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Weakly δ-b-Continuous Functions

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Abstract

The aim of this paper is to introduce and investigate some properties of weakly δ-b-continuous functions. We also define the notion of δ-b-continuous functions by using δ-b-open sets. We obtain that the notion of weak δ-b-continuity is weaker than δ-b-continuity, but stronger than both the weak b-continuity and weak e-continuity. In order to show coincidencies in functions whose range space is regular, we introduce and investigate some properties of the notions of $faint$ δ-b-continuity and strong θ-δ-b-continuity. Finally, we obtain some properties of weakly δ-b-continuous functions related to some separation axioms and graphic functions for δ-b-open sets.

Keywords: *δ-b-open sets, δ-b-continuity, $faint$ δ-b-continuity, strong θ-δ-b-continuity, weak δ-b-continuity.*

1 Introduction

Of course, continuity is one of important topic for study in topological spaces. This notion is based on open sets. So, generalizations of continuity are given by using weaker types of open sets such as α-*open* sets [12], *semi*-open sets [9], *preopen* sets [11], *b-open* sets [1]. On the other hand, the notion of weak continuity is defined by Levine [8] and their modifications are studied by several authors such as [4], [11], [14], [13], [20].

In this paper, first we introduce and give some characterizations of *δ-b-continuous* functions. We also define and investigate some properties of *weakly δ-b-continuous* functions weaker than this notation. Then, we consider two types of continuous functions are called faintly *δ-b-continuous* and *strongly θ-δ-b-continuous*, respectively. In order to show coincidencies among *weak δ-b-continuity*, *δ-b-continuity*, faint *δ-b-continuity* and strong *θ-δ-b-continuity*. Finally, we investigate relations between *weak δ-b-continuity* and covering properties (resp. connetedness).

2 Preliminaries

Throught in this paper, (X, τ) and (Y, φ) denote nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ), the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. $\tau(x)$ represents all open neighbourhoods of the point $x \in X$.

A set A is called *θ-open* [10] (resp. *δ-open* [21]) if every point of A has an open neighbourhood whose closure (resp. interior of closure) is contained in A. The *θ-interior* [10] (resp. *δ-interior* [21]) of A in (X, τ) is the union of all *θ-open* (resp. *δ-open*) subsets of A and is denoted by $Int_\theta(A)$ (resp. $Int_\delta(A)$). Of course, the complement of a *θ-open* (resp. *δ-open*) set is called *θ-closed* [10] (resp. *δ-closed* [21]). That is, $Cl_\theta(A) = \{x \in X \mid \forall U \in \tau(x),\ Cl(U) \cap A \neq \varnothing\}$ (resp. $Cl_\delta(A) = \{x \in X \mid \forall U \in \tau(x),\ Int(Cl(U)) \cap A \neq \varnothing\}$).

A subset A of (X, τ) is called *δ-semi-open* [16] (resp. *preopen* [11], *b-open* [1] or *γ-open* [6], *e-open* [5] and *δ-b-open* [7]) if $A \subseteq Cl(Int_\delta(A))$ (resp. $A \subseteq Int(Cl(A))$, $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$, $A \subseteq Int(Cl_\delta(A)) \cup Cl(Int_\delta(A))$, $A \subseteq Int(Cl(A)) \cup Cl(Int_\delta(A))$).

The complement of *δ-semi-open* (resp. *preopen*, *γ-open*, *e-open* and *δ-b-open*) set is said to be *δ-semi-closed* [16] (resp. *preclosed* [11], *γ-closed* [6], *e-closed* [5] and *δ-b-closed* [7]).

The family of all *δ-b-open* and *δ-b-closed* sets of (X, τ) are denoted by δSO (X, τ) and δSC (X, τ), respectively. The family of all *δ-b-open* sets of (X, τ) containing a point $x \in X$ is denoted by $\delta BO(X, x)$.

If A is a subset of a space (X, τ), then the *δ-b-closure* of A, denoted by $bCl_\delta(A)$, is the smallest *δ-b-closed* set containing A. The *δ-b-interior* of A, denoted by $bInt_\delta(A)$, is the largest *δ-b-open* set contained in A.

We have the following statements related to two operators *δ-b-closure*, *δ-b-interior* and *δ-b-closed* sets according to [7].

Lemma 2.1 *For a subset A of a space (X, τ), the following properties are hold:*

(1) $bCl_\delta(A) = A \cup (Int(Cl_\delta(A)) \cap Cl(Int(A)))$;
(2) $bInt_\delta(A) = A \cap (Int(Cl(A)) \cup Cl(Int_\delta(A)))$;
(3) $bCl_\delta(X - A) = X\text{-}bInt_\delta(A)$;
(4) $x \in bCl_\delta(A)$ *if and only if* $A \cap U \neq \varnothing$ *for every* $U \in \delta BO(X, x)$;
(5) $A \in \delta BC(X)$ *if and only if* $A = bCl_\delta(A)$.

Definition 2.2 *A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be*

(1) γ-continuous [6] (resp. e-continuous [5]) if $f^{-1}(V)$ is b-open (resp. e-open) in (X, τ) for every open set V of (Y, φ);

(2) weakly b-continuous [20] (resp. weakly e-continuous [15]) if for each $x \in X$ and each open set V of (Y, φ) containing $f(x)$, there exists a b-open (resp. an e-open) set U of (X, τ) containing x such that $f(U) \subseteq Cl(V)$.

First of all, we define a new type of continuity whose name is δ-b-continuity and give some characterizations of it.

Definition 2.3 *A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be δ-b-continuous if $f^{-1}(V)$ is δ-b-open in (X, τ) for every open set V of (Y, φ).*

Theorem 2.4 *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is δ-b-continuous;

(2) For each $x \in X$ and each $V \in \varphi(f(x))$, there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq V$;

(3) The inverse image of each closed set in (Y, φ) is δ-b-closed in (X, τ);

(4) $Int\,(Cl_\delta\,(f^{-1}\,(B))) \cap Cl\,(Int\,(f^{-1}\,(B))) \subseteq f^{-1}\,(Cl(B))$ for each $B \subseteq Y$;

(5) $f(Int\,(Cl_\delta\,(A)) \cap Cl\,(Int\,(A))) \subseteq Cl(f\,(A))$ for each $A \subseteq X$.

