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Global Existence, Exponential Decay and Blow-up of Solutions for Coupled a Class of Nonlinear Higher-order Wave Equations

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Abstract

In this paper we consider the decay rate and the blow up of solutions for the initial and Dirichlet boundary value problem for coupled a class of nonlinear higher order wave equations, in a bounded domain.

Keywords: *decay, blow-up, higher-order wave equations, weak damping.*

1 Introduction

In this paper we consider the following initial-boundary value problem

$$\begin{cases} u_{tt} + Pu + u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} + Pv + v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \\ D^\alpha u(x,t) = D^\alpha v(x,t) = 0,\ |\alpha| \le m-1, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $P = (-\triangle)^m$, $m \geq 1$ is a natural number, Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n; $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $D^\alpha = \prod_{i=1}^{n} \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$, $x = (x_1, x_2, ..., x_n)$.

The motivation of our work is due to some results regarding higher-order wave equation. A single wave equation of the problem (1.1) becomes as following

$$u_{tt} + (-\triangle)^m u + g(u_t) = |u|^{p-1} u. \tag{1.2}$$

When $m = 1$ and $g(u_t) = u_t$, the equation in (1.2) becomes a wave equation

$$u_{tt} - \triangle u + u_t = |u|^{p-1} u, \tag{1.3}$$

was considered by many authors, see [2, 4, 5]. In [4], Ikehata showed certain decay rate for the total energy $E(t)$ and L^2 norm of a solution to equation (1.3). Also, in [2, 5], the authors obtained a time-decay result.

When $m = 2$, the equation in (1.2) becomes a Petrovsky equation, studied by many authors [8, 11], for $g(u_t) = |u_t|^{q-1} u_t$ $(q > 1)$.

Recently, Ye [12] investigated equation (1.2) for $m \geq 1$ and $g(u_t) = |u_t|^{q-1} u_t$ $(q > 1)$, and showed the existence of global solutions if the initial energy is sufficiently small. Also, Zhou et. al. [13] extended the results of [12].

In this paper, under some restrictions on the initial data, we establish the uniform decay rates. After that, we show blow up of solution with negative and nonnegative initial energy, using the same techniques as in [7].

This paper is organized as follows. In section 2, we present some lemmas, and the local existence theorem. In section 3, the global existence and the decay of the solution are given. In section 4, we show the blow up properties of solution.

2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this work. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$f_1(u,v) = \left[k\,|u+v|^{2(r+1)}(u+v) + l\,|u|^r u\,|v|^{r+2}\right],$$

$$f_2(u,v) = \left[k\,|u+v|^{2(r+1)}(u+v) + l\,|u|^{r+2}|v|^r v\right],$$

where $k, l > 0$ are constants and r satisfies

$$\begin{cases} -1 < r & \text{if } n \leq 2m, \\ -1 < r \leq \frac{3m-n}{n-2m} & \text{if } n > 2m. \end{cases} \tag{2.1}$$

According to the above equalities can easily verify that

$$u\ f_1(u,v) + v f_2(u,v) = 2(r+2)\,F(u,v)\,,\ \forall (u,v) \in R^2, \tag{2.2}$$

where

$$F\left(u,v\right)=\frac{1}{2\left(r+2\right)}\left[k\left|u+v\right|^{2(r+2)}+2l\left|uv\right|^{r+2}\right]. \tag{2.3}$$

We have the following result.

Lemma 2.1 [9]. There exist two positive constants c_0 and c_1 such that

$$c_0\left(\left|u\right|^{2(r+2)}+\left|v\right|^{2(r+2)}\right)\leq 2\left(r+2\right)F\left(u,v\right)\leq c_1\left(\left|u\right|^{2(r+2)}+\left|v\right|^{2(r+2)}\right) \tag{2.4}$$

is satisfied.

