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A Fixed Point Theorem in Cone Metric Spaces Under Weak Contractions

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Abstract

In this paper, we improve the result of B.S. Choudhury and N. Metiya, Nonlinear Analysis 72 (2010). We remove the restriction of continuity on φ. Supporting examples are also provided. Two open problems are given at the end.

Keywords: *Cone metric space, Weak contraction, Regular cone, Fixed point.*

1 Introduction

The concept of weak contraction in Hilbert space was introduced by Alber and Guerre-Delabriere [4] and a fixed point theorem was proved. Rhoades [2] has shown that the result of Alber and Guerre-Delabriere [4] is valid in complete metric spaces also. We state the result of Rhoades below.

Theorem 1.1. [2] *Let* (X, d) *be a complete metric space. Let* $T : X \to X$ *be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1.1}$$

where $x, y \in X$ *and* $\varphi : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* $\varphi(t) = 0$ *if and only if* $t = 0$*. Then* T *has a unique fixed point in* X*.*

Mappings T satisfying (1.1.1) are called weak contractions. B. S. Choudhury and N. Metiya [1] extended the above result to cone metric spaces introduced by Huang and Zhang [3].

Definition 1.2. [3] *Let* E *be a real Banach space and* P *a subset of* E*.* P *is called a cone if*
(i) P *is nonempty, closed and* $P \neq \{0\}$,
(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ *and* $-x \in P \Rightarrow x = 0$.

A partial ordering $\leq$ with respect to a cone P is defined by $x \leq y$ if and only if $y - x \in P$ for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in Int\ P$ where $Int\ P$ denotes the interior of P.

The cone P is said to be normal, if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \ \Rightarrow \ \| x \| \leq K \| y \|$$

The least positive number K satisfying the above statement is called normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Definition 1.3. [3] *Let* X *be a non empty set. Let the mapping* $d : X \times X \to E$ *satisfy*
(i) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.4. [3] *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$*

(i) If for every $c \in E$ with $0 \ll c$, there exists $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. This limit is denoted by $\lim_n x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every $c \in E$ with $0 \ll c$, there exists $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

B.S. Choudhury and N. Metiya [1] extended the results of Rhoades [2] to cone metric spaces as follows.

Theorem 1.5. [1] *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in Int\ P$, for $x, y \in X$ with $x \neq y$. Let $T : X \to X$ be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for $x, y \in X$, where $\varphi : Int\ P \cup \{0\} \to Int\ P \cup \{0\}$ is a continuous and monotone increasing function with

(i) $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $\varphi(t) \ll t$ for $t \in Int\ P$,

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$ for $t \in Int\ P \cup \{0\}$ and $x, y \in X$.

Then T has a unique fixed point in X.

In this paper, we improve Theorem 1.5 by relaxing the continuity condition on φ. We also provide supporting examples. Two open problems are also given at the end of this paper.

2 Main Results

Theorem 2.1. *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in Int\ P$, for $x, y \in X$ with $x \neq y$. Let $T : X \to X$ be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for $x, y \in X$, where $\varphi : Int\ P \cup \{0\} \to Int\ P \cup \{0\}$ is a monotone increasing function with

(i) $\varphi(t) = 0$ *if and only if* $t = 0$,
(ii) $\varphi(t) \ll t$ *for* $t \in Int\ P$,
(iii) either $\varphi(t) \leq d(x, y)$ *or* $d(x, y) \leq \varphi(t)$ *for* $t \in Int\ P \cup \{0\}$ *and* $x, y \in X$.
Then T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$. We construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}, n \geq 1$
If $x_{n+1} = x_n$ for some n, then trivially T has a fixed point.
Assume that $x_{n+1} \neq x_n$ for $n \in N$
By the given condition, we have

$$d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})),\ n = 0, 1, 2, \cdots$$

Hence $\varphi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2}),\ n = 0, 1, 2, \cdots$
Consequently,

$$\begin{aligned} \sum_{i=0}^{n} \varphi(d(x_i, x_{i+1})) &\leq d(x_0, x_1) - d(x_{n+1}, x_{n+2}) \\ &\leq d(x_0, x_1) \end{aligned}$$

So that $\sum\limits_{i=0}^{\infty} \varphi(d(x_i, x_{i+1})) < \infty$ in P.
Hence

$$\varphi(d(x_i, x_{i+1})) \to 0 \text{ as } i \to \infty \text{ in } P \tag{2.1.1}$$

Also $0 \leq \varphi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})$
$\Rightarrow 0 \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})$
$\Rightarrow d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$
Thus the sequence$\{d(x_n, x_{n+1})\}$ is a decreasing sequence and hence converges, since P is regular.