Proof: (1) $\Longrightarrow$ (2) : Let $x \in X$ and $V \in \varphi(f(x))$. Then $f^{-1}\,(V) \in \delta BO(X, x)$. If we consider $U = f^{-1}\,(V)$, we obtain $f(U) \subseteq V$.

(2) $\Longrightarrow$ (1) : Let $V \subseteq Y$ be *open* and $x \in f^{-1}\,(V)$. Then $f\,(x) \in V$ and thus there exists $U_x \in \delta BO(X, x)$ such that $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}\,(V)$, and so $f^{-1}\,(V) = \cup_{x \in f^{-1}(V)} U_x$. According to [7], since the union of any family ofδ-b-*open* sets is a δ-b-*open* set, we have $\cup_{x \in f^{-1}(V)} U_x \in \delta BO(X)$ and then $f^{-1}\,(V) \in \delta BO(X)$. This shows that f is δ-b-*continuous*.

(1) $\Longrightarrow$ (3) : Obvious.

(3) $\Longrightarrow$ (1) : Obvious.

(3) $\Longrightarrow$ (4) : Let $B \subseteq Y$.Then, $f^{-1}\,(Cl(B))$ is δ-b-*closed* in (X, τ). Really, we have

$Int\,(Cl_\delta\,(f^{-1}\,(B))) \cap Cl\,(Int\,(f^{-1}\,(B))) \subseteq Int\,(Cl_\delta\,(f^{-1}\,(Cl(B))))$

$\cap Cl\,(Int\,(f^{-1}\,(Cl(B)))) \subseteq f^{-1}\,(Cl(B))$ is obtained.

(4) $\Longrightarrow$ (5) : Let $A \subseteq X$.If we consider $B = f\,(A)$ in (4), then we have $Int\,(Cl_\delta\,(f^{-1}\,(f\,(A)))) \cap Cl\,(Int\,(f^{-1}\,(f\,(A)))) \subseteq f^{-1}\,(Cl(f\,(A)))$. Since for every subset A of X, $A \subseteq f^{-1}(f\,(A))$ is true, we obtain $Int\,(Cl_\delta\,(A)) \cap Cl\,(Int\,(A)) \subseteq f^{-1}\,(Cl(f\,(A)))$ and hence $f\,(Int\,(Cl_\delta\,(A)) \cap Cl\,(Int\,(A))) \subseteq Cl(f\,(A))$.

(5) $\Longrightarrow$ (1) : Let $V \in \varphi$. If we consider $W = Y$-V and $A = f^{-1}\,(W)$, we have $f\,(Int\,(Cl_\delta\,(f^{-1}\,(Y{-}V))) \cap Cl\,(Int\,(f^{-1}\,(Y{-}V)))) \subseteq Cl(f\,(f^{-1}\,(Y{-}V))) \subseteq Cl\,(Y{-}V) = Y$-$V$ by using " for every $B \subseteq Y$, $f\,(f^{-1}\,(B)) \subseteq B$" and $V \in \varphi$. Therefore, $f^{-1}\,(W) = f^{-1}\,(Y{-}V)$ is δ-b-*closed* in (X, τ). This shows that f is δ-b-*continuous*.

3 Weakly δ-b-Continuous Functions

Definition 3.1 *A function $f : (X,\tau) \longrightarrow (Y,\varphi)$ is said to be weakly δ-b-continuous (briefly w.δ.b.c.) at $x \in X$ if for each open set V of (Y,φ) containing $f(x)$, there exists a δ-b-open set U of (X,τ) containing x such that $f(U) \subseteq Cl(V)$. The function f is w.δ.b.c. iff f is w.δ.b.c. for all $x \in X$.*

We have the following Diagram from Definitions 1, 2 and 3.

$$\begin{array}{ccccc}
\text{e-continuity} & \longleftarrow & \delta\text{-}b\text{-continuity} & \longrightarrow & b\text{-continuity} \\
\downarrow & & \downarrow & & \downarrow \\
\text{weak e-continuity} & \longleftarrow & \text{weak } \delta\text{-}b\text{-continuity} & \longrightarrow & \text{weak } b\text{-continuity}
\end{array}$$

Diagram

We state that the converses of these implications are not true in generally, as shown in the [5] and the following examples.

Example 3.2 *Let (X,τ) and (Y,φ) are two topological space such that $X = \{a,b,c,d\}$, $\tau = \{\varnothing, X, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}\}$, $Y = \{a,b\}$ and $\varphi = \{\varnothing, Y, \{a\}\}$. A function $f : (X,\tau) \longrightarrow (Y,\varphi)$ defined as follows: $f(a) = f(b) = a$ and $f(c) = f(d) = b$. Then, f is b-continuous but not δ-b-continuous.*

Example 3.3 *Let (X,τ) and (Y,φ) are two topological spaces such that $X = \{a,b,c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$, $Y = \{a,b\}$ and $\varphi = \{\varnothing, Y, \{a\}\}$. A function $f : (X,\tau) \longrightarrow (Y,\varphi)$ defined as follows: $f(c) = a$ and $f(a) = f(b) = b$. Then, f is e-continuous but not δ-b-continuous.*

Example 3.4 *Let (X,τ) and (X,φ) are two topological spaces such that $X = \{a,b,c,d\}$, $\tau = \{\varnothing, X, \{a\}, \{d\}, \{a,c\}, \{a,c,d\}\}$ and $\varphi = \{\varnothing, X, \{a,b\}, \{c,d\}\}$. Then, the identity function $f : (X,\tau) \longrightarrow (X,\varphi)$ is weakly b-continuous but not weakly δ-b-continuous.*

Example 3.5 *Let (X,τ) and (X,φ) are two topological spaces such that $X = \{a,b,c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$ and $\varphi = \{\varnothing, X, \{a\}, \{b,c\}\}$. A function $f : (X,\tau) \longrightarrow (X,\varphi)$ defined as follows: $f(c) = a$ and $f(a) = f(b) = b$. Then, f is weakly e-continuous but not weakly δ-b-continuous.*

Example 3.6 *Let $X = \{a,b,c,d,e\}$, $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}, \{c,d\}, \{a,c,d\}\}$ and $\varphi = \{\varnothing, X, \{b,c,d\}\}$. Then, the identity function $f : (X,\tau) \longrightarrow (X,\varphi)$ is weakly δ-b-continuous but not δ-b-continuous.*

Now, we give some characterizations of *weak δ-b-continuity* at one point.