Let's define

$$J\left(t\right)=\frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right)-\int_\Omega F\left(u,v\right)dx, \tag{2.5}$$

$$I\left(t\right)=\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2-2\left(r+2\right)\int_\Omega F\left(u,v\right)dx, \tag{2.6}$$

and also the energy function as follows

$$E\left(t\right)=\frac{1}{2}\left(\left\|u_t\right\|^2+\left\|v_t\right\|^2\right)+\frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right)-\int_\Omega F\left(u,v\right)dx. \tag{2.7}$$

Lemma 2.2. $E\left(t\right)$ is a nonincreasing function for $t\geq 0$ and

$$E'\left(t\right)=-\left(\left\|u_t\right\|^2+\left\|v_t\right\|^2\right)\leq 0. \tag{2.8}$$

Proof. Multiplying first equation of (1.1) by u_t and second equation by v_t, integrating over Ω using integrating by parts and summing up the product results, we get

$$E\left(t\right)-E\left(0\right)=-\int_0^t\left(\left\|u_\tau\right\|^2+\left\|v_\tau\right\|^2\right)d\tau \text{ for } t\geq 0. \tag{2.9}$$

Moreover, the following energy inequality holds:

$$E\left(t\right)+\int_s^t\left(\left\|u_\tau\right\|^2+\left\|v_\tau\right\|^2\right)d\tau\leq E\left(s\right), \text{ for } 0\leq s\leq t<T. \tag{2.10}$$

Lemma 2.3 (Sobolev-Poincare inequality) [1]. If $2\leq p\leq\frac{2q}{[n-2q]^+}$ $(2\leq p<\infty$ if $n=2q)$, then

$$\left\|u\right\|_p\leq C_*\left\|\left(-\triangle\right)^{\frac{q}{2}}u\right\| \text{ for } u\in H_0^q\left(\Omega\right) \tag{2.11}$$

holds with some constant C_*, where we put $[a]^+=\max\left\{0,a\right\}$, $\frac{1}{[a]^+}=\infty$ if $[a]^+=0$.

The following integral inequality plays an important role in our proof of the energy decay of the solutions to problem (1.1).

Lemma 2.4 [6]. Let $h : [0,\infty) \longrightarrow [0,\infty)$ be a nonincreasing function and assume that there exists a constant $c > 0$ such that

$$\int_t^{\infty} h(\tau)\, d\tau \leq ch(t), \ \forall t \in [0,\infty).$$

Then we have

$$h(t) \leq h(0)\, e^{1-tc^{-1}}, \ \forall t \geq c.$$

Lemma 2.5 [7]. Let $\delta > 0$ and $B(t) \in C^2(0,\infty)$ be a nonnegative function satisfying

$$B''(t) - 4(\delta+1) B'(t) + 4(\delta+1) B(t) \geq 0.$$

If

$$B'(0) > r_2 B(0) + K_0,$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, then $B'(t) > K_0$ for $t > 0$, where K_0 is a constant.

Lemma 2.6 [7]. If $H(t)$ is a nonincreasing function on $[t_0,\infty)$ and satisfies the differential inequality

$$[H'(t)]^2 \geq a + b[H(t)]^{2+\frac{1}{\delta}}, \ \text{for } t \geq t_0,$$

where $a > 0,\ b \in R$, then there exists a finite time T^* such that

$$\lim_{t \longrightarrow T^{*-}} H(t) = 0.$$

Upper bounds for T^* are estimated as follows:
(i) If $b < 0$ and $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$ then

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$

(ii) If $b = 0$, then

$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$

(iii) If $b > 0$, then

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \text{ or } T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[1 - (1 + cH(t_0))^{-\frac{1}{2\delta}}\right],$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.

Next, we state the local existence theorem that can be established by combining arguments of [3, 8].

Theorem 2.1. (Local existence) Suppose that (2.1) holds, and further $(u_0, v_0) \in H_0^m(\Omega) \times H_0^m(\Omega)$, $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$. Then problem (1.1) has a unique local solution

$$u, v \in C\left([0,T); H_0^m(\Omega)\right),$$

$u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^2(\Omega \times [0,T))$ and $v_t \in C\left([0,T); L^2(\Omega)\right) \cap L^2(\Omega \times [0,T))$.

Moreover, at least one of the following statements holds true:

i) $T = \infty$,

ii) $\|u_t\|^2 + \|v_t\|^2 + \left\|P^{\frac{1}{2}}u\right\|^2 + \left\|P^{\frac{1}{2}}v\right\|^2 \longrightarrow \infty$ as $t \longrightarrow T^-$.

Remark 2.1. We denote by C various positive constants which may be different at different occurrences.

