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On Degree of Approximation by Product Means $(E,q)(N,p_n)$ of Fourier Series

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Abstract

In this paper a theorem on degree of Approximation of a function $f \in Lip\ \alpha$ by product summability $(E,q)(N,p_n)$ of Fourier series associated with f.

Keywords: *Degree of Approximation, $f \in Lip\ \alpha$ class of function, (E,q) mean, (N,p_n) mean, $(E,q)(N,p_n)$ product mean, Fourier series, Lebesgue integral*

1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

(1.1) $$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty \quad \text{as} \quad n \to \infty\ ,\ (P_{-i} = p_{-i} = 0\ , i \geq 0)$$

The sequence –to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_{n-\upsilon} s_\upsilon$$

defines the sequence $\{t_n\}$ of the (N, p_n) -mean of the sequence $\{s_n\}$ generated by the by sequence of coefficient $\{p_n\}$. If

(1.3) $$t_n \to s \quad , \text{as} \quad n \to \infty\ ,$$

then the series $\sum a_n$ is said to be (N, p_n) summable to s .
The conditions for regularity of Nörlund summability (N, p_n) are easily seen to be

(i) $$\frac{p_n}{P_n} \to 0 \quad \text{as} \quad n \to \infty$$

(ii) $$\sum_{k=0}^{n} p_k = O(P_n) \quad \text{as} \quad n \to \infty$$

The sequence –to-sequence transformation, [1]

(1.4) $$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad .$$

defines the sequence $\{T_n\}$ of the (E, q) mean of the sequence $\{s_n\}$.
If

(1.5) $$T_n \to s \quad \text{as} \quad n \to \infty$$

then the series $\sum a_n$ is said to be (E, q) summable to s .
Clearly (E, q) method is regular.

Further, the (E, q) transform of the (N, p_n) transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\, T_k$$

(1.6)
$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}s_\upsilon\right\}$$

If

(1.7) $\tau_n \to s$ as $n \to \infty$

then $\sum a_n$ is said to be $(E,q)(N,p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π, L-integrable over (-π,π), The Fourier series associated with f at any point x is defined by

(1.8)
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)$$

Let $s_n(f;x)$ be the n-th partial sum of (1.8).

The L_∞-norm of a function $f : R \to R$ is defined by

(1.9)
$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and the L_υ-norm is defined by

(1.10)
$$\|f\|_\upsilon = \left(\int_0^{2\pi}|f(x)|^\upsilon\right)^{\frac{1}{\upsilon}}, \ \upsilon \ge 1$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\,.\,\|_\infty$ is defined by [4].

(1.11)
$$\|P_n - f\|_\infty = \sup\{|p_n(x) - f(x)| : x \in R\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by

(1.12)
$$E_n(f) = \min_{P_n}\|P_n - f\|_\upsilon$$

This method of approximation is called Trigonometric Fourier approximation.
A function $f \in Lip\alpha$ if

(1.13) $$|f(x+t) - f(x)| = O(|t|^{\alpha}) \; ,0 < \alpha \leq 1$$

We use the following notation throughout this paper :

(1.14) $$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

and

(1.15) $$K_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} \frac{\sin\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}.$$

Further, the method $(E,q)(N,p_n)$ is assumed to be regular and this case is supposed through out the paper.

2 Known Theorem

Dealing with The degree of approximation by the product $(E,q)\,(c,1)$-mean of Fourier series, Nigam [2] proved the following theorem:

Theorem 2.1 *If a function* f *,*2π *- periodic, belonging to class* $Lip\alpha$*, then its degree of approximation by* $(E,q)\,(c,1)$ *summability mean on its Fourier series*

$$\sum_{n=0}^{\infty} A_n(t) \text{ is given by } \left\| E_n^q c_n^1 - f \right\|_{\infty} = o\left(\frac{1}{(n+1)^{\alpha}} \right) \;,0 < \alpha < 1,$$

where $E_n^q c_n^1$ *represents the* (E,q) *transform of* $(c,1)$ *transform of* $s_n(f;x)$**.**

3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)(N,p_n)$ of Fourier series (1.8) .we prove

Theorem 3.1 *If* f *is a* $2\pi-$ *Periodic function of class* $Lip\alpha$*, then degree of approximation by the product* $(E,q)(N,p_n)$ *summability means on its Fourier series (1.8) is given by*

$$\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right) \quad ,0<\alpha<1$$ where τ_n on defined in (1.6) .