Proposition 3.7 *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is $w.\delta.b.c.$ at $x \in X$;

(2) $x \in Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$ for each open neighbourhood V of $f(x)$;

(3) $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$ for each $V \in \varphi$.

Proof: (1) $\Longrightarrow$ (2). Let V be any *open* subset of (Y, φ) such that containing $f(x)$. Since f is $w.\delta.b.c.$ at x, there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq Cl(V)$ and hence $U \subseteq f^{-1}(Cl(V))$. Since U is *δ-b-open*, $x \in U \subseteq Cl(Int_\delta(U)) \cup Int(Cl(U)) \subseteq Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$.

(2) $\Longrightarrow$ (3). Let $x \in f^{-1}(V)$. Then, we have $f(x) \in V$. Since $V \subseteq Cl(V)$ for every subset of (Y, φ), we have $x \in f^{-1}(Cl(V))$. By hypothesis since $x \in Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$, we have $x \in (f^{-1}(Cl(V)) \cap [Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))])$ and hence $x \in bInt_\delta(f^{-1}(Cl(V)))$. Consequently, we obtain $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$.

(3) $\Longrightarrow$ (1). Let V be any *open* neighbourhood of $f(x)$. Then, $x \in f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$. If we consider $U = bInt_\delta(f^{-1}(Cl(V)))$, we have $U \in \delta BO(X, x)$ and $f(U) \subseteq Cl(V)$. Consequently, this shows that f is $w.\delta.b.c.$ at $x \in X$.

The following three theorems are related to some characterizations of weak δ-b-continuity.

Theorem 3.8 *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is $w.\delta.b.c.$;

(2) $bCl_\delta(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$ for every subset B of (Y, φ);

(3) $bCl_\delta(f^{-1}(Int(F))) \subseteq f^{-1}(F)$ for every regular closed set F of (Y, φ);

(4) $bCl_\delta(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open set V of (Y, φ);

(5) $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$ for every open set V of (Y, φ);

(6) $f^{-1}(V) \subseteq Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$ for every open set V of (Y, φ).

Proof: (1) $\Longrightarrow$ (2). Let B be any subset of (Y, φ). Suppose that $x \in (X - f^{-1}(Cl(B)))$. Then, $f(x) \in (Y - Cl(B))$ and there exists an *open* set V containing $f(x)$ such that $V \cap B = \varnothing$; hence $Cl(V) \cap Int(Cl(B)) = \varnothing$. Since f is $w.\delta.b.c.$, there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq Cl(V)$. Therefore, we obtain $U \cap f^{-1}(Int(Cl(B))) = \varnothing$ and hence $x \in (X - bCl_\delta(f^{-1}(Int(Cl(B)))))$. So, we have $(X - bCl_\delta(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$.

(2) $\Longrightarrow$ (3). Let F be any *regular closed* set of (Y, φ). Then, we have $bCl_\delta(f^{-1}(Int(F))) = bCl_\delta(f^{-1}(Int(Cl(Int(F))))) \subseteq f^{-1}(Cl(Int(F))) = f^{-1}(F)$.

(3) $\Longrightarrow$ (4). For any *open* set V of (Y, φ). Therefore, we have $bCl_\delta(f^{-1}(V)) \subseteq bCl_\delta(f^{-1}(Int(Cl(V)))) \subseteq f^{-1}(Cl(V))$.

(4) $\Longrightarrow$ (5). Let V be any *open* set of (Y, φ). Then, $(Y - Cl(V))$ is *open* in (Y, φ) and by using Lemma 1, we have $(X - Int_\delta(f^{-1}(Cl(V)))) = bCl_\delta(f^{-1}(Y\text{-}Cl(V))) \subseteq f^{-1}(Cl(Y\text{-}Cl(V))) \subseteq (X - f^{-1}(V))$. Hence, we obtain $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$.

(5) $\Longrightarrow$ (6). Let V be any open set of (Y, φ). By Lemma 1, we have $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V))) \subseteq Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$.

(6) $\Longrightarrow$ (1). Let x be any point of (X, τ) and V be any *open* set of (Y, φ) such that containing $f(x)$. Then, $x \in f^{-1}(V) \subseteq Cl(Int_\delta(f^{-1}(Cl(V)))) \cup Int(Cl(f^{-1}(Cl(V))))$. So, we obtain f is $w.\delta.b.c.$ from Proposition 3.

Theorem 3.9 *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is $w.\delta.b.c.$;

(2) $bCl_\delta(f^{-1}(Int(Cl(V)))) \subseteq f^{-1}(Cl(V))$ for every δ-b-open set V of (Y, φ);

(3) $bCl_\delta(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every preopen set V of (Y, φ);

(4) $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(Cl(V)))$ for every preopen set V of (Y, φ).

Proof: (1) $\Longrightarrow$ (2). This is obvious from Theorem 4.2.

(2) $\Longrightarrow$ (3). Since every *preopen* set is *δ-b-open* set and $V \subseteq Int(Cl(V))$, this is obvious.

(3) $\Longrightarrow$ (4). This proof is similar to the proof of the implication (4) $\Longrightarrow$ (5) in Theorem 2.

(4) $\Longrightarrow$ (1). Since every *open* set is *preopen*, it is obtained from Theorem 4.

Theorem 3.10 *For a function $f : (X, \tau) \longrightarrow (Y, \varphi)$, the following properties are equivalent:*

(1) f is $w.\delta.b.c.$;

(2) $f(\, bCl_\delta(A)) \subseteq Cl_\theta(f(A))$ for each subset A of (X, τ);

(3) $bCl_\delta(f^{-1}(B)) \subseteq f^{-1}(Cl_\theta(B))$ for each subset B of (Y, φ);

(4) $bCl_\delta(f^{-1}(Int(Cl_\theta(B)))) \subseteq f^{-1}(Cl_\theta(B))$ for each subset B of (Y, φ).

Proof: (1) $\Longrightarrow$ (2). Let $x \in bCl_\delta(A)$, V be any open set of (Y, φ) containing $f(x)$. Then, there exists $U \in \delta BO(X,\ x)$ such that $f(U) \subseteq Cl(V)$. Then, we have $U \cap A \neq \varnothing$ and $\varnothing \neq f(U) \cap f(A) \subseteq Cl(V) \cap f(A)$, so that $f(x) \in Cl_\theta(f(A))$. The proof is completed.

