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Approach Merotopies and Near Filters* Theory and Application

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Abstract

This article considers the problem of how approach spaces can be used in the study of near filters, in general, and descriptively near filters, in particular. The solution to the problem stems from recent work on approach spaces, approach merotopies, near filters, descriptively near sets, and a specialised form of merotopy defined in terms of a variation of the Čech gap functional in measuring the distance between nonempty sets. A nonempty set equipped with a distance function satisfying certain properties is an approach space. This article investigates the theory and application of merotopies and near filters in terms of the nearness of digital images.

Keywords *Approach space, filter, gap functional, merotopy, near sets.*

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1 Introduction

The problem considered in this paper is how to formulate a framework for the study of near filters, in general, and descriptively near filters, in particular. The solution to this problem stems from recent work on near families [25] and near sets [10, 26, 28, 29, 36]. The collection of all subsets of a nonempty set X is denoted by $\mathcal{P}X$ (or by $\mathcal{P}(X)$, for clarity). In addition, let $\mathcal{A} \in \mathcal{P}^2(X)$ denote a collection of subsets in $\mathcal{P}(\mathcal{P}(X))$. This paper on near filters grew out of recent work on the nearness of collections of subsets of $\mathcal{P}X$ viewed in the context of approach spaces [1, 11, 16, 14, 15, 19, 21, 34] and the practical implications of approach merotopies and near filters in classifying digital images.

2 Approach Spaces

An **approach space** (X, ρ) [1, 19, 21] is a nonempty set X equipped with a distance function $\rho : \mathcal{P}X \times \mathcal{P}X :\rightarrow [0, \infty]$ if, and only if, for all nonempty subsets $A, B, C \in \mathcal{P}X$, ρ satisfies properties (A.1)-(A.4), *i.e.*,

(A.1) $\rho(A, A) = 0$,
(A.2) $\rho(A, \emptyset) = \infty$,
(A.3) $\rho(A, B \cup C) = \min\{\rho(A, B), \rho(A, C)\}$,
(A.4) $\rho(A, B) \leq \rho(A, C) + \underset{C \in \mathcal{P}X}{sup}\ \rho(C, B)$.

It has been observed that the notion of distance in an approach space is closely related to the notion of nearness [14].

Remark 2.1. Gap functional
For a nonempty subset $A \in \mathcal{P}X$ and a nonempty set $B \in \mathcal{P}X$, define a **gap functional** $D_\rho(A, B)$, a variation of the distance function introduced by E. Čech in his 1936–1939 seminar on topology [6] (see, also, [18, 3, 8]), where $D_\rho : \mathcal{P}X \times \mathcal{P}X \rightarrow [0, \infty]$ is defined by

$$D_\rho(A, B) = \begin{cases} \inf \{\rho(a, b) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

In general, hyperspace topologies arise from topologies determined by families of gap functionals [2].

Remark 2.2. Norm
In this article, the distance function $\rho_{\|\cdot\|}$ is defined in the context of a normed space. Let X be a linear space over the reals with origin 0. A **norm** on X is a function $\| \cdot \|: X \rightarrow [0, \infty]$ satisfying several properties for a normed space [32]. Each norm on X induces a metric d on X defined by $d(x, y) =\| x - y \|$ for

$x, y \in \mathbb{R}$ [2]. For Example, let $\vec{a}, \vec{b}$ denote a pair of n-dimensional vectors of numbers that are positive real values representing *perceived* intensities of light reflected from objects in a visual field, *i.e.*, $\vec{a} = (a_1, \ldots, a_i, \ldots, a_n), \vec{b} = (b_1, \ldots, b_i, \ldots, b_n)$ such that $a_i, b_i \in \mathbb{R}$. Then, the distance function $\rho_{\|\cdot\|} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ is defined by the $\| \cdot \|_1$ norm called the taxicab distance, *i.e.*,

$$\rho_{\|\cdot\|}(\vec{a}, \vec{b}) = \| \vec{a} - \vec{b} \|_1 = \sum_{i=1}^{n} |a_i - b_i|.$$