Now, by (2.1.1), $\{\varphi(d(x_n, x_{n+1}))\}$ decreases to 0 as $n \to \infty$.
Suppose $\{d(x_n, x_{n+1})\}$ decreases to l. Then
$\varphi(l) \leq \varphi(d(x_n, x_{n+1}))$ decreases to 0 as $n \to \infty$
$\Rightarrow \varphi(l) = 0 \quad \Rightarrow l = 0$. Therefore $\{d(x_n, x_{n+1})\} \to 0$ as $n \to \infty$.
Let $c \in E$ with $0 \ll c$ be arbitrary. Since $\{d(x_n, x_{n+1})\} \to 0$ as $n \to \infty$, there exists $m \in N$ such that

$$d(x_m, x_{m+1}) \ll \varphi(\varphi(c/2)) \tag{2.1.2}$$

Let $B(x_m, c) = \{x \in X : d(x, x_m) \ll c\}$
Clearly $x_m \in B(x_m, c)$ and $x_{m+1} \in B(x_m, c)$.
Suppose for $k \geq 1, x_{m+k} \in B(x_m, c)$ we have two cases by property (iii) of φ

Case (i): $d(x_m, x_{m+k}) \leq \varphi(c/2)$

Then

$$\begin{aligned} d(x_{m+k+1}, x_m) &\le d(Tx_{m+k}, Tx_m) + d(Tx_m, x_m) \\ &\le d(x_{m+k}, x_m) - \varphi(d(x_{m+k}, x_m)) + d(Tx_m, x_m) \\ &\le \varphi(c/2) + \varphi(c/2) \\ &\ll c/2 + c/2 = c \end{aligned}$$

Hence $x_{m+k+1} \in B(x_m, c)$.

Case (ii): $\varphi(c/2) \le d(x_m, x_{m+k}) \ll c$ (2.1.3)
Now

$$\begin{aligned} d(x_m, x_{m+k+1}) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) \\ &\le d(x_m, x_{m+1}) + d(Tx_m, Tx_{m+k}) \\ &\le d(x_m, x_{m+1}) + d(x_m, x_{m+k}) - \varphi(d(x_m, x_{m+k})) \\ &\le \varphi(\varphi(c/2)) + d(x_m, x_{m+k}) - \varphi(\varphi(c/2)) \qquad \text{(by (2.1.3))} \\ &\le d(x_m, x_{m+k}) \ll c \end{aligned}$$

Therefore $x_{m+k+1} \in B(x_m, c)$.
Thus, by induction, $x_n \in B(x_m, c)$ for $n \ge m$
Consequently, $\{x_n\}$ is a Cauchy sequence . By the completeness of X, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.
Now

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\le d(x_n, x) - \varphi(d(x_n, x)) \\ &\le d(x_n, x) \end{aligned}$$

On letting $n \to \infty$ we have $d(x, Tx) \le 0$
Therefore $d(x, Tx) = 0$ i.e. $Tx = x$
Hence x is the fixed point of T.

Uniqueness: If y is another fixed point of T, then

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\le d(x, y) - \varphi(d(x, y)) \end{aligned}$$

$$\Rightarrow \varphi(d(x, y)) \le 0 \quad \text{so that } x = y$$

Therefore T has a unique fixed point.

The following two examples are in support of our result.

Example 2.2. Let $X = [0,1]; E = R^2$ with usual norm, is a real Banach space. Let $P = \{(x,y) \in E : x, y \geq 0\}$. Then P is a regular cone and the partial ordering $\leq$ with respect to the cone P, is the usual component wise partial ordering in E.

Define $d : X \times X \to E$ by $d(x,y) = (\mid x-y \mid, \mid x-y \mid)$ for $x, y \in X$. Then (X,d) is a complete cone metric space with $d(x,y) \in Int\ P$ for $x, y \in X$ and $x \neq y$.