(2) $\Longrightarrow$ (3). Let B be any subset of (Y, φ). Set $A = f^{-1}(B)$ in (2), then we have $f(\, bCl_\delta(f^{-1}(B))) \subseteq Cl_\theta(B)$ and $bCl_\delta(f^{-1}(B)) \subseteq f^{-1}(f(bCl_\delta(f^{-1}(B)))) \subseteq f^{-1}(Cl_\theta(B))$.

(3) $\Longrightarrow$ (4). Let B be any subset of (Y, φ). Since $Cl_\theta(B)$ is *closed* in (Y, φ), we have $bCl_\delta(f^{-1}(Int(Cl_\theta(B)))) \subseteq f^{-1}(Cl_\theta(Int(Cl_\theta(B)))) \subseteq f^{-1}(Cl_\theta(B))$.

(4) $\Longrightarrow$ (1). Let V be any *open* set of (Y,φ). Then, we have $V \subseteq Int(Cl(V)) = Int(Cl_\theta(V))$ and so $bCl_\delta(f^{-1}(V)) \subseteq bCl_\delta(f^{-1}(Int(Cl_\theta(V)))) \subseteq f^{-1}(Cl_\theta(V)) \subseteq f^{-1}(Cl(V))$. Consequently, we obtain f is *w.δ.b.c.* from Theorem 2.

We have the following two properties as results of Theorems 5 and 6.

Corollary 3.11 *If $f : (X,\tau) \longrightarrow (Y,\varphi)$ is w.δ.b.c., then $f^{-1}(V)$ is δ-b-closed (resp. δ-b-open) in (X,τ) for every θ-closed (resp. θ-open) set V of (Y,φ).*

Proof: (a) If V is θ-closed, we obtain $bCl_\delta(f^{-1}(V)) \subseteq f^{-1}(Cl_\theta(V)) = f^{-1}(V)$ and hence $f^{-1}(V)$ is *δ-b-closed* from Theorem 6.3.

(b) Although this is obvious considering complement of (a), we prove it alternatively as the following.

If V is θ-open, then $(Y-V)$ is θ-closed and so $bCl_\delta(f^{-1}(Y\text{-}V)) \subseteq f^{-1}(Cl_\theta(Y\text{-}V)) = f^{-1}(Y\text{-}V)$ from Theorem 5. Therefore, we have $bCl_\delta(X\text{-}f^{-1}(V)) \subseteq X\text{-}f^{-1}(V)$ and hence $X\text{-}bInt_\delta(f^{-1}(V)) \subseteq X\text{-}f^{-1}(V)$. Consequently, we have $f^{-1}(V) \subseteq bInt_\delta(f^{-1}(V))$ and $f^{-1}(V)$ is *δ-b*-open.

Corollary 3.12 *Let $f : (X,\tau) \longrightarrow (Y,\varphi)$ be a function. If $f^{-1}(Cl_\theta(B))$ is δ-b-closed in (X,τ) for every subset B of (Y,φ), then f is w.δ.b.c..*

Proof: Since $f^{-1}(Cl_\theta(B))$ is *δ-b-closed* in (X,τ), we have $bCl_\delta(f^{-1}(B)) \subseteq bCl_\delta(f^{-1}(Cl_\theta(B))) = f^{-1}(Cl_\theta(B))$. So, f is *w.δ.b.c.* by using Theorem 6.

Now, we define a new type of *faintly continuity* by using *δ-b-open* sets.

Definition 3.13 *A function $f : (X,\tau) \longrightarrow (Y,\varphi)$ is said to be faintly δ-b-continuous if for each $x \in X$ and each θ-open set V of (Y,φ) containing $f(x)$, there exists a δ-b-open set U of (X,τ) containing x such that $f(U) \subseteq V$.*

We give some characterizations of *faintly δ-b-continuity*.

Proposition 3.14 *For a function $f : (X,\tau) \longrightarrow (Y,\varphi)$, the following properties are equivalent:*

(1) f is faintly δ-b-continuous;

(2) The inverse image of every θ-open set in (Y,φ) is δ-b-open set in (X,τ);

(3) The inverse image of every θ-closed set in (Y,φ) is δ-b-closed set in (X,τ).

Proof: (1) $\Longrightarrow$ (2). Let V be any *θ-open* set of (Y,φ) and $x \in f^{-1}(V)$. Then, we have $f(x) \in V$ and so there exists $U_x \in \delta BO(X,x)$ such that $f(U_x) \subseteq V$. Then, we have $x \in U_x \subseteq f^{-1}(V)$ and so $f^{-1}(V) = \cup_{x\in f^{-1}(V)} U_x$. Since the union of any family of *δ-b-open* sets is a *δ-b-open* set from Theorem

2 in [7], we have $\left(\cup_{x\in f^{-1}(V)}U_x\right) \in \delta BO(X)$. Therefore, we obtain $f^{-1}(V)$ is a δ-b-*open* set of (X, τ).

(2) $\Longrightarrow$ (1). Let $x \in X$ and V be any θ-*open* subset of (Y, φ) containing $f(x)$. Then, we have $f^{-1}(V) \in \delta BO(X, x)$. If we consider $U = f^{-1}(V)$, then we obtain $f(U) \subseteq V$. This shows that f is $faintly$ δ-b-continuous.

(1) $\Longrightarrow$ (3). This proof is similar to the proof of (1) $\Longrightarrow$ (2).

(3) $\Longrightarrow$ (1). This proof is similar to the proof of (2) $\Longrightarrow$ (1).

(2) $\Longleftrightarrow$ (3). Since the complement of every θ-*closed* set is θ-*open*, proofs are obvious.

Now, we give a new type of *strongly* θ-b-*continuous* by using δ-b-*open* set.

Definition 3.15 *A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be strongly θ-δ-b-continuous (briefly st.θ.δ.b.c.) if for each $x \in X$ and each open set V of (Y, φ) containing $f(x)$, there exists a δ-b-open set U of (X, τ) containing x such that $f(bCl_\delta(U)) \subseteq V$.*

Immediately, we give the following equivalence.