3 Global Existence and Energy Decay

In this section, we consider the global existence and energy decay of solutions for problem (1.1).

Lemma 3.1 [9]**.** Suppose that (2.1) holds. Then there exists $\eta > 0$ such that for any $(u, v) \in H_0^m(\Omega) \times H_0^m(\Omega)$, the inequality

$$\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \leq \eta \left(\left\|P^{\frac{1}{2}}u\right\|^2 + \left\|P^{\frac{1}{2}}v\right\|^2 \right)^{r+2} \tag{3.1}$$

is satisfied.

To prove our result and for the sake of simplicity, we take $k = l = 1$ and introduce

$$B = \eta^{\frac{1}{2(r+2)}},\ \alpha_* = B^{-\frac{r+2}{r+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)}\right)\alpha_*^2, \tag{3.2}$$

where η is the optimal constant in (3.1). Next, we will state and prove a lemma which similar to the one introduced firstly by Vitillaro in [10] to study a class of a single wave equation.

Lemma 3.2. Suppose that (2.1) holds. Let (u, v) be the solution of system (1.1). Assume further that $E(0) < E_1$ and

$$\left(\left\|P^{\frac{1}{2}}u_0\right\|^2 + \left\|P^{\frac{1}{2}}v_0\right\|^2 \right)^{\frac{1}{2}} < \alpha_*. \tag{3.3}$$

Then

$$\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)^{\frac{1}{2}}<\alpha_{*}, \tag{3.4}$$

for all $t\in[0,T)$.

Proof. First have from (2.7), (2.11), (3.1) and the definition of B,

$$\begin{aligned}
E(t) &\geq \frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)-\int_{\Omega}F(u,v)\,dx\\
&= \frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)-\frac{1}{2(r+2)}\left(\|u+v\|_{2(r+2)}^{2(r+2)}+2\|uv\|_{r+2}^{r+2}\right)\\
&\geq \frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)-\frac{1}{2(r+2)}\eta\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)^{r+2}\\
&\geq \frac{1}{2}\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)-\frac{B^{2(r+2)}}{2(r+2)}\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)^{r+2}
\end{aligned} \tag{3.5}$$

So we get

$$E(t)\geq G\left(\left\|P^{\frac{1}{2}}u\right\|^{2}+\left\|P^{\frac{1}{2}}v\right\|^{2}\right)\quad \text{for } t\geq 0, \tag{3.6}$$

where $G(\alpha)=\frac{1}{2}\alpha^{2}-\frac{B^{2(r+2)}}{2(r+2)}\alpha^{2(r+2)}$. Note that $G(\alpha)$ has the maximum at $\alpha_{*}=\frac{1}{B^{\frac{r+2}{r+1}}}$ and maximum value is

$$E_{1}=G(\alpha_{*})=\left(\frac{1}{2}-\frac{1}{2(r+2)}\right)\alpha_{*}^{2}. \tag{3.7}$$

Now we establish (3.4) by contradiction. Suppose (3.4) does not hold, then it follows from the continuity of $(u(t),v(t))$ that there exists $t_0\in(0,T)$ such that

$$\left(\left\|P^{\frac{1}{2}}u(t_0)\right\|^{2}+\left\|P^{\frac{1}{2}}v(t_0)\right\|^{2}\right)^{\frac{1}{2}}=\alpha_{*}. \tag{3.8}$$

By (3.5), we see that

$$E(t_0)\geq G\left[\left(\left\|P^{\frac{1}{2}}u(t_0)\right\|^{2}+\left\|P^{\frac{1}{2}}v(t_0)\right\|^{2}\right)^{\frac{1}{2}}\right]=G(\alpha_{*})=E_{1}. \tag{3.9}$$

This is impossible since $E(t)\leq E(0)<E_1,\ t\geq 0$. Hence (3.4) is established.

Theorem 3.1. (Global existence and energy decay) Assume that (2.1) hold. If the initial data $(u_0,u_1)\in H_0^m(\Omega)\times L^2(\Omega)$, $(v_0,v_1)\in H_0^m(\Omega)\times L^2(\Omega)$, satisfy $E(0)<E_1$ and

$$\left(\left\|P^{\frac{1}{2}}u_0\right\|^{2}+\left\|P^{\frac{1}{2}}v_0\right\|^{2}\right)^{\frac{1}{2}}<\alpha_{*}, \tag{3.10}$$

where the constants α_* and E_1 are defined in (3.2), then the corresponding solution to system (1.1) globally exists, i.e., $T=\infty$.

