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Connectivity in a Fuzzy Graph and its Complement

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Abstract

Connectivity has important role in the area of applications of fuzzy graphs such as fuzzy neural networks and clustering. In this paper criterion for connectivity of a fuzzy graph and its complement is analysed. The structure of the complement of a fuzzy cycle is also discussed.

Keywords: *Fuzzy relations, Complement of fuzzy graph, Fuzzy cycle, Connectivity in fuzzy graphs, m-strong arcs.*

1 Introduction

The notion of fuzzy graph was introduced by Rosenfeld in the year 1975 [6]. Fuzzy analogues of many structures in crisp graph theory, like bridges, cut nodes, connectedness, trees and cycles etc were developed after that. Fuzzy trees were characterized by Sunitha and Vijayakumar [3]. The authors have characterized fuzzy trees using its unique maximum spanning tree. A sufficient condition for a node to be a fuzzy cut node is also established. Center problems in fuzzy graphs, blocks in fuzzy graphs and properties of self complementary fuzzy graphs were also studied by the same authors. They have obtained a characterization for blocks in fuzzy graphs using the concept of strongest paths [5]. Bhutani and Rosenfeld have introduced the concepts of strong arcs, fuzzy end nodes and geodesics in fuzzy graphs [1] .The authors have defined the concepts of strong arcs and strong paths. As far as the applications are concerned (information networks, electric circuits, etc.), the reduction of flow between pairs of nodes is more relevant and may frequently occur than the total disruption of the flow or the disconnection of the entire network. In this paper we put forward the conditions under which a fuzzy graph and its complement will be connected.

2 Preliminaries

The following basic definitions are taken from[2]. A fuzzy graph is a pair G:(σ, μ), where σ is a fuzzy subset of a set V and μ is a fuzzy relation on σ, i.e, μ(x,y) $\leq$ σ(x) $\wedge$ σ(y), $\forall$ x,y $\in$ V. We assume that V is finite and non empty, μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also we denote the underlying crisp graph by G* : (σ*, μ*), where $\sigma^* = \{$u$\in$V / σ(u) >0$\}$ and $\mu^* = \{$(u,v) $\in$ VxV : μ(u,v)>0$\}$. H = (τ,ν) is called a partial fuzzy subgraph of G if $\tau \leq \sigma$ and $\nu \leq \mu$. We call H = (τ,ν) a spanning fuzzy subgraph of G = (σ, μ) if $\tau = \sigma$.

A path P of length n is a sequence of distinct nodes $u_0, u_1, u_2, \ldots u_n$ such that $\mu(u_{i-1}, u_i) > 0$ and degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and it is a fuzzy cycle if there is more than one weak arc.

The strength of connectedness between two nodes x,y is defined as the maximum of strengths of all paths between x and y and is denoted by $CONN_G(x, y)$.

An arc (x,y) is called a fuzzy bridge in G if the removal of (x,y) reduces the strength of connectedness between some pair of nodes in G. A connected fuzzy graph is called a fuzzy tree if it contains a spanning subgraph F which is a tree such that, for all arcs (x,y) not in F, $\mu(x, y) < CONN_F(x, y)$.

An arc (u,v) of G is called m-strong if $\mu(u, v) = \wedge[\sigma(u), \sigma(v)]$. Suppose G:$(\sigma, \mu)$ be a fuzzy graph.

The complement of G [4] is denoted as G^c : (σ^c, μ^c), where $\sigma^c = \sigma$ and $\mu^c(x, y) = \wedge[\sigma(x), \sigma(y)] - \mu(x, y)$ [*figure 1*].
If (u,v) is m-strong, then $\mu^c(u, v) = 0$.

In all the examples of this paper we assume that the membership value of each node is 1 unless otherwise specified.

3 Connectivity in G^c

It is a fact that the class of fuzzy graphs are so wide and needs great effort to understand and analyze the structural properties of fuzzy graphs. It was noticed that there are fuzzy graphs which are connected but their complements become disconnected. There may be cases when a fuzzy graph will be a fuzzy tree or fuzzy cycle and this structural property may not be satisfied by its complement. In this paper we propose a criterion by which a fuzzy graph and its complement will be connected simultaneously. We also discuss a result regarding the fuzzy cycle and its complement.

Example 1 A fuzzy graph on 3 vertices and its complement.

Figure 1: A fuzzy graph and its complement

Proposition 1 *Let G =(σ,μ) be a connected fuzzy graph with no m-strong arcs then G^c is connected.*

Proof. The fuzzy graph G is connected and contain no m-strong arcs.Suppose u ,v be two arbitrary nodes of G^c. Then they are also nodes of G. Since G is connected there exist a path between u and v in G. Let this path be P.
Then P = $(u_0,u_1)(u_1,u_2)\ldots(u_{n-1},u_n)$ where $\mu(u_{i-1},u_i)>0\ \forall$ i.
Since G contain no m-strong arcs, $\mu^c(u_{i-1},u_i) > 0\ \forall$ i.
Hence P will be a (u,v) path in G^c also. Therefore G^c is connected.