Proposition 3.16 *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function and (Y, φ) be a regular space. Then f is st.θ.δ.b.c. if and only if f is δ.b.continuous.*

Proof: Because of necessity is obvious, we only prove sufficiency. Let $x \in X$ and V be any *open* subset of (Y, φ) containing $f(x)$. Since (Y, φ) is $regular$, there exists an *open* set G such that $f(x) \in G \subseteq Cl\,(G) \subseteq V$. If f is δ.*b.continuous*, there exists $U \in \delta BO(X, x)$ such that $f\,(U) \subseteq G$. Now we shall show that $f(bCl_\delta(U)) \subseteq Cl\,(G)$. Assume that $y \notin Cl\,(G)$. There exists an *open* set W containing y such that $W \cap G = \varnothing$. Since f is δ.*b.continuous*, $f^{-1}(W) \in \delta BO(X)$ and $f^{-1}(W) \cap U = \varnothing$, and hence $f^{-1}(W) \cap bCl_\delta(U) = \varnothing$. Therefore, we have $W \cap f\,(bCl_\delta(U)) = \varnothing$ and $y \notin f\,(bCl_\delta(U))$. As a result, we have $f\,(bCl_\delta(U)) \subseteq Cl\,(G) \subseteq V$ and f is *st.θ.δ.b.c.*

The next theorem is important. If the range space (Y, φ) of a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is $regular$, then it is stated that *st.θ.δ.b.c.*, *δ.b.c.*, *w.δ.b.c.* and *f.δ.b.c.* are coincide each other.

Theorem 3.17 *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function and (Y, φ) be a regular space. Then, the following properties are equivalent:*

(1) f is st.θ.δ.b.c.;

(2) f is δ.b.c.;

(3) $f^{-1}(Cl_\theta(B))$ is δ-b-closed set in (X, τ) for every subset B of (Y, φ);

(4) f is w.δ.b.c.;

(5) f is f.δ.b.c..

Proof: (1) $\Longrightarrow$ (2). Let $x \in X$ and V be any *open* subset of (Y, φ) containing $f(x)$. Then, there exists $U \in \delta BO(X, x)$ such that $f(bCl_\delta(U)) \subseteq V$. Since $U \subseteq bCl_\delta(U)$, we have $f(U) \subseteq f(bCl_\delta(U)) \subseteq V$ and hence f is *δ-b-continuous*.

(2) $\Longrightarrow$ (3). Since $Cl_\theta(B)$ is *closed* in (Y, φ) for every subset B of (Y, φ), $f^{-1}(Cl_\theta(B))$ is *δ-b-closed* in (X, τ) by using Theorem 2.

(3) $\Longrightarrow$ (4). It is obvious from Corollary 8.

(4) $\Longrightarrow$ (5). It is obtained from Corollary 7 and Proposition 9.

(5) $\Longrightarrow$ (1). Let V be any *open* subset of (Y, φ). Since (Y, φ) is *regular*, V is *θ-open* in (Y, φ). According to the *faint δ-b-continuity* of f, $f^{-1}(V)$ is *δ-b-open* in (X, τ). Therefore, f is *δ-b-continuous*. Besides since (Y, φ) is *regular*, f is $st.\theta.\delta.b.c.$ by using Proposition 10.

We state that *faint δ-b-continuity* doesn't imply *strong θ-δ-b-continuity* as shown the following example.

Example 3.18 *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ function is as same as in Example 5. Then, f is faintly δ-b-continuous but not strongly θ-δ-b-continuous.*

4 Some Properties and Some Separation Axioms

In this section, we investigate properties of $w.\delta.b.c.$ functions. Of course, these properties are related to some separation axioms and graphic functions for *δ-b-open* sets. So, we have recalled and introduced some definitions.

Theorem 4.1 *Let $g \circ f : (X, \tau) \longrightarrow (Z, \psi)$ be the composition for two functions $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \psi)$. Then, the following properties are hold:*

(1) If f is $w.\delta.b.c.$ and g is continuous, then the composition $g \circ f$ is $w.\delta.b.c.$.

(2) If f is open continuous surjection and $g \circ f$ is $w.\delta.b.c.$, then g is $w.\delta.b.c.$.

Proof: (1) Let $x \in X$ and G be any *open* subset of (Z, ψ) containing $g(f(x))$. Then, $g^{-1}(G)$ is an *open* set of (Y, φ) containing $f(x)$ and there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq Cl(g^{-1}(G))$ by using hypothesis. Since g is *continuous*, we obtain $(g \circ f)(U) \subseteq g(Cl(g^{-1}(G))) \subseteq g(g^{-1}(Cl(G))) \subseteq Cl(G)$. This shows that $g \circ f$ is $w.\delta.b.c.$.

(2) Let G be an *open* set of (Z, ψ). By hypothesis because of $g \circ f : (X, \tau) \longrightarrow (Z, \psi)$ is $w.\delta.b.c.$ and f is *continuous*, it is obvios that $(g \circ f)^{-1}(G) \subseteq Cl(Int_\delta((g \circ f)^{-1}(Cl(G)))) \cup Int(Cl((g \circ f)^{-1}(Cl(G))) = Cl(Int_\delta(f^{-1}(g^{-1}(Cl(G))))) \cup Int(Cl(f^{-1}(g^{-1}(Cl(G)))))$. Since f is *open continuous surjection*, there exist the following relations: $g^{-1}(G) = f(f^{-1}(g^{-1}(G)))$

$\subseteq f\left(Cl(Int_{\delta}(f^{-1}(g^{-1}(Cl(G))))))\cup f\left(Int(Cl(f^{-1}(g^{-1}(Cl(G))))))\right.$
$\subseteq Cl(Int_{\delta}(f(f^{-1}(g^{-1}(Cl(G))))))\cup Int(Cl(f(f^{-1}(g^{-1}(Cl(G))))))$
$\subseteq Cl(Int_{\delta}(g^{-1}(Cl(G))))\cup Int(Cl(g^{-1}(Cl(G))))$. Consequently, from Theorem 2, we obtain g is $w.\delta.b.c.$

Let $\{X_\alpha \mid \alpha\in\Delta\}$ and $\{Y_\alpha \mid \alpha\in\Delta\}$ be any two families of spaces with the some index set Δ. Let $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ be a function for each $\alpha\in\Delta$. The product space $\Pi\{X_\alpha \mid \alpha\in\Delta\}$ will be denoted by ΠX_α and $f:\Pi X_\alpha\longrightarrow\Pi Y_\alpha$ will be denote the product function defined by $f(\{x_\alpha\})=\{f_\alpha(x_\alpha)\}$ for every $\{x_\alpha\}\in\Pi X_\alpha$. Morever, let $\rho_\beta:\Pi X_\alpha\longrightarrow X_\beta$ and $q_\beta:\Pi Y_\alpha\longrightarrow Y_\beta$ be the natural projections.

