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Differential Subordinations for New Subclass of Multivalent Functions Defined by Generalized Derivative Operator

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Abstract

In this paper, we introduce new subclass of multivalent functions by making use of the principle of subordination between these functions and generalized derivative operator. We study several properties like, coefficient estimates, distortion theorem, partial sums and integral means inequalities'.

Keywords: *Multivalent function, Differential subordination, Fractional calculus, Integral means.*

1 Introduction

Let $\Sigma_{p,\delta}$ denote the class of functions of the form:

$$f(z) = z^{p+\delta} + \sum_{k=2}^{\infty} a_k z^{k+p+\delta}, \qquad (0 \le \delta < 1) \tag{1}$$

which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$.

Let A_p denote the subclass of $\Sigma_{p,\delta}$ consisting of functions of the form:

$$f(z) = z^{p+\delta} - \sum_{k=2}^{\infty} a_k z^{k+p+\delta}, \qquad (a_k \ge 0, 0 \le \delta < 1) \tag{2}$$

which are analytic in the unit disk U.For the Hadamard product or (convolution) of two power series f defined in (2) and the function g where:

$$g(z) = z^{p+\delta} - \sum_{k=2}^{\infty} b_k z^{k+p+\delta}, \qquad (b_k \geq 0, 0 \leq \delta < 1, z \in U)$$

Is:

$$(f * g)(z) = z^{p+\delta} - \sum_{k=2}^{\infty} a_k b_k z^{k+p+\delta} = (g * f)(z), \qquad (z \in U) \tag{3}$$

Note that the authors defined and studied some classes of analytic functions take form (1) in [3]

We consider the following definitions of fractional integrals and fractional derivatives are due to Owa [5] and Srivastava and Owa [8]

Definition 1. *The fractional integral of order λ is defined for a function f by:*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \tag{4}$$

where $\lambda > 0$, f is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 2. *The fractional derivative of order λ is defined for a function f by:*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt \tag{5}$$

where $0 \leq \lambda < 1$, f is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1 above.

Definition 3. *Under the hypothesis of Definition 2 , the fractional derivative of order $n+\lambda$ is defined , for a function f , by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \qquad (0 \leq \lambda < 1; n \in N_0 = N \cup \{0\}).$$

It readily follows from Definition 2 that

$$D_z^{\lambda} z^{k+p+\delta} = \frac{\Gamma(k+p+\delta+1)}{\Gamma(k+p+\delta-\lambda+1)} z^{k+p+\delta-\lambda}, (0 \leq \lambda < 1). \tag{6}$$

Motivated with the definition of salagean operator [1], we need to introduce a generalized derivative operator such that class can be defined by means of this derivative operator. For a function $f \in A_p$ given by (2) , we define the derivative operator $(D_z^{\lambda,p,\delta})^n f(z)$ by:

$(D_z^{\lambda,p,\delta})^0 f(z) = \frac{\Gamma p+\delta-\lambda+1}{\Gamma p+\delta+1} z^\lambda D_z^\lambda f(z)$

$(D_z^{\lambda,p,\delta})^1 f(z) = \frac{z}{p+\delta}\left(\frac{\Gamma p+\delta-\lambda+1}{\Gamma p+\delta+1} z^\lambda D_z^\lambda f(z)\right)'$

$$\begin{aligned}(D_z^{\lambda,p,\delta})^n f(z) &= \left((D_z^{\lambda,p,\delta})(D_z^{\lambda,p,\delta})^{n-1} f(z)\right)\\ &= z^{p+\delta} - \sum_{k=2}^{\infty} (\phi(k,p,\delta,\lambda))^n a_k z^{k+p+\delta}, \end{aligned} \tag{7}$$

where $\phi(k,p,\delta,\lambda) = \frac{\Gamma(k+p+\delta+1)\Gamma(p+\delta-\lambda+1)}{\Gamma(p+\delta+1)\Gamma(k+p+\delta-\lambda+1)}\left(\frac{k+p+\delta}{p+\delta}\right)$. $\quad (p \in N; 0 \le \delta < 1)$
Note that ,when $\delta = 0$ the operator $(D_z^{\lambda,p,0})^n$ was studied recently by S.Porwal ,P.Dixit and V.Kumar [6] .
Clearly, (7) yields :