Moreover, if

$$1-\eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}>0, \tag{3.11}$$

then we have the following decay estimates

$$E(t)\leq E(0)\,e^{1-\mu C_4^{-1}t} \tag{3.12}$$

for every $t\geq \frac{C_4}{\mu}$, where C_4 is positive constant.

Proof. First, we prove that $T=\infty$. Since $E(0)<E_1$ and

$$\left(\left\|P^{\frac{1}{2}}u_0\right\|^2+\left\|P^{\frac{1}{2}}v_0\right\|^2\right)^{\frac{1}{2}}<\alpha_*,$$

it follows from Lemma 3.2 that

$$\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2<\alpha_*^2=\eta^{-\frac{1}{r+1}}$$

which implies that

$$\begin{aligned} I(t) &= \left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2-2(r+2)\int_\Omega F(u,v)\,dx \\ &\geq \left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2-\eta\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right)^{r+2}\geq 0 \end{aligned}$$

for $t\in[0,T)$. Furthermore, (2.5) and (2.6), we get

$$\begin{aligned} J(t) &= \frac{r+1}{2(r+2)}\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right)+\frac{1}{2(r+2)}I(t) \\ &\geq \frac{r+1}{2(r+2)}\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right). \end{aligned}$$

From (2.7) and $E(t)\leq E(0)$, we deduce that

$$\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\leq\frac{2(r+2)}{r+1}J(t)\leq\frac{2(r+2)}{r+1}E(t)\leq\frac{2(r+2)}{r+1}E(0) \tag{3.13}$$

for $t\in[0,T)$. So it follows from (3.13) and Lemma 2.2

$$\begin{aligned} \frac{r+1}{2(r+2)}\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right)+\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right) &\leq J(t)+\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right) \\ &= E(t)\leq E(0)<E_1,\ \forall t\in[0,T) \end{aligned}$$

which implies

$$\|u_t\|^2 + \|v_t\|^2 + \left\|P^{\frac{1}{2}}u\right\|^2 + \left\|P^{\frac{1}{2}}v\right\|^2 < C'E_1, \tag{3.14}$$

where $C' = \max\left\{\frac{1}{2}, \frac{2(r+2)}{r+1}\right\}$. Then, by Theorem 2.1, we have the global existence result.

Next, we want to derive the decay rate of energy function for problem (1.1). By multiplying first equation of system (1.1) by u and second equation of system (1.1) by v, integrating them over $\Omega \times [t_1, t_2]$ $(0 \le t_1 \le t_2)$, using integration by parts and summing up, we have

$$\begin{aligned}
& \int_\Omega uu_t dx|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|u_t\|^2 dt + \int_\Omega vv_t dx|_{t_1}^{t_2} - \int_{t_1}^{t_2} \|v_t\|^2 dt \\
& + \int_{t_1}^{t_2} \left\|P^{\frac{1}{2}}u\right\|^2 dt + \int_{t_1}^{t_2} \left\|P^{\frac{1}{2}}v\right\|^2 dt + \int_{t_1}^{t_2}\int_\Omega uu_t dxdt + \int_{t_1}^{t_2}\int_\Omega uu_t dxdt \\
= & \int_{t_1}^{t_2}\int_\Omega \left[uf_1(u,v) + vf_2(u,v)\right] dxdt.
\end{aligned}$$

It follows from (2.7)

$$\begin{aligned}
& 2\int_{t_1}^{t_2} E(t)\, dt - 2(r+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\, dxdt \\
= & -\int_\Omega (uu_t + vv_t)\, dx|_{t_1}^{t_2} + 2\int_{t_1}^{t_2} \left(\|u_t\|^2 + \|v_t\|^2\right) dt - \int_{t_1}^{t_2}\int_\Omega (uu_t + uu_t)\, dxdt \\
= & A_1 + A_2 + A_3.
\end{aligned} \tag{3.15}$$

