_n)\|x_n-p\|^2+b_n\|z_n-p\|^2\\
&+\mu_n\|v_n-p\|^2-b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|),
\end{aligned}
$$

and therefore we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\|^2+\alpha_n\|r_n-p\|^2+\beta_n\|t_n-p\|^2\\
&+\lambda_n\|w_n-p\|^2-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\|^2+\alpha_n H(Ty_n,Tp)^2+\beta_n H(Tz_n,Tp)^2\\
&+\lambda_n\|w_n-p\|^2-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2+\beta_n\|z_n-p\|^2\\
&+\lambda_n\|w_n-p\|^2-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & (1-\alpha_n-\beta_n-\lambda_n)\|x_n-p\|^2+\alpha_n\Big[(1-b_n-\mu_n)\|x_n-p\|^2\\
&+b_n\|z_n-p\|^2+\mu_n\|v_n-p\|^2-b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\Big]\\
&+\beta_n\|z_n-p\|^2+\lambda_n\|w_n-p\|^2-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & \|x_n-p\|^2+[\alpha_n(1-b_n-\mu_n)-\alpha_n-\beta_n-\lambda_n]\|x_n-p\|^2\\
&(\alpha_n b_n+\beta_n)[(1-\gamma_n)\|x_n-p\|^2+\gamma_n\|u_n-p\|^2]+\alpha_n\mu_n\|v_n-p\|^2\\
&+\lambda_n\|w_n-p\|^2-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\\
&-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
=\ & \|x_n-p\|^2-(\alpha_n\mu_n+\lambda_n+\alpha_n b_n\gamma_n+\beta_n\gamma_n)\|x_n-p\|^2\\
&+(\alpha_n b_n+\beta_n)\gamma_n\|u_n-p\|^2+\alpha_n\mu_n\|v_n-p\|^2+\lambda_n\|w_n-p\|^2\\
&-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\\
&-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & \|x_n-p\|^2+(\alpha_n b_n+\beta_n)\gamma_n\|u_n-p\|^2+\alpha_n\mu_n\|v_n-p\|^2+\lambda_n\|w_n-p\|^2\\
&-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\\
&-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|)\\
\leq\ & \|x_n-p\|^2+2\gamma_n\|u_n-p\|^2+\mu_n\|v_n-p\|^2+\lambda_n\|w_n-p\|^2\\
&-\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|)\\
&-\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|x_n-r_n\|).
\end{aligned}
$$

From the assumption we have

$$
\begin{aligned}
\alpha_n b_n(1-b_n-c_n-\mu_n)g(\|x_n-t_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&+S(\gamma_n+\mu_n+\lambda_n),
\end{aligned}
\tag{9}
$$

$$\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(||x_n-r_n||) \le ||x_n-p||^2-||x_{n+1}-p||^2 \\ +S(\gamma_n+\mu_n+\lambda_n), \tag{10}$$

where $S = \max\{\sup\{2\|u_n-p\|^2, n\ge 0\}, \sup\{\|v_n-p\|^2, n\ge 0\}, \sup\{\|w_n-p\|^2, n\ge 0\}\}$. In the sequence we prove the (i). If $\liminf_n \alpha_n > 0$ and $0 < \liminf_n b_n \le \limsup_n(b_n+c_n+\mu_n) < 1$, then there exist a positive integer n_0 and $\nu,\eta,\theta \in (0,1)$ such that

$$0<\nu<\alpha_n \textit{ and } 0<\eta<b_n \textit{ and } b_n+c_n+\mu_n<\theta<1, \textit{ for all } n\ge n_0.$$

From (9)we get that

$$\nu\eta(1-\theta)g(\|x_n-t_n\|) \le \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\ +S(\gamma_n+\mu_n+\lambda_n) \tag{11}$$

for all $n\ge n_0$. It follows from (11) that for $m\ge n_0$,

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|x_n-t_n\|) &\le \frac{1}{\nu\eta(1-\theta)}\Big(\sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\ &\quad S\sum_{n=n_0}^{m}(\gamma_n+\mu_n+\lambda_n)\Big) \\ &\le \frac{1}{\nu\eta(1-\theta)}\Big(\|x_{n_0}-p\|^2+S\sum_{n=n_0}^{m}(\gamma_n+\mu_n+\lambda_n)\Big).\end{aligned}$$