The focus of this work is on measuring the nearness the descriptions of objects in disjoint sets. For this reason, we consider the gap functional in terms of the greatest lower bound of the distances between feature vectors $\vec{a}, \vec{b}$ for pairs of objects $a, b \in A, B \in \mathcal{P}X$ such that $A \cap B = \emptyset$, *i.e.*, A and B are disjoint. For this reason, we introduce $D_{\rho_{\|\cdot\|}} : \mathcal{P}X \times \mathcal{P}X \to [0, \infty]$ defined by

$$D_{\rho_{\|\cdot\|}}(A, B) = \begin{cases} \inf \{\rho_{\|\cdot\|}(\vec{a}, \vec{b}) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

Then $D_{\rho_{\|\cdot\|}}(A, B)$ is a **norm gap functional** that is defined in terms of $\rho_{\|\cdot\|}$ to measure the lower distance between the descriptions of objects in a pair of non-empty sets A, B.

Lemma 2.3. *Suppose X is a metric space with distance function ρ, $x \in X$ and $\mathcal{A} \subset \mathcal{P}X$. Then*

$$\rho(x, \bigcup \mathcal{A}) = \inf\{\rho(x, A) : A \in \mathcal{A}\}.$$

Proof. The proof appears in [30, p. 25].

Lemma 2.4. $D_{\rho_{\|\cdot\|}} : \mathcal{P}X \times \mathcal{P}X \to [0, \infty]$ *satisfies (A.1)-(A.4) for ρ in an approach space.*

Proof. The proof appears in [25].

Theorem 2.5. $(X, D_{\rho_{\|\cdot\|}})$ *is an approach space.*

3 Merotopies

From an interest in determining the nearness of sets and collections rather the nearness of points and collections, the distance function $\rho : X \times \mathcal{P}X :\to [0, \infty]$ in [19, 21] is here defined in terms of a mapping from $\mathcal{P}X \times \mathcal{P}X$ to $[0, \infty]$. The basic approach is to consider a nonempty set X equipped with a distance function $\rho : \mathcal{P}X \times \mathcal{P}X :\to [0, \infty]$ satisfying certain properties. In that case, (X, ρ) is an approach space. Considered in the context of this form of an approach space, one can consider different forms of what are known as approach merotopies.

A collection $\mathcal{A}$ **corefines** a collection $\mathcal{B}$ (denoted $\mathcal{A} \prec \mathcal{B}$), if and only if, for all $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $B \subseteq A$. Let

$$\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

Let $\varepsilon \in (0, \infty]$. Then, in a manner similar to [34], a function $\nu : \mathcal{P}^2 X \times \mathcal{P}^2 X :\to [0, \infty]$ is an **ε-approach merotopy** on X, if and only if, for any collections $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}^2 X$, properties (AM.1)-(AM.5) are satisfied:

(AM.1) $\mathcal{A} \prec \mathcal{B} \Rightarrow \nu(\mathcal{C}, \mathcal{A}) \leq \nu(\mathcal{C}, \mathcal{B})$,
(AM.2) $(\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \Rightarrow \nu(\mathcal{A}, \mathcal{B}) < \varepsilon$,
(AM.3) $\nu(\mathcal{A}, \mathcal{B}) = \nu(\mathcal{B}, \mathcal{A})$ and $\nu(\mathcal{A}, \mathcal{A}) = 0$,
(AM.4) $\emptyset \in \mathcal{A} \Rightarrow \nu(\mathcal{C}, \mathcal{A}) = \infty$,
(AM.5) $\nu(\mathcal{C}, \mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{C}, \mathcal{A}) \wedge \nu(\mathcal{C}, \mathcal{B})$.

The pair (X, ν) is called an **ε-approach merotopic space**. Recent work has focused on approach merotopies that measure to what degree a collection of sets contains near members [14]. In the current work, the focus is on ε-approach merotopies that measure to what degree disjoint collections of sets are near each other. This work has grown out recent studies of the nearness of sets, especially the nearness of disjoint sets (something that is possible, if measurement of the distance between descriptions of elements of nonempty sets is considered).

Lemma 3.1. *Let D_ρ be a gap functional. Then the function $\nu_{D_\rho} : \mathcal{P}^2 X \times \mathcal{P}^2 X :\to [0, \infty]$ defined as*

$$\nu_{D_\rho}(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_\rho(A, B)$$

is an ε-approach merotopy on X.