As a result of Theorem 12, we give the following theorem.

Theorem 4.2 *If a function $f:\Pi X_\alpha\longrightarrow\Pi Y_\alpha$ is $w.\delta.b.c.$, then $f_\alpha : X_\alpha\longrightarrow Y_\alpha$ for each $\alpha\in\Delta$.*

Proof: Assume that f is $w.\delta.b.c.$. Since q_β is continuous, we have $q_\beta\circ f=f_\beta\circ\rho_{\beta\alpha}$ is $w.\delta.b.c.$ by using Theorem 12.1. Besides, since ρ_β is is open continuous surjection, we have f_β is $w.\delta.b.c.$ from Theorem 12.2.

Definition 4.3 *A topological space (X,τ) is said to be*

(1) Urysohn [23] if for each pair of distinct points x and y in (X,τ), there exist open sets U and V such that $x\in U$, $y\in V$ and $Cl(U)\cap Cl(V)=\varnothing$;

(2) δ-b-T_1 [3] if for each pair of distinct points x and y in (X,τ), there exist δ-b-open sets U and V of (X,τ) containing x and y, respectively, such that $y\notin U$ and $x\notin V$;

(3) δ-b-T_2[3], [17] if for each pair of distinct points x and y in (X,τ), there exist δ-b-open sets U and V of (X,τ) containing x and y, respectively, such that $U\cap V=\varnothing$.

Theorem 4.4 *Let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a $w.\delta.b.c.$ injective function. Then, the following properties hold:*

(1) If (Y,φ) is Urysohn, then (X,τ) is δ-b-T_2;

(2) If (Y,φ) is Hausdorff, then (X,τ) is δ-b-T_1.

Proof: (1) Let x_1 and x_2 be any distinct points in (X,τ). Then $f(x_1)\neq f(x_2)$ and there exist *open* sets V_1 and V_2 of (Y,φ) containing $f(x_1)$ and $f(x_2)$, resectively, such that $Cl(V_1)\cap Cl(V_2)=\varnothing$. Since f is $w.\delta.b.c.$, there exists $U_i\in\delta BO(X,x_i)$ such that $f(U_i)\subseteq Cl(V_i)$, for $i=1,\,2$. Since $f^{-1}(Cl(V_1))$ and $f^{-1}(Cl(V_2))$ are disjoint, we have $U_1\cap U_2=\varnothing$. Therefore, (X,τ) is δ-b-T_2.

(2) Let x_1 and x_2 be any distinct points in (X,τ). Then $f(x_1)\neq f(x_2)$ and there exist *open* sets V_1 and V_2 of (Y,φ) such that $f(x_1)\in V_1$ and $f(x_2)\in V_2$. Then, we have $f(x_1)\notin Cl(V_2)$ and $f(x_2)\notin Cl(V_1)$. Since f is $w.\delta.b.c.$, there esists $U_i\in\delta BO(X,x_i)$ such that $f(U_i)\subseteq Cl(V_i)$, for $i=1,\,2$. Therefore, we obtain $x_1\notin U_2$ and $x_2\notin U_1$. Consequently, (X,τ) is δ-b-T_1.

Theorem 4.5 *If $f:(X,\tau)\longrightarrow(Y,\varphi)$ is w.δ.b.c. and A is θ-closed set of $X\times Y$, then $\rho_x(A\cap G_f)$ is δ-b-closed in (X,τ) where ρ_x represents the projection of $X\times Y$ onto (X,τ) and G_f denotes the graph of f.*

Proof: Let A be a θ-closed set of $X\times Y$ and $x\in bCl_\delta(\rho_x(A\cap G_f))$. Let U be any *open* set of (X,τ) containing x and V any *open* set of (Y,φ) containing $f(x)$. Since f is *w.δ.b.c.*, by Theorem 4, we have $x\in f^{-1}(V)\subseteq bInt_\delta(f^{-1}(Cl(V)))$ and $U\cap bInt_\delta(f^{-1}(Cl(V)))\in\delta BO(X,x)$. Since $x\in bCl_\delta(\rho_x(A\cap G_f))$ by Lemma 1, $[U\cap bInt_\delta(f^{-1}(Cl(V)))]\cap\rho_x(A\cap G_f)$ contains some points y of (X,τ). This shows that $(y,f(y))\in A$ and $f(U)\subseteq Cl(V)$. Hence we obtain $\varnothing\neq(U\times Cl(V))\cap A\subseteq Cl(U\times V)\cap A$ and hence $(x,f(x))\in Cl_\theta(A)$. Since A is *θ-closed*, $(x,f(x))\in(A\cap G_f)$ and $x\in\rho_x(A\cap G_f)$. Then by using Lemma 1, $\rho_x(A\cap G_f)$ is *δ-b-closed*.

Corollary 4.6 *If $f:(X,\tau)\longrightarrow(Y,\varphi)$ has θ-closed graph and $g:(X,\tau)\longrightarrow(Y,\varphi)$ is w.δ.b.c., then the set $\{x\in X\mid f(x)=g(x)\}$ is δ-b-closed in (X,τ).*

Proof: Since G_f is *θ-closed* and $\rho_x(G_f\cap G_g)=\{x\in X\mid f(x)=g(x)\}$, we have that $\{x\in X\mid f(x)=g(x)\}$ is *δ-b-closed* by using Theorem 15.