$$f \in A_p \Rightarrow (D_z^{\lambda,p,\delta})^n f \in A_p.$$

For two functions f and g analytic in U , we say that the function f is subordinate to g in U ,and write $f \prec g \quad (z \in U)$ if there exists a Schwarz function $w(z)$,which is analytic in U with

$$w(0) = 0 \quad , |w(z)| < 1 \qquad (z \in U) \tag{8}$$

Such that

$$f(z) = g(w(z)) \qquad (z \in U). \tag{9}$$

Indeed ,it is know that

$$f(z) \prec g(z) \qquad ,(z \in U) \Rightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{10}$$

Furthermore ,if the function g is univalent in U ,then we have the following equivalence :

$$f(z) \prec g(z) \quad ,(z \in U) \Leftrightarrow f(0) = g(0) \quad , f(U) \subset g(U) \tag{11}$$

By making use of the operator $(D_z^{\lambda,p,\delta})^n$ and the above mentioned principle of subordination between analytic functions , we introduce and investigate the following new subclass of the class A_p of $(p+\delta)$-valent analytic functions.

Definition 4. *A function $f \in A_p$ be given by (2) is said to be in the class $A^{\alpha}_{p,\delta}(k, \lambda, A, B)$ if it satisfies the following subordination condition:*

$$(p+\delta)\left(\frac{(1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z)\right)+\alpha z^{p+\delta}}{z\left((D_z^{\lambda,p,\delta})^n f(z)\right)'}\right) \prec \frac{1+Bz}{1+Az}, \tag{12}$$

where $0 \leq \alpha \leq 1, p \in N, 0 \leq \delta < 1, n \in N_0, 0 \leq \lambda < 1, -1 \leq A < B < 1$ and $(D_z^{\lambda,p,\delta})^n f$ given by (7).

2 Coefficient Estimates

Theorem 1. *Let the function $f \in A_p$ be defined by (2). Then f is in the class $A^{\alpha}_{p,\delta}(k, \lambda, A, B)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{(\phi(k,p,\delta,\lambda))^n \left((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B)\right)}{(p+\delta)(B+A)} \leq 1. \tag{13}$$

where $0 \leq \alpha \leq 1, p \in N, 0 \leq \delta < 1, n \in N_0, 0 \leq \lambda < 1, -1 \leq A < B < 1$.

Proof. Let $f \in A^{\alpha}_{p,\delta}(k, \lambda, A, B)$, then by the definition of subordination , we can write (12) as

$$(p+\delta)\left(\frac{(1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z)\right)+\alpha z^{p+\delta}}{z\left((D_z^{\lambda,p,\delta})^n f(z)\right)'}\right) = \frac{1+Bw(z)}{1+Aw(z)} \qquad (w(z) \in U),$$

which gives

$$(p+\delta)\left(\frac{(1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z)\right)+\alpha z^{p+\delta}}{z\left((D_z^{\lambda,p,\delta})^n f(z)\right)'}\right) - 1$$

$$= \left[B - A(p+\delta)\left(\frac{(1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z)\right)+\alpha z^{p+\delta}}{z\left((D_z^{\lambda,p,\delta})^n f(z)\right)'}\right)\right] w(z). \tag{14}$$

From (14) , we obtain

$$\frac{(p+\delta)z^{p+\delta} - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n ((p+\delta)(1-\alpha)a_k z^{k+p+\delta}}{(p+\delta)z^{p+\delta} - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n (k+p+\delta)a_k z^{k+p+\delta}} - 1$$

$$= \left[B - A(p+\delta)\left(\frac{(p+\delta)z^{p+\delta} - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(p+\delta)(1-\alpha)a_k z^{k+p+\delta}}{(p+\delta)z^{p+\delta} - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(k+p+\delta)a_k z^{k+p+\delta}}\right)\right] w(z)$$

which yields

$$\frac{-\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)-(k+p+\delta))a_k z^k}{(p+\delta) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(k+p+\delta)a_k z^k}$$

$$= \left[\frac{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k}{(p+\delta) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(k+p+\delta)a_k z^k}\right] w(z).$$