Proof. We will show only (AM.1) and (AM.5). Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset \mathcal{P}X$. (AM.1) If $\mathcal{A} \prec \mathcal{B}$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then $D_\rho(C, A) \leq D_\rho(C, B)$. Hence, $\nu_{D_\rho}(\mathcal{C}, \mathcal{A}) \leq \nu_{D_\rho}(\mathcal{C}, \mathcal{B})$. (AM.5) follows by noting that $D_\rho(A, B \cup C) = \min\{D_\rho(A, B), D_\rho(A, C)\}$.

Lemma 3.2. *A norm gap functional $D_{\rho_{\|\cdot\|}}$ defines an ε-approach merotopy $\nu_{D_{\rho_{\|\cdot\|}}}$ on X, where*

$$\nu_{D_{\rho_{\|\cdot\|}}}(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_{\rho_{\|\cdot\|}}(A, B).$$

Proof.
Immediate from the definition of $D_{\rho_{\|\cdot\|}}$ and Lemma 3.1.

The first part of the term *merotopy* comes from the Greek *meros* (part). This makes sense, since merotopies are defined in terms of the parts (subsets) of a collection. Merotopies were introduced by M. Katětov [12] and elaborated in [20, 21, 34].

Example 3.3. Distance Between Sets and Collections
Consider a pair of collections $\mathcal{B}, \mathcal{A} \in \mathcal{P}^2(X)$. Assume that $\mathcal{B} \cap \mathcal{A} = \emptyset$ (the collections are disjoint).
Let $B \in \mathcal{B}$ and $A \in \mathcal{A}$, *i.e.*, B, A are subsets in $\mathcal{B}, \mathcal{A}$, respectively. Put

$$\rho(B, A) = D_{\rho_{\|\cdot\|}}(B, A),$$

i.e., the distance function ρ is defined in terms of the gap functional $D_{\rho_{\|\cdot\|}}$ in measuring the distance between a pair of sets. For simplicity, we omit reference to a collection $\mathcal{B} \in \mathcal{P}^2X$ that contains subset B and only consider a fixed subset $B \in \mathcal{B}$. Then $\nu : \mathcal{P}^2X \times \mathcal{P}^2X :\to [0, \infty]$ is defined by

$$\nu(\{B\}, \mathcal{A}) := \sup_{A \in \mathcal{A}} D_{\rho_{\|\cdot\|}}(B, A).$$

For simplicity, we will write $\nu(\{B\}, \mathcal{A})$ as $\nu(B, \mathcal{A})$.

Remark 3.4. *Observe that in Example 3.3, the function ν is an ε-approach merotopy on X.*

Proof. Immediate from Lemma 3.2.

A nonempty collection $\mathcal{A}$ is **near** a set $B \in \mathcal{B}$, if and only if $\nu(B, A) = \nu(\{B\}, \{A\}) = 0$ for at least one $A \in \mathcal{A}$. In practice, this seldom occurs. Hence, a nonempty collection $\mathcal{A}$ is considered **weakly near** (or ε-near) to a set $B \in \mathcal{B}$, provided there is a $A \in \mathcal{A}$ such that B and A are close enough or not far apart. This is the main reason why we have defined the notion of an ε-approach merotopy. That is, for $\varepsilon \in (0, \infty]$, a collection $\mathcal{A}$ is **ε-near** a set $B \in \mathcal{B}$ if, and only if there is a subset $A \in \mathcal{A}$ such that $\nu(B, A) < \varepsilon$. The study of near sets and near collections is directly related to work on approach, metric, proximity and topological spaces (see, *e.g.*, [16, 14, 15, 19, 21, 22, 23, 24, 27, 28, 26, 29, 25, 34]).

4 Descriptively Near Collections

Descriptively near sets are disjoint sets that resemble each other. Feature vectors (vectors of numbers representing feature values extracted from objects) provide a rigorous basis for set descriptions (see, *e.g.*, [28, 26, 25, 23]). A feature-based gap functional defined for the norm on a pair of nonempty sets was introduced in [29]. Let $B \subset X$. Let $\Phi_n(x) = (\phi_1(x), \ldots, \phi_n(x))$ for $x \in B$ denote a **feature vector**, where $\phi_i : B \to \Re$. In addition, let $\Phi_B = \{\Phi_1(x), \ldots, \Phi_{|X|}(x)\}$ denote a set of feature vectors for objects $x \in B$. Assume A is a subset in a collection $\mathcal{A} \in \mathcal{P}^2X$, *i.e.*, $A \in \mathcal{A}$. In this article, a description-based **norm gap functional** $D_{\Phi,\rho_{\|\cdot\|}}$ is defined in terms of the Hausdorff lower distance [9] relative to the norm on $\mathcal{P}(\Phi_B) \times \mathcal{P}(\Phi_A)$ for sets B, A, *i.e.*,