Definition 4.7 *A function $f:(X,\tau)\longrightarrow(Y,\varphi)$ is said to have a δ-b-strongly closed graph if for each $(x,y)\in(X\times Y)$-G_f, there exist a δ-b-open subset U of (X,τ) and an open subset V of (Y,φ) such that $(x,y)\in(U\times V)$ and $(U\times Cl(V))\cap G_f=\varnothing$.*

Theorem 4.8 *If (Y,φ) is Urysohn space and $f:(X,\tau)\longrightarrow(Y,\varphi)$ is w.δ.b.c., then G_f is δ-b-strongly closed.*

Proof: Let $(x,y)\in(X\times Y)$-G_f. Then $y\neq f(x)$ and there exist *open* set V_1 and V_2 of (Y,φ) containing $f(x)$ and y, respectively, such that $Cl(V_1)\cap Cl(V_2)=\varnothing$. Since f is *w.δ.b.c.*, there exists a *δ-b-open* subset U of (X,τ) containing x such that $f(U)\subseteq Cl(V_1)$. Hence, we have $f(U)\cap Cl(V_2)=\varnothing$ and hence $(U\times Cl(V_2))\cap G_f=\varnothing$. This shows that G_f is *δ-b-strongly closed.*

Theorem 4.9 *Let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a w.δ.b.c. function such that have a δ-b-strongly closed graph G_f. If f is injective, then (X,τ) is δ-b-T_2.*

Proof: Let x_1 and x_2 be any distinct points in (X,τ). Since f is *injective*, $f(x_1)\neq f(x_2)$ and $(x_1,f(x_2))\notin G_f$. Since G_f is *δ-b-strongly closed*, there exist $U\in\delta BO(X,x_1)$, and so $f(U)\cap Cl(V)=\varnothing$. Since f is *w.δ.b.c.*, there exists a $G\in\delta BO(X,x_2)$ such that $f(G)\subseteq Cl(V)$. Hence, we have $f(U)\cap f(G)=\varnothing$ and hence $U\cap G=\varnothing$. This shows that (X,τ) is *δ-b-T_2*.

From now on, we investigate covering properties which is another separation axiom. Recall that a *Hausdorff* space (X,τ) is called *semicompact* [24] at a point x if every neighbourhood U_x contains a V_x such that $B(V_x)$, the boundary of V_x, is *compact*. Of course, it is called *semicompact*, if it has this property at every point.

Theorem 4.10 *If $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a w.δ.b.c. and (Y,φ) be a semicompact Hausdorff space, then f is δ.b.c..*

Proof: Since every semicompact Hausdorff space is regular, we obtain f is is δ.*b.c.* by using Theorem 11.

It is well known that Velićko [21] introduced the notion of *H-set* as the following. A subset A of a space (X,τ) is said to be an *H-set* if for every cover $\{U_\alpha \mid \alpha\in\Delta\}$ of A by open sets of (X,τ), there exists a finite subset Δ_0 of Δ such that $A\subseteq\cup\{Cl(U_\alpha)\mid\alpha\in\Delta_0\}$. This notion is renamed as *quasi-H-closed* relative to X by Porter et al. [18].

Definition 4.11 *A topological space (X,τ) is said to be*

(1) almost compact [13] or quasi-H-closed [18] (resp. almost lindelöf [13]) if every open cover of X has a finite resp. countable) subcover whose closures cover X;

(2) δb-compact (resp. δb-lindelöf) if every δ-b-open cover of X has a finite (resp. countable) subcover.

Now, we have the following theorem.

Theorem 4.12 *For a w.δ.b.c. surjection function $f:(X,\tau)\longrightarrow(Y,\varphi)$, then the following properties hold:*

(1) If (X,τ) is δb-compact, then (Y,φ) is almost compact;

(2) If (X,τ) is δb-lindelöf , then (Y,φ) is almost lindelöf .

Proof: (1) Let $\{V_\alpha\mid\alpha\in\Delta\}$ be a cover of Y by *open* subset of (Y,φ). For each point $x\in X$, there exists $\alpha(x)\in\Delta$ such that $f(x)\in V_{\alpha(x)}$. Since f is *w.δ.b.c.*, there exists a *δ-b-open* set U_x of X containing x such that $f(U_x)\subseteq Cl\left(V_{\alpha(x)}\right)$. The family $\{U_x\mid x\in X\}$ is a cover of X by *δ-b-open* subset of X, and so there exists a finite subset X_0 of X such that $X=\cup_{x\in X_0}U_x$. Hence, we have $Y=f(X)=\cup_{x\in X_0}Cl\left(V_{\alpha(x)}\right)$. Consequently, this shows that (Y,φ) is *almost compact*.

(2) This proof is similar to the proof of (1).

Theorem 4.13 *If a function $f:(X,\tau)\longrightarrow(Y,\varphi)$ has a δb-strongly closed graph G_f, then $f(A)$ is θ-closed in (Y,φ) for each subset A which is δ-b-compact relative to X.*

Proof: Let A be *δ-b-compact* relative to X and $y \notin f(A)$. Then, $y \in (Y - f(A))$ and for each $x \in A$ we have $(x, y) \notin G_f$. So, there exist $U_x \in \delta BO(X, x)$ and an open V_x of (Y, φ) containing y such that $f(U_x) \subseteq Cl(V_x) = \varnothing$. The collection $\{U_x \mid x \in A\}$ is a cover of A by *δ-b-open* subsets of X. Since A is *δ-b-compact* relative to X, there exists a finite subset A_0 of A such that $A \subseteq \cup\{U_x \mid x \in A_0\}$. If we consider $V = \cap_{x \in A_0} V_x$, then we obtain that V is an *open* set in (Y, φ), $y \in V$ and $f(A) \cap Cl(V) \subseteq [\cup_{x \in A_0} f(U_x)] \cap Cl(V) \subseteq [\cup_{x \in A_0} f(U_x) \cap Cl(V)] = \varnothing$. Hence $y \notin Cl_\theta(f(A))$ and hence $f(A)$ is *θ-closed* in (Y, φ).

Definition 4.14 *A topological space (X, τ) is said to be δ-b-connected (resp. γ-connected [6]) if it cannot be written as the union of two nonempty disjoint δ-b-open (resp. γ-open) sets.*

Lemma 4.15 *[7] For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is γ-connected,

(2) X cannot be expressed as the union of two nonempty disjoint δ-b-open sets.

It is obvious that Lemma 22 states a topological space is *γ-connected* if and only if it is *δ-b-connected.*

Theorem 4.16 *If $f : (X, \tau) \longrightarrow (Y, \varphi)$ is w.δ.b.c. surjection and (X, τ) is δ-b-connected (γ-connected), then (Y, φ) is connected.*

Proof: Assume that (Y, φ) is not *connected.* There exist nonempty *open* sets V_1 and V_2 of (Y, φ) such that $V_1 \cup V_2 = Y$ and $V_1 \cap V_2 = \varnothing$. Then V_1 and V_2 are *clopen* in (Y, φ). In this state, we obtain $f^{-1}(V_1) \subseteq bInt_\delta(f^{-1}(Cl(V_1))) = bInt_\delta(f^{-1}(V_1))$ and hence $f^{-1}(V_1)$ is *δ-b-open* in (X, τ) by using Proposition 3.3. Similarly, we have $f^{-1}(V_2)$ is *δ-b-open* in (X, τ). Besides, we have $f^{-1}(V_1) \cup f^{-1}(V_2) = X$, $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$ and $f^{-1}(V_1)$, $f^{-1}(V_2)$ are nonempty. So, (X, τ) is not *δ-b-connected.*

We give the following result by using Theorem 23.