Let us define $\varphi : Int\ P \cup \{0\} \to Int\ P \cup \{0\}$ as follows:

$\varphi(0) = 0$

For $t = (\alpha, \beta) \in Int\ P$. Let $\gamma = min\ \{\alpha, \beta\} > 0$

$\varphi(t) = (1/2(n+1), 1/2(n+1))$ if $1/(n+1) < \gamma \leq 1/n,\ n \geq 1$

and $\varphi(t) = (n/2, n/2)$ if $n < \gamma \leq n+1,\ n \geq 1$

Clearly $\varphi(t) \ll t$ for $t \in Int\ P$. φ is not continuous, since φ is a step function. φ satisfies all the required properties of Theorem 2.1.

Define $T : X \to X$ by $Tx = x/2$

Now $d(Tx, Ty) = d(x/2, y/2) = (\mid x-y \mid /2, \mid x-y \mid /2)$

(i) $1/(n+1) < \mid x-y \mid \leq 1/n$

$$\Rightarrow d(x,y) - \varphi(d(x,y)) = (\mid x-y \mid, \mid x-y \mid) - (1/2(n+1), 1/2(n+1)) \geq (\mid x-y \mid /2, \mid x-y \mid /2)$$

Thus

$$d(Tx, Ty) \leq d(x,y) - \varphi(d(x,y)) \text{ for } x, y \in X \tag{2.1.4}$$

(ii) if $n < \mid x-y \mid \leq n+1$, we can show similarly that (2.1.4) holds.

Also 0 is the unique fixed point of T.

The following example is a generalized version of example 2.2.

Example 2.3. Let $X = [0,1]; E = R^2$ with usual norm is a real Banach space. Let $P = \{(x,y) \in E : x, y \geq 0\}$. Then P is a regular cone and the partial ordering $\leq$ with respect to the cone P, is the usual component wise partial ordering in E. Let $m > 0$.

Define $d : X \times X \to E$ by $d(x,y) = (\mid x-y \mid,\ m \mid x-y \mid)$ for $x, y \in X$. Then (X,d) is a complete cone metric space with $d(x,y) \in Int\ P$ for $x, y \in X$ and $x \neq y$.

Let us define $\varphi : Int\ P \cup \{0\} \to Int\ P \cup \{0\}$ as follows:

$\varphi(0) = 0$

For $t = (\alpha, \beta) \in Int\ P$, let $\gamma = min\ \{\alpha, \beta/m\} > 0$.

$\varphi(t) = (1/2(n+1),\ m/2(n+1))$ if $1/(n+1) < \gamma \leq 1/n,\ n \geq 1$

and $\varphi(t) = (n/2,\ mn/2)$ if $n < \gamma \leq n+1,\ n \geq 1$

Clearly $\varphi(t) \ll t$ for $t \in Int\ P$. φ is not continuous, since φ is a step function. φ satisfies all the required properties of Theorem 2.1.

Define $T : X \to X$ by $Tx = x/2$

Now $d(Tx,\ Ty) = d(x/2,\ y/2) = (\mid x-y \mid /2,\ m \mid x-y \mid /2))$

(i) $1/(n+1) <| x-y |\leq 1/n$

$$\begin{aligned}\Rightarrow d(x,\ y)-\varphi(d(x,\ y)) &= (| x-y |,\ m \mid x-y \mid)-(1/2(n+1),\ m/2(n+1)) \\ &= (| x-y | -1/2(n+1),\ m(| x-y | -1/2(n+1))) \\ &\geq (| x-y | /2,\ m \mid x-y \mid /2) = d(Tx,\ Ty)\end{aligned}$$

Thus

$$d(Tx,\ Ty) \leq d(x,\ y)-\varphi(d(x,\ y)) \quad \text{for} \quad x, y \in X \tag{2.1.5}$$

(ii) if $n <| x-y |\leq n+1$, we can show similarly that (2.1.5) holds.
Also 0 is the unique fixed point of T.

Open Problems

(i) Is Theorem 2.1 valid without (*iii*)?

(ii) Is Theorem 2.1 valid if the restriction $d(x,y) \in Int\ P$ for $x, y \in X, x \neq y$ is removed?

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References

[1] B.S. Choudhury and N. Metiya, Fixed points of weak contractions in cone metric spaces, *Nonlinear Analysis*, 72(2010), 1589-1593.

[2] B.E. Rhoades, Some Theorems on weakly contractive maps, *Non Linear Analysis*, 47(2001), 2683-2693.

[3] L-G Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.* 332(2007), 1467-1475.

[4] Ya.I. Alber and S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, *I. Gohberg, Yu Lyubich(Eds): in New Results in Operator Theory in: Advances and Appl.*, Birkhuser, Basel 98(1997), 7-22.