$$D_{\Phi,\rho_{\|\cdot\|}}(B,A) = \begin{cases} \inf\left\{\rho_{\|\cdot\|}(\Phi_B, \Phi_A)\right\}, & \text{if } \Phi_B \text{ and } \Phi_A \text{ are not empty,} \\ \infty, & \text{if } \Phi_B \text{ or } \Phi_A \text{ is empty.} \end{cases}$$

Theorem 4.1. $(X, D_{\Phi,\rho_{\|\cdot\|}})$ *is an approach space.*

Proof. Immediate from the definition of $D_{\Phi,\rho_{\|\cdot\|}}$ and Lemma 2.4.

Given an approach space (X, ϕ), define $\nu : \mathcal{P}(\mathcal{P}X) :\to [0, \infty]$ by

$$\nu(\mathcal{A}) = \inf_{x\in X} \sup_{A\in\mathcal{A}} \rho(x, A).$$

The collection $\mathcal{A} \in \mathcal{P}^2X$ is near, if and only if, $\nu(\mathcal{A}) = 0$ for $x \in X$ [21]. The function ν is another form of an approach merotopy [34], if we assume that ρ satisfies the conditions for a distance function in an approach space. In the sequel, consider a function $\nu : \mathcal{P}^2X \times \mathcal{P}^2X \to [0, \infty)$ defined by a function $\nu(\mathcal{B}, \mathcal{A})$ for $\mathcal{B}, \mathcal{A} \in \mathcal{P}^2X$. Then define a ν in the following way.

$$\nu(\mathcal{B}, \mathcal{A}) = \sup_{A\in\mathcal{A}, B\in\mathcal{B}} D_{\Phi,\rho_{\|\cdot\|}}(B, A).$$

For sake of clarity, $\underset{\Phi,\rho_{\|\cdot\|}}{\nu}(\mathcal{B}, \mathcal{A})$ is also written to denote a description merotopy.

Remark 4.2. Nearness of Collections

A nonempty collection $\mathcal{A}$ is **ε-near** a collection $\mathcal{B}$ if, and only if there are $B \in \mathcal{B}, A \in \mathcal{A}$ such that $\nu(B, A) < \varepsilon$. A nonempty collection $\mathcal{A}$ is **descriptively near** a nonempty collection $\mathcal{B}$ (denoted by $\mathcal{A} \underset{\Phi}{\bowtie} \mathcal{B}$), if and only, if a description merotopy

$$\underset{\Phi,\rho_{\|\cdot\|}}{\nu}(B, A) = D_{\Phi,\rho_{\|\cdot\|}}(B, A) = 0,$$

for at least one $B \in \mathcal{B}$ that is near at least one $A \in \mathcal{A}$. By varying the choice of features (with corresponding probe functions) used to construct feature vectors used to compare A and B, the precision of the estimate of the closeness of a pair of collections will either increase or decrease. A collection $\mathcal{A}$ is

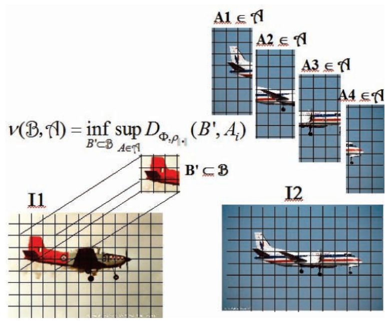

Figure 1: Near Collection

descriptively ε**-near** a nonempty collection $\mathcal{B}$ (denoted by $\mathcal{A} \underset{\Phi,\varepsilon}{\bowtie} \mathcal{B}$), if and only if, a description merotopy

$$\underset{\Phi,\rho_{\|\cdot\|}}{\nu}(B, A) = D_{\Phi,\rho_{\|\cdot\|}}(B, A) < \varepsilon,$$

for at least one $B \in \mathcal{B}$ that is near at least one $A \in \mathcal{A}$. By varying the choice ε and choice of features for feature vectors used to compare A and B, the precision of the estimate of the closeness of a pair of collections will either increase or decrease.

