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Upper and Semisimple Radical Class

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Abstract

We prove here some useful equivalent conditions for a subclass of a fixed universal class to be a semisimple radical class and give some consequences of Upper radical class.

Keywords: *Semirings, Ideal, Radical class, Upper radical class, Semisimple class.*

1 Introduction

The paper is concerned with generalizing some results in ring theory. In correspondence to the Kurosh-Amitsur radical theory for associative rings, an abstract concept of radical classes and radicals for semirings has been introduced and investigated in a series of publications [4]-[8] by D. M. Olson and several coauthors.

Semirings, additively cancellative semirings, commutative semirings, semimodules, additively cancellative semimodules, ideals, k-ideals (subtractive ideals), homomorphisms are as defined in [2].

Each homomorphism $\phi\colon S \to T$ of semirings corresponds to a congruence k of S and the homomorphic image $\phi(S)$ is isomorphic to the semiring S/k of congruence classes. In this paper we mainly use congruences that are determined by an ideal I of S according to $sk_Is' \Leftrightarrow$ there are

$$a_i \in I \text{ satisfying } s + a_1 = s' + a_2.$$

In this case one usually denotes S/k_I by S/I. Moreover, $k_I = k_{\overline{I}}$ and thus $S/I = S/\overline{I}$ hold for all ideals I of S with the same k-closure $\overline{I}, S/I$ has always an absorbing zero, namely the congruence class $\overline{I} = [a]_I = [a]_{\overline{I}}$ determined by each $a \in I$. We also mention that a semiring has in general much more congruences than those determined by its ideals. For a last concept of this kind, let $\phi\colon S \to T$ be a surjective homomorphism for semirings which have a zero. Then ϕ is called a semi-isomorphism and denoted by $\phi\colon S \tilde{\to} T$ if $\phi(0_S) = 0_T$ and $\phi^{-1}(0_T) = 0_S$ are satisfied. We emphasize here that such a semi-isomorphism, despite of misleading name, has in general very little in common with an isomorphism.

Convention: Throughout $R \longmapsto S$ is a surjective homomorphism.

Theorem 1.1. *[3] Let S be a semiring, T a semiring with an absorbing zero 0_T, and $\phi\colon S \to T$ a surjective homomorphism. Then $K = \phi^{-1}(0_T)$ is a k-ideal of S (also called the kernel of ϕ) and $\phi([s]_K) = \phi(s)$ for all $s \in S$ defines a semi-isomorphism $\phi\colon S/K \tilde{\to} T$ which satisfies $\phi \circ {k_K}^{\#} = \phi$, where ${k_K}^{\#}$ denotes the natural homomorphism of S onto $S/K = S/k_K$.*

Theorem 1.2. *[3] For a semiring S with an absorbing zero 0 let S be a subsemiring which contains 0 and B an ideal of S. Then $\phi([a]_{A\cap\overline{B}}) = [a]_B$ for all $a \in A \subseteq A + B$ defines a semi-isomorphism*

$$\phi\colon A/A \cap \overline{B} \tilde{\to} A + B/B.$$

Theorem 1.3. *[3] Let A, B be ideals of a semiring S with the additional condition $A \subseteq B$. Then $\overline{\phi}([s]_B) = [[s]_A]_{\overline{B}/A}$ for all $s \in S$ defines an isomorphism*

$$\overline{\phi}\colon S/B \to (S/A)/(\overline{B}/A).$$

2 Radical Class

Definition 2.1. *[1] Let $\mathcal{R}$ be a class of semirings. A semiring (ideal) belonging to the class $\mathcal{R}$, will be called a $\mathcal{R}$-semiring ($\mathcal{R}$-ideal).*

Definition 2.2. *[1] A class $\mathcal{R}$ of semirings is called a radical class whenever the following three conditions are satisfied:*

(a) $\mathcal{R}$ is homomorphically closed; i.e. if S is a homomorphic image of a $\mathcal{R}$-semiring R then S is also a $\mathcal{R}$-semiring

(b) Every semiring R contains a $\mathcal{R}$-ideal $\mathcal{R}(R)$ which in turn contains every other $\mathcal{R}$-ideal of R.

(c) *The factor semiring $R/\mathcal{R}(R)$ does not contain any nonzero $\mathcal{R}$-ideal; i.e. $\mathcal{R}(R/\mathcal{R}(R)) = 0$.*

Proposition 2.3. *[9] Assuming conditions (a) and (b) on a class $\mathcal{R}$ of semirings, condition (c) is equivalent to*
(c') If I is an ideal of the semiring R and if both I and R/I are in $\mathcal{R}$, then R itself is in $\mathcal{R}$.