Remark There are fuzzy graphs which contain m-strong arcs such that G and G^c are connected [*figure 2*]

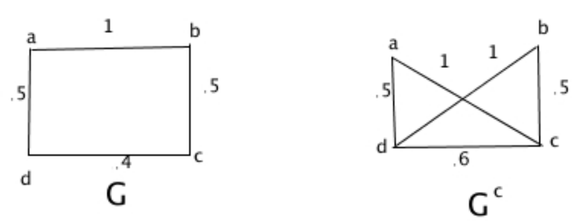

Figure 2: (a,b) is an m- strong arc in G and still G^c is connected

Theorem 1 *Let G = (σ.μ) be a fuzzy graph. G and G^c are connected if and only if G contains at least one connected spanning fuzzy subgraph with no* *m-strong arcs.*

Proof. Suppose that G contains a spanning subgraph H that is connected, having no m-strong arcs. Since H contain no m-strong arcs and is connected using proposition-1, H^c will be a connected spanning fuzzy subgraph of G^c and thus G^c is also connected.

Conversely assume that G and G^c are connected. We have to find a connected spanning subgraph of G that contain no m-strong arcs.

Let H be an arbitrary connected spanning subgraph of G. If H contain no m-strong arcs then H is the required subgraph. Suppose H contain one m-strong arc say (u,v). Then arc (u,v) will not be present in G^c. Since G^c is connected there will exist a u-v path in G^c.Let this path be P_1. Let $P_1 =$ (u_1,u_2), $(u_2,u_3)\ldots,(u_{n-1},u_n)$, where u$_1$ = u and u$_n$ = v.

If all the arcs of P_1 are present in G then H-(u,v) together with P_1 will be the required spanning subgraph. If not, there exist at least one arc say (u_1,v_1) in P_1 which is not in G. Since G is connected we can replace (u_1,v_1) by another u_1-v_1 path in G. Let this path be P_2. If P_2 contain no m-strong arcs then H-(u,v) - (u_1, v_1)together with P$_1$ and P$_2$ will be the required spanning subgraph. If P_2 contain an m-strong arc then this arc will not be present in G^c. Then replace this arc by a path connecting the corresponding vertices in G^c and proceed as above and since G contain only finite number of arcs finally we will get a spanning subgraph of that contain no m-strong arcs.

If more than one m-strong arc is present in H, then the above procedure can be repeated for all other m-strong arcs of H to get the required spanning subgraph of G.

Corollory *Let $G = (\sigma, \mu)$ be a fuzzy graph. G and G^c are connected if and only if G contains at least one fuzzy spanning tree having no m- strong arcs.*

Proof. Using previous theorem we will get a connected fuzzy spanning subgraph of G which contain no m -strong arcs. The maximum spanning fuzzy tree of this subgraph will be a spanning fuzzy tree of G that contain no m - strong arcs.

4 Complement Of Fuzzy Cycles

Next we examine the case of the complement of fuzzy cycles. By choosing the membership values of arcs and nodes suitably we can construct the complement of fuzzy cycles on 3, 4,5 vertices as fuzzy cycles.

Example 2 n = 3, G and G^c are both fuzzy cycles [*figure-3*].

Figure 3: C_3,C_3^c

Example 3 n = 4, G and G^c are both fuzzy cycles [*figure-4*].

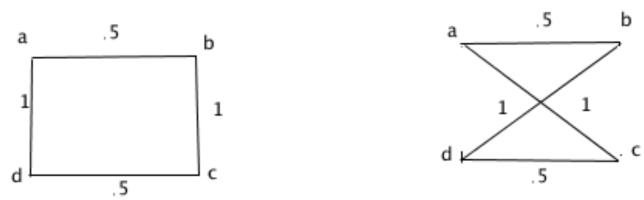

Figure 4: C_4,C_4^c

Example 4 n = 5, G and G^c are both fuzzy cycles [*figure-5*].

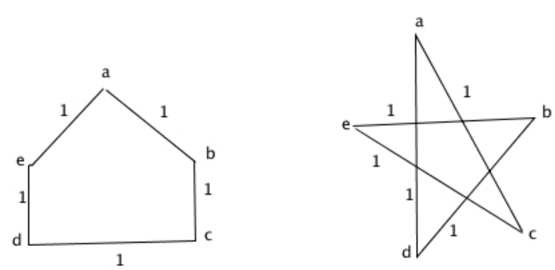

Figure 5: C_5,C_5^c

Theorem 2 *Let G :(σ,μ) be a fuzzy graph such that G* is a cycle with more than 5 vertices. Then (G*)^c cannot be a cycle.*

Proof. Given G* is a cycle having n nodes where $n \geq 6$. Then G* will have exactly n arcs. Since all the nodes of G are also present in G^c number of nodes of G^c is n . Let the nodes of G and G^c be $v_1,v_2,\ldots v_n$.
Then G^c must contain at least the following edges.
$(v_1, v_3), (v_1, v_4) \ldots (v_1, v_n)$; $(v_2, v_4), (v_2, v_5) \ldots (v_2, v_n)$; $(v_3, v_5), (v_3, v_6) \ldots (v_3, v_n)$

Since $n \geq 6$ the total number of edges in G^c will be greater than n. Thus G^c will not be a cycle.

Corollory *Let G be fuzzy cycle with 6 or more nodes. Then G^c will not be fuzzy cycle.*

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