Example 4.3. Sample Near Image Collections
A digital image can be viewed as a set of points. In this case, a point is either a picture element (pixel) or $p \times p$ subimage for $p \in [1, n], n \in \mathbb{N}$. In this Example, an approach space (X, ρ) is defined for a set of digital images X and distance $\rho = D_{\Phi,\rho_{\|\cdot\|}}$. For Example, let $I1, I2$ denote the pair of aircraft images in Fig. 1. Put $X = I1 \cup I2$ and assume the collection $\mathcal{A}$ always comes from $\mathcal{P}^2(I2)$ and the collection $\mathcal{B}$ always comes from $\mathcal{P}^2(I1)$, *i.e.*,

$$\mathcal{A} \in \mathcal{P}^2(I2) \text{ and } \mathcal{B} \in \mathcal{P}^2(I1),$$

where

$$X = I1 \cup I2 \text{ and } I1 \cap I2 = \emptyset.$$

In this illustration, let $B' \in \mathcal{B}$ denote the set of subimages contained in the tail section for the small aircraft shown in Fig. 1 and let $\mathcal{A} \in \mathcal{P}^2(I2)$ denote a collection of subsets containing subimages of the passenger plane in the same figure. Then

$$\nu(B', \mathcal{A}) = \sup_{A \in \mathcal{A}} D_{\Phi,\rho_{\|\cdot\|}}(B', A).$$

The important thing to notice here is that B' represents either a region of interest (ROI) in image $I1$ in Fig. 1 or B' is a subset containing subimages chosen arbitrarily (*i.e.*, B' is any random selection of subimages in $I1$). However, if we want to determine that some part of image $I1$ is similar (near to) one or more parts of image $I2$, then subset B' is specifically chosen because there is something about the subimages in B' that is interesting. In the case of the small aircraft in image I1 in Fig. 1, the subset B' is just the aircraft tail section and we wish to determine if the description of B' matches any portion of image $I2$ shown in Fig. 1. In the case of image $I2$, the comparison between B' and a collection $\mathcal{A}$ using a merotopy ρ is made in terms of an arbitrary selection of a collection $\mathcal{A}$ found in $I2$. In this Example, we use the ε-approach merotopy from Example 3.3.

Basically, the description of each subimage in the nonempty subset $B' \in \mathcal{B} \in \mathcal{P}^2(I1)$ is compared with the description of subimages contained in nonempty subsets A in $\mathcal{A} \in \mathcal{P}^2(I2)$. For simplicity, we consider subimage description in terms of feature values extracted from a subimage using a single probe function[ψ]. That is, let $\Phi(x) = (\phi_{eo}(x))$ contain a single probe function $\phi_{eo}(x) =$ that determines the average edge gradient direction of pixels in a subimage x. A method that can be used to determine edge gradient direction is given in [31, §5.3.2, p. 133] and not repeated, here. In that case, the similarities in the gradient directions of the two tail sections of the aircraft in Fig. 1 may lead to $\nu(B', \mathcal{A}) = 0$ for some subset $B' \in \mathcal{B} \in \mathcal{P}^2(I1)$. It is often the case that $\nu(B', \mathcal{A}) < \varepsilon$ for some small ε. Then the set B' and the collection $\mathcal{A}$ are descriptively ε-near. The smaller the value of ε, the closer the description of B' is to collection $\mathcal{A}$, when $\nu(B', \mathcal{A}) < \varepsilon$. There are a number of different cases to consider in determining the nearness or apartness of digital images. In the following cases, assume ε-approach merotopy ν is defined in terms of distance $\rho = D_{\Phi,\rho_{\|\cdot\|}}$.

(n.1) **Descriptively ε-near ROI-to-single collection case**. A single region of interest (ROI) $B' \in \mathcal{B}$ from image $I1$ is near $\mathcal{A}$ from image $I2$ when $\nu(B', \mathcal{A}) < \varepsilon$, where the distance $\nu(B', \mathcal{A})$ is defined to be

$$\nu(B', \mathcal{A}) = \sup_{A \in \mathcal{A}} D_{\Phi,\rho_{\|\cdot\|}}(B', A).$$

Then the ROI B' is descriptively ε-near collection $\mathcal{A}$ when $\nu(B', \mathcal{A}) < \varepsilon$ for $\varepsilon > 0$. In other words, region B' in image $I1$ is descriptively similar to one or more parts of image $I2$ represented by the collection $\mathcal{A}$. It is this case that is illustrated in Fig. 1, if we assume that $\nu(B', \mathcal{A}) < \varepsilon$ for some small ε.