In what follows we will estimate $A_1,\ A_2,\ A_3$ in (3.15). Firstly, by Hölder, Young and Sobolev Poincare inequalities, we have

$$\begin{aligned}
\int_\Omega (|uu_t| + |vv_t|)\, dx &\le \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 \\
&\le \frac{C}{2}\left\|P^{\frac{1}{2}}u(t)\right\|^2 + \frac{1}{2}\|u_t(t)\|^2 + \frac{C}{2}\left\|P^{\frac{1}{2}}v(t)\right\|^2 + \frac{1}{2}\|v_t(t)\|^2.
\end{aligned}$$

Then, by (3.13), we have

$$A_1 \le \int_\Omega (|uu_t| + |vv_t|)\, dx|_{t_1}^{t_2} \le 2C_1E(t_1). \tag{3.16}$$

For A_2 in (3.15), applying $\|u_t\|^2 + \|v_t\|^2 \le -E'(t)$ from (2.8), we have

$$A_2 = 2\int_{t_1}^{t_2} \left(\|u_t\|^2 + \|v_t\|^2\right) dt \le 2C_2E(t_1). \tag{3.17}$$

We also have the following estimate

$$
\begin{aligned}
A_3 &= \int_{t_1}^{t_2}\int_\Omega (uu_t + vv_t)\,dxdt \\
&= \frac{1}{2}\int_{t_1}^{t_2}\frac{d}{dt}\|u\|^2\,dt + \frac{1}{2}\int_{t_1}^{t_2}\frac{d}{dt}\|v\|^2\,dt \\
&= \frac{1}{2}\left(\|u(t_2)\|^2 - \|u(t_1)\|^2\right) + \frac{1}{2}\left(\|v(t_2)\|^2 - \|v(t_1)\|^2\right) \\
&\le \frac{2(r+2)}{r+1}E(t_1) = C_3E(t_1). \qquad (3.18)
\end{aligned}
$$

Inserting estimates (3.16)-(3.18) into (3.15), we arrive at

$$
2\int_{t_1}^{t_2}E(t)\,dt - 2(r+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\,dxdt \le C_4E(t_1), \qquad (3.19)
$$

where $C_4 = 2C_1 + 2C_2 + C_3$.

On the other hand, from (3.1) and (3.13), we have

$$
\begin{aligned}
2(r+1)\int_\Omega F(u,v)\,dx &= \frac{r+1}{r+2}\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right) \\
&\le \frac{r+1}{r+2}\eta\left(\left\|P^{\frac{1}{2}}u\right\|^2 + \left\|P^{\frac{1}{2}}v\right\|^2\right)^{r+2} \\
&\le 2\eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}E(t)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&2\int_{t_1}^{t_2}E(t)\,dt - 2(r+1)\int_{t_1}^{t_2}\int_\Omega F(u,v)\,dxdt \\
\ge\ & 2\left(1-\eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}\right)\int_{t_1}^{t_2}E(t)\,dt. \qquad (3.20)
\end{aligned}
$$

Noting that $E(0) < E_1$, we see

$$
1-\eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1} > 0.
$$

Thus, combining (3.19) and (3.20), we have

$$
2\left(1-\eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}\right)\int_{t_1}^{t_2}E(t)\,dt \le C_4E(t_1),
$$

that is

$$\mu \int_{t_1}^{t_2} E(t)\,dt \leq C_4 E(t_1), \tag{3.21}$$

where $\mu = 2\left(1 - \eta \left(\frac{2(r+2)}{r+1} E(0)\right)^{r+1}\right)$.

We rewrite (3.21) as

$$\mu \int_{t}^{\infty} E(t)\,dt \leq C_4 E(t) \tag{3.22}$$

for every $t \in [0, \infty)$.

Since $\mu > 0$ from the assumption of conditions, by Lemma 2.4, we have

$$E(t) \leq E(0)\, e^{1-\mu C_4^{-1} t}$$

for every $t \geq \frac{C_4}{\mu}$. The proof is completed.

4 Blow up of Solution

In this section, we deal with the blow up of solution of problem (1.1).