Since $|w(z)| \leq 1$,

$$\left|-\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)-(k+p+\delta))a_k z^k\right|$$
$$\leq \left|(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k\right|$$

Letting $|z| \to 1^-$, through real values ,we have

$$\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))a_k \leq (p+\delta)(B-A)$$

and , therefore ,

$$\sum_{k=2}^{\infty}\frac{(\phi(k,p,\delta,\lambda))^n\left((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B)\right)}{(p+\delta)(B+A)} \leq 1.$$

Conversely ,assume that (13) be true .From (14) we see that since $|w(z)| \leq 1$,

$$\left|\frac{(p+\delta)\left((1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z) + \alpha z^{p+\delta}\right)\right) - z\left((D_z^{\lambda,p,\delta})^n f(z)\right)'}{Bz\left((D_z^{\lambda,p,\delta})^n f(z)\right)' - A(p+\delta)\left((1-\alpha)\left((D_z^{\lambda,p,\delta})^n f(z) + \alpha z^{p+\delta}\right)\right)}\right|$$
$$= \left|\frac{-\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)-(k+p+\delta))a_k z^k}{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k}\right| \leq 1. \quad (15)$$

We need to prove that (15) is true .

By applying the hypothesis (13) and letting $|z| = 1$, we find that

$$\left|\frac{-\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)-(k+p+\delta))a_k z^k}{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k}\right|$$
$$\leq \frac{\sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)-(k+p+\delta))a_k z^k}{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k}$$
$$\leq \frac{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k}{(p+\delta)(B-A) - \sum_{k=2}^{\infty}(\phi(k,p,\delta,\lambda))^n(B(k+p+\delta) - A(p+\delta)(1-\alpha)a_k z^k} \leq 1.$$

Hence we find that (15) is true . Therefore $f \in A_{p,}^{\alpha}(k,\lambda,A,B)$.

This completes the proof of the theorem.

3 Distortion Theorem

Theorem 2. *If* $f \in A^{\alpha}_{p,\delta}(k, \lambda, A, B)$ *, then*

$$r^{p+\delta} - r^{p+\delta+2}\left\{\frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}\right\}$$
$$\leq \quad |f(z)| \leq r^{p+\delta} + r^{p+\delta+2}\left\{\frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}\right\}$$

$(|z| = r < 1)$.
Equality is attained for

$$f(z) = z^{p+\delta} - z^{p+\delta+2}\left\{\frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}\right\}.$$

Proof. From Theorem 1, we have that

$$\sum_{k=2}^{\infty} a_k \leq \frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}. \qquad (16)$$

From (2) and (16) , it follows that

$$\begin{aligned} |f(z)| &= \left| z^{p+\delta} - \sum_{k=2}^{\infty} a_k z^{k+p+\delta} \right| \\ &\leq |z|^{p+\delta} + \sum_{k=2}^{\infty} a_k |z|^{k+p+\delta} \\ &\leq r^{p+\delta} + r^{p+\delta+2} \sum_{k=2}^{\infty} a_k \\ &\leq r^{p+\delta} + r^{p+\delta+2}\left\{\frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}\right\}. \end{aligned}$$

Similarly ,

$$\begin{aligned} |f(z)| &\geq |z|^{p+\delta} - \sum_{k=2}^{\infty} a_k |z|^{k+p+\delta} \\ &\geq r^{p+\delta} - r^{p+\delta+2} \sum_{k=2}^{\infty} a_k \\ &\geq r^{p+\delta} - r^{p+\delta+2}\left\{\frac{(p+\delta)(B-A)}{((p+\delta)(1-\alpha)(1-A)-(p+\delta+2)(1-B))\,(\phi(2,p,\delta,\lambda))^n}\right\}. \end{aligned}$$

This completes the proof.