(n.2) **Descriptively ε-near ROI-to-image case**. A single ROI $B' \in \mathcal{B}$ from image $I1$ is compared with every collection $\mathcal{A} \in \mathcal{P}^2(I2)$ and $\nu(B', \mathcal{A}) < \varepsilon$

[ψ]For a detailed view of multiple features in descriptions of descriptions subimages, see [10].

for $\varepsilon > 0$. Then region $B' \in \mathcal{B}$ is descriptively ε-near image $I2$, where the distance $\nu(B', \mathcal{A})$ for all collections $\mathcal{A} \in \mathcal{P}^2(I2)$ is defined to be

$$\nu(B', \mathcal{A}) = \sup_{A \in \mathcal{A}} D_{\Phi, \rho_{\|\cdot\|}}(B', A).$$

To conclude that ROI B' is sufficiently close to some part of image $I2$, it is only necessary that B' is near *at least one* part of image $I2$ represented by a collection $\mathcal{A}$. In other words, B' in image $I1$ is descriptively similar to one or more parts of image $I2$, if there is a collection $\mathcal{A}$ such that $\nu(B', \mathcal{A}) < \varepsilon$.

(n.3) **Descriptively ε-near image-to-collection case**. Every nonempty subset $B \in \mathcal{B} \in \mathcal{P}^2(I1)$ is compared with a single, arbitrary collection $\mathcal{A}$ from image $I2$ such that $\nu(B, \mathcal{A}) < \varepsilon$, where $\nu(B, \mathcal{A})$, for all collections $\mathcal{B} \in \mathcal{P}^2(I1)$ and a single collection $\mathcal{A} \in \mathcal{P}^2(I2)$, is defined to be

$$\nu(B, \mathcal{A}) = \sup_{A \in \mathcal{A}} D_{\Phi, \rho_{\|\cdot\|}}(B, A).$$

That is, there is at least one subset B from $I1$ that is descriptively ε-near a collection $\mathcal{A}$ in $\mathcal{P}^2(I2)$. In other words, one or more subsets B in image $I1$ resemble some part of image $I2$.

(n.4) **Descriptively ε-near image-to-image case**. This is the extreme case, where every part of one image is compared with every part of a second image. In other words, every subset $\mathcal{B} \in \mathcal{P}^2(I1)$ is compared with every subset in the collection $\mathcal{A} \in \mathcal{P}^2(I2)$. Image $I1$ is considered ε-near image $I2$ (*i.e.*, $\mathcal{P}(I1) \underset{\Phi,\varepsilon}{\bowtie} \mathcal{P}(I2)$) if, and only if there is at least one pair $\mathcal{B} \in \mathcal{P}^2(I1), \mathcal{A} \in \mathcal{P}^2(I2)$ such that $\nu(\mathcal{B}, \mathcal{A}) < \varepsilon$, where, for all collections $\mathcal{B} \in \mathcal{P}^2(I1)$ and all collections $\mathcal{A} \in \mathcal{P}^2(I2)$, $\nu(\mathcal{B}, \mathcal{A})$ is defined by

$$\nu(\mathcal{B}, \mathcal{A}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_{\Phi, \rho_{\|\cdot\|}}(B, A).$$

If $\nu(\mathcal{B}, \mathcal{A}) > \varepsilon$ for one of the comparison cases (n.1)-(n.4), then the images $I1, I2$ used to define X are considered *far* apart (not near each other) for the particular choice of probe function(s) in the descriptions of the subimages used in computing ν.

Theorem 4.4. *Given an approach space $(X, D_{\Phi, \rho_{\|\cdot\|}})$, a collection $\mathcal{A} \in \mathcal{P}^2 X$ is near a subset $B \in \mathcal{B} \in \mathcal{P}^2 X$, if and only if, $D_{\Phi, \rho_{\|\cdot\|}}(B, A) = 0$ for $B \in \mathcal{B}$ and for at least one $A \in \mathcal{A}$.*

Proof.
$\Rightarrow$ Given that an ε-approach merotopy ν and a collection $\mathcal{A} \in \mathcal{P}^2(X)$ that is near a set $B \in \mathcal{B}$, then $\nu(B, \mathcal{A}) = 0$. Hence, there is at least one $A \in \mathcal{A}$ such

that $D_{\Phi,\rho_{\|\cdot\|}}(B, A) = 0$.
$\Leftarrow$ Given that $D_{\Phi,\rho_{\|\cdot\|}}(B, A) = 0$ for $B \in \mathcal{B}$ and for at least one $A \in \mathcal{A}$, it follows that the collection $\mathcal{A}$ is near $B \in \mathcal{B}$.