Definition 2.4. *$\mathcal{R}$ is said to be closed under extensions. If I is an ideal of the semiring R and if both I and R/I are in $\mathcal{R}$, then R itself is in $\mathcal{R}$.*

Proposition 2.5. *[9] Assuming conditions (a) and (c') on a class $\mathcal{R}$ of semirings, condition (b) is equivalent to*
(b') if $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \ldots$ is an ascending chain of ideals of a semiring R and if each I_λ is in $\mathcal{R}$, then $\bigcup I_\lambda$ is in $\mathcal{R}$.

Theorem 2.6. *[9] A non-empty sub class $\mathcal{R}$ of a universal class $\mathbb{U}$ is a radical class if and only if*
a) $\mathcal{R}$ is homomorphically closed.
b') $\mathcal{R}$ has the inductive property.
c') $\mathcal{R}$ is closed under extensions.

Theorem 2.7. *[9] For any sub class $\mathcal{R}$ of a fixed universal class $\mathbb{U}$, the following conditions are equivalent*

I. *$\mathcal{R}$ is a radical class.*

II. *(R1) If $R \in \mathcal{R}$ then every $R \mapsto S \neq 0$ there is a $I \triangleleft S$ such that $0 \neq I \in \mathcal{R}$.*

   (R2) If R is a semiring of a universal class $\mathbb{U}$ and for every $R \mapsto S \neq 0$ there is a $I \triangleleft S$ such that $0 \neq I \in \mathcal{R}$, then $R \in \mathcal{R}$.

III. *$\mathcal{R}$ satisfies condition (R1), has the inductive property and closed under extensions.*

3 Semisimple Class and Upper Radical Class

The definition of Semisimple classes deals with the definition of radical classes and for that purpose we characterized conditions (R1) and (R2) of Theorem 2.7.

Definition 3.1. *[3] A subclass ϱ of a universal class $\mathbb{U}$ is called a semisimple class of $\mathbb{U}$ if ϱ satisfies following two axioms which refer to*

$$\forall\ I\ (I \triangleright R)\ \exists J\ (I \mapsto J \quad and \quad J \in \varrho) \tag{1}$$

(Si) For all $R \in \mathbb{U}$, 1 implies $R \in \varrho$.

(Sii) Each $R \in \sigma$, satisfies 1.

Conditions (Si) and (Sii), are the dual to (Ri) and (Rii) where the conditions $\longmapsto$ and $\triangleleft$ are interchanged (R1) and (R2) of Theorem 2.7. Since the relation $\longmapsto$ is transitive one can show that every radical class is homomorphically closed. However $\triangleleft$ is not transitive in general, therefore it is very difficult to describe semisimple classes.

Proposition 3.2. *[3] If $\mathcal{R}$ is radical class, then $\sigma = \{R/\mathcal{R}(R) = 0\}$ is a semisimple class.*

Theorem 3.3. *[3] For any radical $\mathcal{R}$ and any semiring R, if $I \triangleleft R$, then $\mathcal{R}(I) \triangleleft R$.*

Definition 3.4. *[3] A class $\mathcal{R}$ of semirings is a hereditary radical class if $R \in \mathcal{R}$ and I is an ideal of R, then $I \in \mathcal{R}$.*

Definition 3.5. *[3] A class $\mathcal{R}$ is said to be regular if for every semiring $R \in \mathcal{R}$, every nonzero ideal of R has a nonzero homomorphic image in $\mathcal{R}$.*

In particular, every hereditary class is regular.

Theorem 3.6. *If $\mathcal{R}$ is a regular class of semirings, then the class*

$$\mathscr{U}_{\mathcal{R}} = \{\ R \mid R \text{ has no nonzero homomorphic image in } \mathcal{R}\}$$

is a radical class, $\mathcal{R} \cap \mathscr{U}_{\mathcal{R}} = \{0\}$ and $\mathscr{U}_{\mathcal{R}}$ is largest radical having zero intersection with $\mathcal{R}$.

Convention: The operator $\mathscr{U}$ is called upper radical operator and $\mathscr{U}_{\mathcal{R}}$ is called the upper radical of the class $\mathcal{R}$.