4 Partial Sums of the Function Class $f \in A^{\alpha}_{p,\delta}(k, \lambda, A, B)$

Following the earlier work by Silverman [7] and recently Atshan and Joudah [2],we investigate the ratio of real parts of functions involving (2) and its sequence of partial sums defined by:

$$L_m(z) = \begin{cases} z^{p+\delta} & ,(m=1) \\ z^{p+\delta} - \sum_{k=2}^{m} a_k z^{k+p+\delta} & ,(m=2,3,...;k \geq 2; p \in N) \end{cases} \tag{17}$$

and determine sharp lower bounds for $R\left\{\frac{f(z)}{L_m(z)}\right\}$, $R\left\{\frac{L_m(z)}{f(z)}\right\}$.

Theorem 3. *Let $f \in A_p$ and $L_m(z)$ be given (2) and (17), respectively. Suppose also that*

$$\sum_{k=2}^{\infty} \varphi_k a_k \leq 1,$$

where

$$\left(\varphi_k = \frac{((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))(\phi(k,p,\delta,\lambda))^n}{(p+\delta)(B-A)}\right). \tag{18}$$

Then for, we have

$$R\left\{\frac{f(z)}{L_m(z)}\right\} > 1 - \frac{1}{\varphi_{m+1}}, \tag{19}$$

and

$$R\left\{\frac{L_m(z)}{f(z)}\right\} > \frac{\varphi_{m+1}}{1+\varphi_{m+1}}. \tag{20}$$

The results are sharp for every m with the extremely functions given by

$$f(z) = z^{p+\delta} - \frac{1}{\varphi_{m+1}} z^{m+1} \tag{21}$$

Proof. Under the hypothesis of the theorem , we can see from (18) that

$$\varphi_{k+1} > \varphi_k > 1 \qquad (k \geq 2).$$

Therefore , we have

$$\sum_{k=2}^{m} a_k + \varphi_{m+1} \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} \varphi_k a_k \leq 1, \tag{22}$$

by using hypothesis (18) again
Upon setting

$$h_1(z) = \varphi_{m+1}\left[\frac{f(z)}{L_m(z)} - \left(1 - \frac{1}{\varphi_{m+1}}\right)\right] = 1 - \frac{\varphi_{m+1}\sum_{k=m+1}^{\infty} a_k z^k}{1-\sum_{k=2}^{m} a_k z^k}. \tag{23}$$

By applying (22) and (23) ,we find that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| = \left|\frac{-\varphi_{m+1}\sum_{k=m+1}^{\infty} a_k z^k}{2-2\sum_{k=2}^{m} a_k z^k - \varphi_{m+1}\sum_{k=m+1}^{\infty} a_k z^k}\right| \le \frac{\varphi_{m+1}\sum_{k=m+1}^{\infty} a_k z^k}{2-2\sum_{k=2}^{m} a_k z^k - \varphi_{m+1}\sum_{k=m+1}^{\infty} a_k z^k} \le 1 \quad (z \in U; k \ge 2), \tag{24}$$

which shows that $R\{h_1(z)\} > 0 \quad (z \in U)$. From (23), we immediately obtain the inequality (19).

To see that the function f given by (21) gives the sharp result ,we observe for $z \to 1^-$ that

$$\frac{f(z)}{L_m(z)} = 1 - \frac{1}{\varphi_{m+1}} z^{m-(p+\delta)+1} \to 1 - \frac{1}{\varphi_{m+1}},$$

which shows that the bound in (19) is the best possible .
Similarly , if we put

$$h_2(z) = (1+\varphi_{m+1})\left[\frac{L_m(z)}{f(z)} - \frac{\varphi_{m+1}}{1-\varphi_{m+1}}\right] = 1 + \frac{(1+\varphi_{m+1})\sum_{k=m+1}^{\infty} a_k z^k}{1-\sum_{k=2}^{m} a_k z^k}, \tag{25}$$