5 Filters

Filters were introduced by H. Cartan in 1937 [4, 5]. A theory of convergence stems from the notion of a filter (see, *e.g.*, [13], [33, p. 78ff]) and the completion of uniform spaces by Cauchy clusters [6, 17, 22]. A collection $\mathcal{F} \in \mathcal{P}^2(X)$ containing subsets of X is a **filter** [7, p. 56], if and only if, for all nonempty $A, B \subset \mathcal{F}$, the collection $\mathcal{F}$ satisfies conditions (F.1)-(F.3)[33].

(F.1) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(F.2) if $A \in \mathcal{F}$ and $A \subset B \subset \mathcal{P}(X)$ implies $B \in \mathcal{F}$,
(F.3) $\emptyset \notin \mathcal{F}$.

In other words, a filter is a collection of ascending subsets, *e.g.*, starting with the smallest nonempty subset $A_0 \in \mathcal{F}$ and next smallest subset $A_1 \in \mathcal{F}$, property (F2) guarantees that $A_0 \subset A_1$. This view of a filter is an example of the traditional view of filters (see, *e.g.*, [35]), since a filter is now defined in terms of a collection in an approach space.

In an approach space X, let $x_0 \in X$ and put $\varepsilon \in (0, \infty]$. The set

$$B_\rho(x_0, \varepsilon) = \{y \in X : \rho(x_0, y) < \varepsilon\},$$

is called an **open ball** with center x_0 and radius ε. A nonempty subset $E \subset X$ in an approach space (X, ρ) is termed as an **open set** if, and only if, for each $x \in E$, there is an open ball $B(x_0, \varepsilon) \subset E$. In other words, an open set is the union of open balls. A subset $N \subset X$ is a **neighbourhood of a point** $x \in X$ (denoted N_x) in an approach space (X, ρ), if and only if, there exists an open set $E \subset N$ such that $x \in E$.

For a neighbourhood N_x in an approach space X, point x is called a **limit of a filter** $\mathcal{F}$. This is a specialization of the notion of a neighbourhood in a topology [33] in terms of approach spaces. J.L. Kelley [13] observes that a filter $\mathcal{F}$ converges to a point $x \in X$, if and only if, each neighbourhood of x is a member of $\mathcal{F}$.

Theorem 5.1. *Let $\mathcal{F}$ be a filter in an approach space (X, ρ). A point $x \in X$ is a limit of the filter, if and only if, $N_x \supset \mathcal{F}$.*

Proof. See proof in [33].

Corollary 5.2. *Given an approach space $(X, D_{\Phi_X,\rho_{\|\cdot\|}})$, a filter $\mathcal{F} \in \mathcal{P}^2 X$ is near $B \subset X$, if and only if, $D_{\Phi,\rho_{\|\cdot\|}}(A, B) = 0$ for at least one $A \in \mathcal{F}$.*

Proof. Symmetric with the proof of Theorem 4.4.

Corollary 5.3. *Given a neighbourhood N_x in an approach space $(X, D_{\Phi,\rho_{\|\cdot\|}})$, a filter $\mathcal{F} \in \mathcal{P}^2(X)$ is descriptively near N_x, if and only if, $D_{\Phi,\rho_{\|\cdot\|}}(N_x, A) = 0$ for at least one $A \in \mathcal{F}$.*

Assume a merotopy ν is defined in terms of $D_{\Phi,\rho_{\|\cdot\|}}$. In keeping with an interest

Figure 2: Sample Image Filter in Image I2

in applying filter theory to digital images, define a collection $\mathcal{F}$ to be a filter that is descriptively ε-near to a subset $B \in \mathcal{B}$, if and only if, there is at least one $A \in \mathcal{F}$ such that $\nu(B, F) < \varepsilon$. This definition of a near filter differs from the usual definition of a near collection, since a filter $\mathcal{F}$ is considered near B when a subset $A \in \mathcal{F}$ is close enough to B, *i.e.*, when the distance between B and the filter $\mathcal{F}$ is less than some (usually small) ε.