Definition 4.1. A solution (u, v) of (1.1) is called blow-up if there exists a finite time T^* such that

$$\lim_{t \longrightarrow T^{*-}} \left\{ \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dx ds \right\} = \infty. \tag{4.1}$$

Let

$$a(t) = \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dx ds, \text{ for } t \geq 0. \tag{4.2}$$

Lemma 4.1. Assume (2.1) holds and that $0 < \delta \leq \frac{r}{4}$, then we have

$$a''(t) \geq 4(\delta + 1) \int_\Omega \left(u_t^2 + v_t^2\right) dx + (-4 - 8\delta) E(0) + (4 + 8\delta) \int_0^t \left(\|u_t\|^2 + \|v_t\|^2\right) dt. \tag{4.3}$$

Proof. From (4.2), we have

$$a'(t) = 2 \int_\Omega (u u_t + v v_t)\, dx + \|u\|^2 + \|v\|^2. \tag{4.4}$$

By (1.1) and Divergence theorem, we get

$$\begin{aligned} a''(t) &= 2 \int_\Omega \left(u_t^2 + v_t^2\right) dx + 2 \int_\Omega (u u_{tt} + v v_{tt})\, dx + 2 \int_\Omega (u u_t + v v_t)\, dx \\ &= 2 \int_\Omega \left(u_t^2 + v_t^2\right) dx - 2 \left(\left\|P^{\frac{1}{2}} u\right\|^2 + \left\|P^{\frac{1}{2}} v\right\|^2\right) \\ &\quad + 2(r+2) \int_\Omega F(u, v)\, dx. \end{aligned} \tag{4.5}$$

Then from (2.9) and (4.5), we have

$$\begin{aligned} a''(t) &\geq 4(\delta+1)\int_\Omega \left(u_t^2+v_t^2\right)dx + (-4-8\delta)E(0) + (4+8\delta)\int_0^t \left(\|u_t\|^2+\|v_t\|^2\right)dt \\ &\quad +4\delta\left(\left\|P^{\frac{1}{2}}u\right\|^2+\left\|P^{\frac{1}{2}}v\right\|^2\right) + (2r-8\delta)\int_\Omega F(u,v)\,dx. \end{aligned}$$

Since $0<\delta\leq\frac{r}{4}$, $2r-8\delta\geq 0$, we obtain (4.3).

Lemma 4.2. Assume (2.1) holds and one of the following statements are satisfied:

(i) $E(0)<0$,
(ii) $E(0)=0$, and $\int_\Omega (u_0u_1+v_0v_1)\,dx>0$,
(iii) $E(0)>0$, and

$$a'(0) > r_2\left[a(0)+\frac{K_1}{4(\delta+1)}\right] + \left(\|u_0\|^2+\|v_0\|^2\right) \tag{4.6}$$

holds.

Then $a'(t)>\|u_0\|^2+\|v_0\|^2$ for $t>t^*$, where $t_0=t^*$ is given by (4.7) in case (i) and $t_0=0$ in cases (ii) and (iii).

Where K_1 and t^* are defined in (4.12) and (4.7), respectively.

Proof. (i) If $E(0)<0$, then from (4.3), we have

$$a'(t)\geq a'(0)-4(1+2\delta)E(0)\,t,\ \ t\geq 0.$$

Thus we get $a'(t)>\|u_0\|^2+\|v_0\|^2$ for $t>t^*$, where

$$t^*=\max\left\{\frac{a'(0)-\left(\|u_0\|^2+\|v_0\|^2\right)}{4(1+2\delta)E(0)},0\right\}. \tag{4.7}$$

(ii) If $E(0)=0$, and $\int_\Omega (u_0u_1+v_0v_1)\,dx>0$, then $a''(t)\geq 0$ for $t\geq 0$. We have $a'(t)>\|u_0\|^2+\|v_0\|^2,\ t\geq 0$.

(iii) If $E(0)>0$, we first note that

$$2\int_0^t\int_\Omega uu_t\,dxdt = \|u\|^2-\|u_0\|^2. \tag{4.8}$$

By Hölder inequality and Young inequality, we have from (4.8)

$$\|u\|^2\leq \|u_0\|^2+\int_0^t\|u\|^2\,dt+\int_\Omega \|u_t\|^2\,dt. \tag{4.9}$$

Similarly,

$$\|v\|^2 \leq \|v_0\|^2 + \int_0^t \|v\|^2 \, dt + \int_\Omega \|v_t\|^2 \, dt. \tag{4.10}$$