and make use of (22) , we can deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| = \left|\frac{(1+\varphi_{m+1})\sum_{k=m+1}^{\infty} a_k z^k}{2-2\sum_{k=2}^{m} a_k z^k + (\varphi_{m+1}-1)\sum_{k=m+1}^{\infty} a_k z^k}\right| \le \frac{(1+\varphi_{m+1})\sum_{k=m+1}^{\infty} a_k z^k}{2-2\sum_{k=2}^{m} a_k z^k + (\varphi_{m+1}-1)\sum_{k=m+1}^{\infty} a_k z^k} \le 1 \quad (z \in U; k \ge 2), \tag{26}$$

which leads us immediately to assertion (20) of the theorem .
The bound in (20) is sharp with the extremal function given by (21).
The proof of theorem is thus completed.

5 Weighted Mean and Arithmetic Mean

For f and g belong to A_p, the weighted mean h_j of f and g is given by:

$$h_j(z) = \frac{1}{2}\left[(1-j)f(z) + (1+j)g(z)\right]. \tag{27}$$

In the theorem below we will show the weighted mean for this class.

Theorem 4. *If f and g are in the class $A^{\alpha}_{p,\delta}(k,\lambda,A,B)$, then the weighted mean of f and g is also in $A^{\alpha}_{p,\delta}(k,\lambda,A,B)$.*

Proof. We have for h_j be given (27),

$$\begin{aligned} h_j(z) &= \frac{1}{2}\left[(1-j)\left(z^{p+\delta} - \sum_{k=2}^{\infty} a_k z^{k+p+\delta}\right) + (1+j)\left(z^{p+\delta} - \sum_{k=2}^{\infty} b_k z^{k+p+\delta}\right)\right]. \\ &= z^{p+\delta} - \sum_{k=2}^{\infty} \frac{1}{2}\left[(1-j)a_k + (1+j)b_k\right] z^{k+p+\delta}. \end{aligned}$$

Since f and g are in the class $A^{\alpha}_{p,\delta}(k,\lambda,A,B)$, so by Theorem1 we must prove that

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))}{(p+\delta)(B-A)}\left(\frac{1}{2}\left[(1-j)a_k + (1+j)b_k\right]\right) \\ &= \frac{1}{2}(1-j)\sum_{k=2}^{\infty} \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))}{(p+\delta)(B-A)} \\ &+ \frac{1}{2}(1+j)\sum_{k=2}^{\infty} \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))}{(p+\delta)(B-A)} \\ &\leq \frac{1}{2}(1-j) + \frac{1}{2}(1+j) = 1. \end{aligned}$$

The proof is complete.

Theorem 5. *Let $f_1, f_2, \ldots, f_\ell$ defined by:*

$$f_i(z) = z^{p+\delta} - \sum_{k=2}^{\infty} a_{i,k} z^{k+p+\delta}, \qquad (a_{i,k} \geq 0; i = 1,2,\ldots,\ell; k \geq 2) \tag{28}$$

be in the class $A^{\alpha}_{p,\delta}(k,\lambda,A,B)$, then the arithmetic mean of $f_i (i = 1,2,\ldots,\ell)$ defined by :

$$h(z) = \frac{1}{\ell}\sum_{i=1}^{\ell} f_i(z), \tag{29}$$

is also in the class $A^{\alpha}_{p,\delta}(k, \lambda, A, B)$.

Proof. By (28),(29) we can write

$$h(z) = \frac{1}{\ell}\sum_{i=1}^{\ell}\left(z^{p+\delta} - \sum_{k=2}^{\infty} a_{i,k} z^{k+p+\delta}\right) = z^{p+\delta} - \sum_{k=2}^{\infty}\left(\frac{1}{\ell}\sum_{i=1}^{\ell} a_{i,k}\right) z^{k+p+\delta}.$$

Since $f_i \in A^{\alpha}_{p,\delta}(k, \lambda, A, B)$ for every $i = 1, 2, \ldots, \ell$, so by using Theorem 1,
We prove that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))}{(p+\delta)(B-A)}\left(\frac{1}{\ell}\sum_{i=1}^{\ell} a_{i,k}\right) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell}\left(\sum_{k=2}^{\infty}\frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-\beta))}{(p+\delta)(B-A)} a_{i,k}\right) \\
&\leq \frac{1}{\ell}\sum_{i=1}^{\ell} = 1
\end{aligned}$$

The proof is complete.