Theorem 3.7. *[3] For any Semisimple class ϱ and a radical class $\mathcal{R}$ we have $\mathscr{S}\mathscr{U}_{\varrho} = \varrho$ and $\mathscr{U}\mathscr{S}_{\mathcal{R}} = \mathcal{R}$.*

Proposition 3.8. *Every Semisimple class ϱ is closed under extensions.*

Proof. We take I and R/I in ϱ and we want to show that R is in σ. First we note that

$$(\mathscr{U}_{\varrho}(R) + I)/I \cong \mathscr{U}_{\varrho}(R)/(\mathscr{U}_{\varrho}(R) \cap I) \in \mathscr{U}_{\varrho}.$$

It is also clear that

$$(\mathscr{U}_{\varrho}(R) + I)/I \triangleleft R/I \in \varrho = \mathscr{S}\mathscr{U}_{\varrho}.$$

Therefore $\mathscr{U}_{\varrho}(R) + I)/I$ must be zero and so $\mathscr{U}_{\varrho}(R) \subseteq I$. Now by $\mathscr{U}_{\varrho}(R) \triangleleft R$ also $\mathscr{U}_{\varrho}(R) \triangleleft I$, and since $\mathscr{U}_{\varrho}(R) \in \mathscr{U}_{\varrho}(R)$, we get $\mathscr{U}_{\varrho}(R) \subseteq \mathscr{U}_{\varrho}(I) = 0$. Thus $R \in \mathscr{S}\mathscr{U}_{\varrho} = \varrho$. Thus class ϱ is closed under extension.

Theorem 3.9. *The classes $\mathcal{R}$ and ϱ are corresponding radical and semisimple classes if and only if*

i) $R \in \mathcal{R}$ *and* $R \mapsto S \neq 0$ *imply* $S \notin \varrho$, *that is,* $\mathcal{R} \subseteq \mathscr{U}_\varrho$,

ii) $R \in \varrho$ *and* $0 \neq S \lhd R$ *imply* $S \notin \mathcal{R}$, *that is,* $\varrho \subseteq \mathscr{S}_\mathcal{R}$.

iii) every semiring R *of the universal class* $\mathbb{U}$ *has an ideal* S *such that* $S \in \mathcal{R}$ *and* $R/S \in \varrho$.

Proof. If classes $\mathcal{R}$ and ϱ are corresponding radical and semisimple classes then the three conditions are clear (to get (iii) just take $S = \mathcal{R}(R)$).

Conversely, suppose we have classes $\mathcal{R}$ and ϱ satisfying the three conditions. Let us consider a semiring $R \in \mathscr{U}_\varrho$, by (iii) R has an ideal $S \in \mathcal{R}$ such that $R/S \in \varrho$. Hence by $R \in \mathscr{U}_\varrho$ we conclude that $R/S = 0$, and so $R = S \in \mathcal{R}$ holds, proving $\mathscr{U}_\varrho \subseteq \mathcal{R}$. This and (i) gives $\mathcal{R} = \mathscr{U}_\varrho$. A similar reasoning yields that $\varrho = \mathscr{S}_\mathcal{R}$. Since $\varrho = \mathscr{S}_\mathcal{R} = \mathscr{S}\mathscr{U}_\varrho$, also $\varrho \subseteq \mathscr{S}\mathscr{U}_\varrho$ holds and this is nothing but the regularity of the class ϱ. Hence $\mathcal{R} = \mathscr{U}_\varrho$ is a radical class and $\varrho = \mathscr{S}\mathscr{U}_\varrho = \mathscr{S}_\mathcal{R}$ the corresponding semisimple class.

Proposition 3.10. *The Semisimple class ϱ is hereditary if and only if the corresponding radical class $\mathcal{R} = \mathscr{U}_\varrho$ satisfies*

$$\mathcal{R}(I) \subseteq \mathcal{R}(R) \quad for\ every \quad I \lhd R. \tag{2}$$

Proof. If we have (2), then for any $R \in \varrho$ and $I \lhd R$ we have $\mathcal{R}(I) \subseteq \mathcal{R}(R)) = 0$, and so $I \in \varrho$. Thus ϱ is hereditary. Conversely, suppose that ϱ is hereditary. Then for $I \lhd R$ we have

$$(\mathcal{R}(I) + \mathcal{R}(R))/\mathcal{R}(R) \lhd (I + \mathcal{R}(R))/\mathcal{R}(R) \lhd R/\mathcal{R}(R) \in \varrho.$$

Hence $I + \mathcal{R}(R)/\mathcal{R}(R) \in \varrho$ and $(\mathcal{R}(I) + \mathcal{R}(R))/\mathcal{R}(R) \in \varrho$ because ϱ is hereditary. But this gives us

$$\mathcal{R}(I)/(\mathcal{R}(I) \cap \mathcal{R}(R)) \cong (\mathcal{R}(I) + \mathcal{R}(R))/\mathcal{R}(R) \in \mathcal{R} \cap \varrho = \{0\}.$$

Thus $\mathcal{R}(I) \subseteq \mathcal{R}(R)$ as claimed. □

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