By Hölder inequality, Young inequality and inequalities (4.9) and (4.10), we have

$$a'(t) \leq a(t) + \|u_0\|^2 + \|v_0\|^2 + \int_\Omega \left(u_t^2 + v_t^2\right) dx + \int_0^t \left(\|u_t\|^2 + \|v_t\|^2\right) dt. \tag{4.11}$$

Hence, by (4.3) and (4.11), we obtain

$$a''(t) - 4(\delta+1) a'(t) + 4(\delta+1) a(t) + K_1 \geq 0,$$

where

$$K_1 = (4 + 8\delta) E(0) + 4(\delta+1)\left(\|u_0\|^2 + \|v_0\|^2\right). \tag{4.12}$$

Let

$$b(t) = a(t) + \frac{K_1}{4(\delta+1)}, \; t > 0.$$

Then $b(t)$ satisfies Lemma 2.5. Consequently, from (4.6), $a'(t) > \left(\|u_0\|^2 + \|v_0\|^2\right), \; t > 0$, where r_2 is given in Lemma 2.5.

Theorem 4.1. Assume (2.1) holds and one of the following statements are satisfied (for $0 < \delta \leq \frac{r}{4}$):

(i) $E(0) < 0$,
(ii) $E(0) = 0$, and $\int_\Omega (u_0 u_1 + v_0 v_1) \, dx > 0$,
(iii) $0 < E(0) < \frac{\left(a'(t_0) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]}$, and (4.6) holds.

Then the solution (u, v) blows up in finite time T^* in the sense of (4.1). In case (i),

$$T^* \leq t_0 - \frac{H(t_0)}{H'(t_0)}. \tag{4.13}$$

Furthermore, if $H(t_0) < \min\left\{1, \sqrt{-\frac{a}{b}}\right\}$, we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}, \tag{4.14}$$

where

$$a = \delta^2 H^{2+\frac{2}{\delta}}(t_0) \left[\left(a'(t_0) - \|u_0\|^2 - \|v_0\|^2\right)^2 - 8E(0) H^{-\frac{1}{\delta}}(t_0)\right] > 0, \tag{4.15}$$

$$b = 8\delta^2 E(0). \tag{4.16}$$

In case (ii),

$$T^{*}\leq t_{0}-\frac{H\left(t_{0}\right)}{H'\left(t_{0}\right)}. \tag{4.17}$$

In case (iii),

$$T^{*}\leq\frac{H\left(t_{0}\right)}{\sqrt{a}}\text{ or }T^{*}\leq t_{0}+2^{\frac{3\delta+1}{2\delta}}\left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}\frac{\delta}{\sqrt{a}}\left\{1-\left[1+\left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}H\left(t_{0}\right)\right]^{-\frac{1}{2\delta}}\right\}, \tag{4.18}$$

where a and b are given (4.15), (4.16).

Proof. Let

$$H\left(t\right)=\left[a\left(t\right)+\left(T_{1}-t\right)\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right]^{-\delta},\text{ for }t\in\left[0,T_{1}\right], \tag{4.19}$$

where $T_1>0$ is a certain constant which will be specified later. Then we get

$$\begin{aligned} H'\left(t\right) &= -\delta\left[a\left(t\right)+\left(T_{1}-t\right)\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right]^{-\delta-1}\left[a'\left(t\right)-\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right] \\ &= -\delta H^{1+\frac{1}{\delta}}\left(t\right)\left[a'\left(t\right)-\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right], \end{aligned} \tag{4.20}$$

$$\begin{aligned} H''\left(t\right) &= -\delta H^{1+\frac{2}{\delta}}\left(t\right)a''\left(t\right)\left[a\left(t\right)+\left(T_{1}-t\right)\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right] \\ &\quad +\delta H^{1+\frac{2}{\delta}}\left(t\right)\left(1+\delta\right)\left[a'\left(t\right)-\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right]^{2} \\ &= -\delta H^{1+\frac{2}{\delta}}\left(t\right)V\left(t\right). \end{aligned} \tag{4.21}$$

where

$$V\left(t\right)=a''\left(t\right)\left[a\left(t\right)+\left(T_{1}-t\right)\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right]-\left(1+\delta\right)\left[a'\left(t\right)-\left(\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}\right)\right]^{2}. \tag{4.22}$$