4 Required Lemmas

We require the following Lemmas to prove the theorem.

Lemma 4.1

$$|K_n(t)| = O(n) \quad ,0 \le t \le \frac{1}{n+1}$$

Proof:

For $0 \le t \le \frac{1}{n+1}$, we have sinnt ≤ nsint then

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}(2k+1)\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\right\}\right|$$

$$\le \frac{(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|$$

=O(n).

This proves the lemma.

Lemma 4.2

$$|K_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \le t \le \pi .$$

Proof:

For $\frac{1}{n+1} \le t \le \pi$ **,** we have by Jordan's lemma**,** $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$, $\sin nt \le 1$

Then $$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi\ p_{k-\upsilon}}{t}\right\}\right|$$

$$= \frac{1}{2(1+q)^n\, t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\right\}\right|.$$

$$= \frac{1}{2(1+q)^n\, t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

5 Proof of Theorem 3.1

Using Riemann –Lebesgue theorem, we have for the n-th partial sum $s_n(f;x)$ of the Fourier series **(**1.8) of $f(x)$,

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt\,,$$

following Titechmarch [3],the (N, p_n) transform of $s_n(f;x)$ using (1.2) is given by

$$t_n - f(x) = \frac{1}{2\pi P_n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n} p_{n-k}\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt\,,$$

Directing the $(E,q)(N,p_n)$ transform of $s_n(f;x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{1}{2\pi(1+q)^n}\int_0^\pi \phi(t)\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

$$=\int_0^\pi \phi(t)\, K_n(t)\,dt$$

$$=\left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\}\phi(t)\, K_n(t)\, dt$$

(5.1) $$= I_1 + I_2\,,\ say$$

Now

$$|I_1| = \frac{1}{2\pi(1+q)^n}\left|\int_0^{1/(n+1)} \phi(t)\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_{k-\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right|$$

$$\le O(n)\int_0^{\frac{1}{n+1}} |\phi(t)|\,dt \qquad \text{, using Lemma 4.1}$$

$$= O(n)\int_0^{\frac{1}{n+1}} \left| t^{\alpha}\right| dt$$

$$= O(n)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$$

$$= O(n)\left[\frac{1}{(\alpha+1)(n+1)^{\alpha+1}}\right].$$

(5.2) $$= O\left[\frac{1}{(n+1)^{\alpha+1}}\right]$$

Next

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)|\,|K_n(t)|\,dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} |\phi(t)|\, O\left(\frac{1}{t}\right) dt$$, using Lemma 4.2

$$= \int_{\frac{1}{n+1}}^{\pi} |t^{\alpha}|\, O\left(\frac{1}{t}\right) dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}\, dt$$

(5.3) $$= O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Then from (5.2) and (5.3) , we have

$$|\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha}}\right), \text{ for } 0<\alpha<1$$

$$\|\tau_n - f(x)\|_{\infty} = \sup_{-\pi<x<\pi} |\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha}}\right), 0<\alpha<1 .$$

This completes the proof of the theorem.

6 Corollaries

The following corollaries can be derived from our main theorem.

Corollary 6.1 *If* $p_n = 1$ $, \forall\, n \in N,$ *theorem 2.1 follows from theorem 3.1.*

Corollary 6.2 *If* $p_n = 1, \forall\, n$ *and* $q = 1$ *then the theorem 3.1 reduces to degree of approximation for* $(E,1)\,(C,1)$ *method of Fourier series.*

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