Let $m\to\infty$ in above inequality we get that $\sum_{n=n_0}^{\infty} g(\|x_n-t_n\|)<\infty$ and hence $\lim_{n\to\infty} g(\|x_n-t_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|x_n-t_n\|=0$, therefore we have $0\le \lim_{n\to\infty} d(x_n,Tz_n)\le \lim_{n\to\infty}\|x_n-t_n\|=0$.
(ii) If $0<\liminf_n\alpha_n\le\limsup_n(\alpha_n+\beta_n+\lambda_n)<1$, then using a similar method, together with inequality (10), it can be shown that $\lim_n d(x_n,Ty_n)=0$.
(iii) If $0<\liminf_n b_n\le\limsup_n(b_n+c_n+\mu_n)<1$ and $0<\liminf_n\alpha_n\le\limsup_n(\alpha_n+\beta_n+\lambda_n)<1$, by (i) and (ii) we have

$$\lim_{n\to\infty}\|x_n-t_n\|=0 \textit{ and } \lim_{n\to\infty}\|x_n-r_n\|=0. \tag{12}$$

From (5) we have

$$\|y_n-x_n\|\le b_n\|x_n-t_n\|+c_n\|x_n-s_n\|+\mu_n\|x_n-v_n\|.$$

Hence we have that

$$\begin{aligned}\|s_n-x_n\| &\le \|s_n-r_n\|+\|r_n-x_n\| \\ &\le H(Tx_n,Ty_n)+\|r_n-x_n\| \\ &\le \|x_n-y_n\|+\|r_n-x_n\| \\ &\le b_n\|x_n-t_n\|+c_n\|x_n-s_n\|+\mu_n\|x_n-v_n\|+\|r_n-x_n\|.\end{aligned}$$

It follows that

$$(1-c_n)\|s_n - x_n\| \leq b_n\|x_n - t_n\| + \mu_n\|x_n - v_n\| + \|r_n - x_n\| \tag{13}$$

from (12) and $\limsup_n c_n < 1$ we get $\lim_n \|s_n - x_n\| = 0$, hence $0 \leq \lim_{n\to\infty} d(x_n, Tx_n) \leq \lim_{n\to\infty} \|x_n - s_n\| = 0$.

Theorem 3.3 *Let X, T and $\{x_n\}, \{y_n\}, \{z_n\}$ be the same as in Lemma 3.2, if K be a nonempty compact convex subset of a Banach space X and*

(i) If $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n + \mu_n) < 1$, and

(ii) $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \lambda_n) < 1$,

then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.2, we have $\lim_n d(x_n, Tx_n) = 0$. Since K be a nonempty compact convex subset, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$ for some $q \in K$. Thus,

$$\begin{aligned} d(q, Tq) &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq 2\|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \to 0. \end{aligned}$$

Hence q is a fixed point of T. From Lemma 3.1, now on take on q in place of p, we get that $\lim_{n\to\infty} \|x_n - q\| = 0$. From Lemma 3.2 we get that

$$\|y_n - x_n\| \leq b_n\|t_n - x_n\| + c_n\|s_n - x_n\| + \mu_n\|v_n - x_n\| \to 0 \; as \; n \to \infty,$$

and

$$\|z_n - x_n\| \leq a_n\|x_n - s_n\| + \gamma_n\|x_n - u_n\| \to 0 \; as \; n \to \infty,$$

it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$. So the desired conclusion follows.

Theorem 3.4 *Let X, T, K and $\{x_n\}, \{y_n\}, \{z_n\}$ be the same as in Lemma 3.2, if T satisfies Condition (A) with respect to the sequence $\{x_n\}$,*

(i) If $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n + \mu_n) < 1$, and

(ii) $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \lambda_n) < 1$,

then $\{x_n\}, \{y_n\}, \{z_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.2, we have

$$\lim_n d(x_n, Tx_n) = 0.$$

Since T satisfies Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leq d(x_n, Tx_n) \to 0.$$

Thus we get $\lim_n d(x_n, F(T)) = 0$. The remainder of the proof is the same as Theorem 2.4 in [12] and Theorem 3.3, we omit it.

Theorem 3.5 *Let X, T and $\{x_n\}, \{y_n\}, \{z_n\}$ be the same as in Lemma 3.2 and $T : K \to C(K)$. If K be a nonempty weakly compact convex subset of a Banach space X and X satisfies Opial's condition,*

(i) If $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n + \mu_n) < 1$, and

(ii) $0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \lambda_n) < 1$,

then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. The proof of the Theorem is the same as Theorem 2.5 in [12], we omit it.

Remark. From iterative scheme (5), Theorem 3.3, Theorem 3.4 and Theorem 3.5 which generalized the results obtained by Song [12] and also give some new results are different from the results in [11]. Furthermore the above conclusions hold for iterative scheme (6), (7).

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