For simplicity of calculation, we define

$$\begin{aligned} &P_{u}=\int_{\Omega}u^{2}dx,\quad R_{u}=\int_{\Omega}u_{t}^{2}dx,\quad Q_{u}=\int_{0}^{t}\left\|u\right\|^{2}dt,\quad S_{u}=\int_{0}^{t}\left\|u_{t}\right\|^{2}dt, \\ &P_{v}=\int_{\Omega}v^{2}dx,\quad R_{v}=\int_{\Omega}v_{t}^{2}dx,\quad Q_{v}=\int_{0}^{t}\left\|v\right\|^{2}dt,\quad S_{v}=\int_{0}^{t}\left\|v_{t}\right\|^{2}dt. \end{aligned}$$

From (4.4), (4.8) and Hölder inequality, we get

$$\begin{aligned} a'\left(t\right) &= 2\int_{\Omega}\left(uu_{t}+vv_{t}\right)dx+\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}+2\int_{0}^{t}\int_{\Omega}\left(uu_{t}+vv_{t}\right)dxdt \\ &\leq 2\left(\sqrt{R_{u}P_{u}}+\sqrt{Q_{u}S_{u}}+\sqrt{R_{v}P_{v}}+\sqrt{Q_{v}S_{v}}\right)+\left\|u_{0}\right\|^{2}+\left\|v_{0}\right\|^{2}. \end{aligned} \tag{4.23}$$

If case (i) or (ii) holds, by (4.3) we have

$$a''\left(t\right)\geq\left(-4-8\delta\right)E\left(0\right)+4\left(1+\delta\right)\left(R_{u}+S_{u}+R_{v}+S_{v}\right). \tag{4.24}$$

Thus, from (4.19) and (4.22)- (4.24), we obtain

$$\begin{aligned} V(t) \geq & \left[(-4-8\delta) E(0) + 4(1+\delta)(R_u + S_u + R_v + S_v)\right] H^{-\frac{1}{\delta}}(t) \\ & -4(1+\delta)\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R_v P_v} + \sqrt{Q_v S_v}\right)^2 . \end{aligned}$$

From (4.2)

$$a(t) = \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \int_\Omega \left(u^2 + v^2\right) dx ds = P_u + P_v + Q_u + Q_v .$$

From the above inequality and (4.19), we get

$$\begin{aligned} V(t) \geq & (-4-8\delta) E(0) H^{-\frac{1}{\delta}}(t) + 4(1+\delta) \\ & \left[(R_u + S_u + R_v + S_v)(T_1 - t)\left(\|u_0\|^2 + \|v_0\|^2\right) + \Theta(t)\right], \end{aligned}$$

where

$$\begin{aligned} \Theta(t) = & (R_u + S_u + R_v + S_v)(P_u + Q_u + P_v + Q_v) \\ & -\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{R_v P_v} + \sqrt{Q_v S_v}\right)^2 . \end{aligned}$$

By the Schwarz inequality, and $\Theta(t)$ being nonnegative, we have

$$V(t) \geq (-4-8\delta) E(0) H^{-\frac{1}{\delta}}(t), \ t \geq t_0. \tag{4.25}$$

Therefore, by (4.21) and (4.25), we get

$$H''(t) \leq 4\delta(1+2\delta) E(0) H^{1+\frac{1}{\delta}}(t), \ t \geq t_0. \tag{4.26}$$

By Lemma 4.2, we know that $H'(t) < 0$ for $t \geq t_0$. Multiplying (4.27) by $H'(t)$ and integrating it from t_0 to t, we get

$$H'^2(t) \geq a + bH^{2+\frac{1}{\delta}}(t)$$

for $t \geq t_0$, where a, b are defined in (4.15) and (4.16) respectively.

If case (iii) holds, by the steps of case (i), we get $a > 0$ if and only if

$$E(0) < \frac{\left(a'(t_0) - \left(\|u_0\|^2 + \|v_0\|^2\right)\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|u_0\|^2 + \|v_0\|^2\right)\right]}.$$

Then by Lemma 2.6, there exists a finite time T^* such that $\lim_{t \longrightarrow T^{*-}} H(t) = 0$ and upper bound of T^* is estimated according to the sign of $E(0)$. This means that (4.1) holds.

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