Example 5.4. Sample ε-near Image Filters
In this Example, ε-near filters are considered in an approach space $(X, D_{\Phi,\rho_{\|\cdot\|}})$, where $X = \{I, I'\}$ contains a pair of digital images I, I' and $\mathcal{P}(I1)$ is the set of all subsets of subimages in $I1$ and $\mathcal{P}(I2)$ is the set of all subsets of subimages in $I2$. By contrast with Example 4.3, where separate collections of subsets (not necessarily ascending) $\mathcal{A} \in \mathcal{P}^2(I2)$ are compared with a set $B \in \mathcal{B} \in \mathcal{P}^2(I1)$, now ascending subsets F in a filter $\mathcal{F} \in \mathcal{P}^2(I2)$ are compared with $B \in \mathcal{B}$. Again, it is assumed that the collection $\mathcal{B}$ always comes from a region of image $I1$ and the collection $\mathcal{F}$ always comes from a part of image $I2$. In this Example, only cases (n.1)[**ROI-to-single filter**] and (n.3) [**image-to-filter**] are considered. For case (n.1), recall that a single ROI B in image $I1$ is compared with a single collection $\mathcal{F}$ that is a filter in image $I2$. This case is represented in Fig. 2. For case (n.3), recall that image-to-collection, in general, means every subset B in $I1$ is compared to a single collection, *e.g.*, filter $\mathcal{F}$ from image $I2$. There are two other cases to consider in this Example.

(n.5) **Descriptively ε-near ROI-to-filters case**. In practice, a limited number of filters would be selected either from horizontal or vertical sections of an image. Each section of the image would extend from one edge to another edge of an image. For Example, the sample filter in Fig. 2 is extracted from a horizontal section of the airline image in Fig. 1. A single region of interest B from image $I1$ is compared with more than one filter $\mathcal{F}$ from image $I2$. An image $I1$ is considered ε-near to an image $I2$, if an ROI B in $I1$ is descriptively similar to at least one subset F of a filter $\mathcal{F}$ in $I2$, *i.e.*, $\nu(B, F) < \varepsilon$, where $\nu(B, \mathcal{F})$, for some but not necessarily all filters $\mathcal{F} \in \mathcal{P}^2(I2)$, is defined to be

$$\nu(B, F) = D_{\Phi,\rho_{\|\cdot\|}}(B, F).$$

This case is a variation of case (n.2) and is partially represented in Fig. 2.

(n.6) **Descriptively ε-near image-to-filters case**. Every subset $B \in \mathcal{B}$ from image $I1$ is compared with one or more filters $\mathcal{F}$ from image $I2$. Image $I1$ is considered ε-near image $I2$, provided there is at least one B and at least one subset F of a filter $\mathcal{F}$ in $I2$ such that $\nu(B, F) < \varepsilon$.

Again, let B denote the set of subimages contained in the the tail section for the small aircraft shown in Fig. 1 and let $\mathcal{F} \in \mathcal{P}^2(X)$ denote a collection of ascending subsets containing subimages of the passenger plane in Fig. 2. For simplicity, again let $\Phi(x) = (\phi_{eo}(x))$ contain a single probe function $\phi_{eo}(x)$ that extracts the average edge gradient direction for the pixels in a subimage x. Edge gradient direction has been chosen, since there are obvious similarities between the sample images in terms of the inclination of the aircraft tail structures. In this case, the similarities in the edge directions of the subimages in  (from Fig. 1) are compared with the edge directions of the subimages in the tail section in  (from the filter in Fig. 2). The comparison between the set of subimages in B and the subimages in a subset of the filter $\mathcal{F}$ in Fig. 2 terminates, whenever $\nu(B, F) < \varepsilon$ (*i.e.*, when there is a $F \in \mathcal{F} \in \mathcal{P}^2(I2)$ that is ε-near subset B) *or* there are no further subimages to compare, *i.e.* $\nu(B, \mathcal{F}) > \varepsilon$ and $\mathcal{F}$ is not ε-near B. In that case, the collection of filters $\mathcal{F}$ is descriptively *far* from B.

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