Corollary 4.17 *If $f : (X, \tau) \longrightarrow (Y, \varphi)$ is δ.b.c. surjection and (X, τ) is δ-b-connected (γ-connected), then (Y, φ) is connected.*

Lemma 4.18 *Let A and B be subsets of a space (X, τ). If $A \in \delta BO(X)$ and $B \in \delta O(X)$, then $(A \cap B) \in \delta BO(B)$.*

Proof: Since A is *δ-b-open* set and B is *δ-open* set, we have $A \subseteq Int(Cl(A)) \cup Cl(Int_\delta(A))$ and $B \subseteq Int_\delta(B)$. Then, we obtain

$(A \cap B) \subseteq [Int(Cl(A)) \cup Cl(Int_\delta(A))] \cap Int_\delta(B)$

$$\begin{aligned}
&= [Int\,(Cl\,(A)) \cap Int_\delta\,(B)] \cup [Cl\,(Int_\delta\,(A)) \cap Int_\delta\,(B)] \\
&\subseteq [(Int\,(Cl\,(A)) \cap Int\,(B))] \cup [Cl(Int_\delta\,(A) \cap Int_\delta\,(B))] \\
&= [Int(Cl\,(A) \cap Int\,(B))] \cup Cl(Int_\delta\,(A \cap B) \\
&\subseteq Int(Cl\,(A \cap Int\,(B)) \cup Cl(Int_\delta[(A \cap B) \cap B]) \\
&\subseteq Int(Cl[(A \cap B) \cap B]) \cup Cl(Int_{\delta_B}\,(A \cap B)) \\
&\subseteq Int(Cl_B\,(A \cap B)) \cup Cl[Int_{\delta_B}\,(A \cap B) \cap B] \\
&\subseteq Int[Cl_B\,(A \cap B) \cap B] \cup Cl_B[Int_{\delta_B}\,(A \cap B)] \\
&\subseteq Int_B[Cl_B\,(A \cap B)] \cup Cl_B[Int_{\delta_B}\,(A \cap B)].
\end{aligned}$$

Of course, this shows that $(A \cap B)$ is *δ-b-open* set in subspace (B, τ_B) of (X, τ).

Theorem 4.19 *Let $\{U_\alpha \mid \alpha \in \Delta\}$ be any δ-open cover of a space (X, τ). If a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is w.δ.b.c., then the restriction*

Theorem 4.20 *Let $\{U_\alpha \mid \alpha \in \Delta\}$ be any δ-open cover of a space (X, τ). If a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is w.δ.b.c., then the restriction $f\,|_{U_\alpha}$: $(U_\alpha, \tau_{U_\alpha}) \longrightarrow (Y, \varphi)$ is w.δ.b.c. for each $\alpha \in \Delta$.*

Proof: Let α be an arbitrary fixed index and U_α be δ-open in (X, τ). Let x be any point of U_α and V be any *open* set of (Y, φ) containing $(f\,|_{U_\alpha})\,(x) = f\,(x)$. Since f is $w.\delta.b.$c., there exists $U \in \delta BO(X, x)$ such that $f\,(U) \subseteq Cl\,(V)$. Since U_α is open (*δ-open*) in (X, τ), by Lemma 22, $(U \cap U_\alpha) \in \delta BO(X, x)$ and $(f\,|_{U_\alpha})\,(U \cap U_\alpha) = f\,(U \cap U_\alpha) \subseteq f\,(U) \subseteq Cl\,(V)$. This shows that $f\,|_{U_\alpha}$is *w.δ.b.c.*.

It is well-known that a topological space (X, τ) is said to be

(a) *submaximal* [19], [2] if every dense subset of (X, τ) is open,

(b) *extremally disconnected* [2] if the closure of each open set of (X, τ) is open.

Theorem 4.21 *Let (X, τ) be a submaximal, extremally disconnected space. If*
$f : (X, \tau) \longrightarrow (Y, \varphi)$ has δb-strongly closed graph, then $f^{-1}\,(F)$ is closed in (X, τ) for each subset F which is H-set in (Y, φ).

Proof: Let F be H-set of (Y, φ) and $x \notin f^{-1}\,(F)$. For each $y \in F$, we have $(x, y) \in (X \times Y) \backslash G_f$ and there exist a *δ-b-open* set U_y of (X, τ) containing x and an open set V_y of (Y, φ) containing y such that $f\,(U_y) \cap Cl\,(V_y) = \varnothing$ and hence $U_y \cap f^{-1}(Cl\,(V_y)) = \varnothing$. The collection $\{V_y \mid y \in F\}$ is a cover of F by *open* sets of (Y, φ). Since F is H-set in (Y, φ), there exists a finite subset F_0 of F such that $F \subseteq \cup\{Cl(V_y) \mid y \in F_0\}$. Since (X, τ) is *submaximal* and *extremally disconnected* space, for each U_y is an *open* in (X, τ) we consider $U = \cap_{y \in F_0} U_y$. Then U is an *open* set containing x and $f\,(U) \cap F \subseteq \cup_{y \in F_0}\,[f\,(U) \cap Cl\,(V_y)] \subseteq \cup_{y \in F_0}\,[f\,(U_y) \cap Cl\,(V_y)] = \varnothing$. Therefore, we have $U \cap f^{-1}\,(F) = \varnothing$ and hence $f^{-1}\,(F)$ is *closed* in (X, τ).

Recall that a topological space (X, τ) is said to be *C-compact* [22] if for each closed subset $A \subseteq X$ and each *open* cover $\{U_\alpha \mid \alpha \in \Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{Cl(U_\alpha) \mid \alpha \in \Delta_0\}$.

Corollary 4.22 *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function with a δb-strongly closed graph, from a submaximal, extremally disconnected space (X, τ) into a C-compact space (Y, φ). Then, f is continuous.*

Proof: Let A be a *closed* subset in the *C-compact* space (Y, φ). Then, A is an *H-set* and $f^{-1}(A)$ is *closed* in (X, τ) by Theorem 28. Hence, f is *continuous*.

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