6 Integral Means Inequalities for the Class $A^{\alpha}_{p,\delta}(k, \lambda, A, B)$

In 1925 ,Littlewood [4] proved the following subordination theorem .

Theorem 6. *(Littlewood) If f and g are analytic functions in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\vartheta}(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\vartheta \leq \int_0^{2\pi} |g(z)|^{\mu}\, d\vartheta, \qquad (\mu > 0; 0 < r < 1),$$

we will make use of the following theorem to prove

Theorem 7. *Let $f \in A^{\alpha}_{p,\delta}(k, \lambda, A, B)$ and suppose that f_n is defined by:*

$$f_n(z) = z^{p+\delta} - \frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))} z^{k+p+\delta}.k \geq 2$$

If there exists an analytic function w give by

$$\{w(z)\}^k = \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)} \sum_{k=2}^{\infty} a_k z^k,$$

Then for $z = re^{i\vartheta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\vartheta})\right|^\mu d\vartheta \le \int_0^{2\pi} \left|f_n(re^{i\vartheta})\right|^\mu d\vartheta, \qquad (\mu > 0; 0 < r < 1), \tag{30}$$

Proof. We must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^k\right|^\mu d\vartheta \le \int_0^{2\pi} \left|1 - \frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))} z^k\right|^\mu d\vartheta.$$

By applying Littlewoods subordination theorem ,it would suffice to show that

$$1 - \sum_{k=2}^{\infty} a_k z^k \prec 1 - \frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))} z^k.$$

By setting

$$1 - \sum_{k=2}^{\infty} a_k z^k = 1 - \frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))} \{w(z)\}^k.$$

We find that

$$\{w(z)\}^k = \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)} \sum_{k=2}^{\infty} a_k z^k$$

which readily yields w(0)=0 .
Furthermore, using (13) , we obtain

$$\begin{aligned}
|\{w(z)\}|^k &= \left|\frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)} \sum_{k=2}^{\infty} a_k z^k\right| \\
&\le \frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)} \sum_{k=2}^{\infty} a_k |z|^k \\
&\le |z| \le 1.
\end{aligned}$$

This completes the proof of the theorem.

Theorem 8. *Let* $\mu > 0$ *. If* $f \in A_{p,\delta}^{\alpha}(k,\lambda,A,B)$ *and*

$$f_n(z) = z^{p+\delta} - \frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))} z^{k+p+\delta}.k \ge 2$$

then for $z = re^{i\vartheta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} \left|f'(re^{i\vartheta})\right|^\mu d\vartheta \le \int_0^{2\pi} \left|f_n'(re^{i\vartheta})\right|^\mu d\vartheta, \tag{31}$$

Proof. It is sufficient to show that

$$1-\sum_{k=2}^{\infty}\left(\frac{k+p+\delta}{p+\delta}\right)a_k z^k$$
$$\prec \quad 1-\frac{(p+\delta)(B-A)}{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}\left(\frac{k+p+\delta}{p+\delta}\right)z^k. \quad (32)$$

This follows because

$$\begin{aligned}
|\{w(z)\}|^k &= \left|\sum_{k=2}^{\infty}\frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)}a_k z^k\right| \quad (33)\\
&\le |z|^k\sum_{k=2}^{\infty}\frac{(\phi(k,p,\delta,\lambda))^n((p+\delta)(1-\alpha)(1-A)-(k+p+\delta)(1-B))}{(p+\delta)(B-A)}a_k\\
&\le |z|^k \le |z|\,.
\end{aligned}$$

The proof is complete.

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