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Generalized Homothetic Formula of Two-Parameter Motions

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Abstract

Two-parameter motions and kinematics applications are studied and all one parameter motions obtained from two-parameters motion on the Euclidean plane, are investigated, [5]. Two-parameter motions in there dimensional spaces are defined [1] and [6]. In this study, sliding velocity, pole lines, hodograph and acceleration poles of two-parameter homothetic motions at $\forall(\lambda, \mu)$ positions are obtained. By defining two-parameter homothetic motion along a curve in Euclidean space E^3, the theorems related to this motion and characterizations of the trajectory surface are given.

Keywords: *Two-parameter motion, Planar motion, Euclidean plane and space.*

1 Introduction

The determination of a point or a set of points such that its velocity nor vanishes or that is a minimum has always aroused interest among kinematicians. The explanation of this is two-fold:points whose velocity, or acceleration, vanishes are important for they allow one to write simplified equations for the velocity and acceleration of any other point of the rigid body; and a point or a set of points with a minimum velocity norm locates the connecting place of a kinematic pair, in general a helicoidal pair, that connects the rigid body to the reference body. This connection produces a motion of the rigid body.

Indeed, the search for points of a rigid body with a minimum velocity norm has led to the description of the velocity of rigid body in terms of infinitesimal screws, or helicoidal fields, and therefore to the definition of the instantaneous screw axis.

Muller has introduced one- and two-parameters planar motions and obtained the relations between absolute, relative, sliding velocity and pole curves of these motions [7]. Moreover, two-parameter motions in three-dimensional space are defined by [2] and [6]. In [5] all one-parameter motions obtained from two-parameters motion on the Euclidean plane are investigated.

To investigate the geometry of the motion of a line or a point in the motion of space is important in the study of space kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces [4, 8, 3].

This paper is organised as follows. In this first part, basic concepts have been given in Euclidean plane E^2. Sliding velocity, pole lines, Hodograph and acceleration poles of two-parameter homothetic motions at $\forall(\lambda,\mu)$ positions are obtained. In the second part, by defining two-parameter homothetic motion space E^3, the theorems related to this motion and characterizations of the trajectory surface are given.

2 Two-Parameter Homothetic Motions in Euclidean Plane

The homothetic motion is examined by

$$Y = hAX + C \tag{1}$$

for $h(\lambda,\mu) \neq const.$ Also, there can be given some special results of $(\lambda,\mu) = (0,0)$ and $h(\lambda,\mu)$.

Definition 2.1 *In a Euclidean plane, general two-parameter homothetic motion is defined by*

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = h(\lambda,\mu) \begin{bmatrix} cos\theta(\lambda,\mu) & -\sin\theta(\lambda,\mu) \\ \sin\theta(\lambda,\mu) & cos\theta(\lambda,\mu) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a(\lambda,\mu) \\ b(\lambda,\mu) \end{bmatrix}, \tag{2}$$

where (y_1, y_2) and (x, y) are coordinate functions of the fixed E'^2 plane and moving E^2 planes, respectively. If λ and μ in C^∞ are given by the differential functions of the time parameter t , then homothetic motion M_I are obtained and called homothetic motion M_I obtained from M_{II} homothetic motions.

Here, at the initial time $(\lambda,\mu)=(0,0)$ and $\theta(0,0)=a(0,0)=b(0,0)=(0,0)$, the coordinate systems of the moving E^2 and fixed E'^2 planes are congruent.

Theorem 2.2 *The equation of the pole points of homothetic motions M_I obtained from homothetic motions M_{II} on a moving plane is*

$$\begin{aligned}&\left(h\dot{a}\dot{\theta}\cos\theta+\dot{h}\dot{a}\sin\theta+h\dot{b}\dot{\theta}\sin\theta-\dot{h}\dot{b}\cos\theta\right)x_p\\&+\left(-h\dot{a}\dot{\theta}\sin\theta+\dot{h}\dot{a}\cos\theta+h\dot{b}\dot{\theta}\cos\theta+\dot{h}\dot{b}\sin\theta\right)y_p=0.\end{aligned}\tag{3}$$

Proof. By differentiating equation (2) with respect to t and simplifying it, we obtain

$$\begin{aligned}x_p&=\frac{h\dot{a}\dot{\theta}\sin\theta-\dot{h}\dot{a}\cos\theta-h\dot{b}\dot{\theta}\cos\theta-\dot{h}\dot{b}\sin\theta}{\dot{h}^2+h^2\dot{\theta}^2}\\y_p&=\frac{h\dot{a}\dot{\theta}\cos\theta+\dot{h}\dot{a}\sin\theta+h\dot{b}\dot{\theta}\sin\theta-\dot{h}\dot{b}\cos\theta}{\dot{h}^2+h^2\dot{\theta}^2}.\end{aligned}\tag{4}$$

After some routine calculations, the equation of the pole points (3) is obtained. The pole points of homothetic motions M_I obtained from homothetic motions M_{II} on a moving plane are given by

$$P(x_p,y_p)=\left(-\frac{h\dot{b}\dot{\theta}+\dot{h}\dot{a}}{\dot{h}^2+h^2\dot{\theta}^2}\ ,\ \frac{h\dot{a}\dot{\theta}-\dot{h}\dot{b}}{\dot{h}^2+h^2\dot{\theta}^2}\right)\tag{5}$$

at the position of $(\lambda,\mu)=(0,0)$ and the equation of the pole points is

$$\left(h\dot{a}\dot{\theta}-\dot{h}\dot{b}\right)x_p+\left(h\dot{b}\dot{\theta}+\dot{h}\dot{a}\right)y_p=0\tag{6}$$

The pole points of homothetic motions M_I obtained from homothetic motions M_{II} on a moving plane at the position of $(\lambda,\mu)=(0,0)$ give the following results.

Corollary 2.3 *If $\theta(\lambda,\mu)=const$, then the pole points lie on the line*

$$\left(h_\mu b_\lambda-h_\lambda b_\mu\right)x_p+\left(h_\lambda a_\mu-h_\mu a_\lambda\right)y_p=a_\lambda b_\mu-a_\mu b_\lambda.\tag{7}$$

Corollary 2.4 *If $h(\lambda,\mu)\neq 0$ is a constant, then the pole points lie on the line*

$$\left(a_\mu\theta_\lambda-a_\lambda\theta_\mu\right)x_p+\left(b_\mu\theta_\lambda-b_\lambda\theta_\mu\right)y_p=\frac{1}{h}\left(a_\lambda b_\mu-a_\mu b_\lambda\right).\tag{8}$$

Corollary 2.5 *If $h(\lambda,\mu)=1$, then the pole points lie on the line*

$$\left(a_\mu\theta_\lambda-a_\lambda\theta_\mu\right)x_p+\left(b_\mu\theta_\lambda-b_\lambda\theta_\mu\right)y_p=a_\lambda b_\mu-a_\mu b_\lambda\ \ [1]\tag{9}$$

Theorem 2.6 *The equation of the pole points of homothetic motions M_I obtained from homothetic motions M_{II} on a fixed plane is*

$$\left(h\dot{a}\dot{\theta}-\dot{h}\dot{b}\right)\bar{x}_p+\left(h\dot{b}\dot{\theta}+\dot{h}\dot{a}\right)\bar{y}_p=a\left(\dot{h}\dot{b}-h\dot{a}\dot{\theta}\right)-b\left(\dot{h}\dot{a}+h\dot{b}\dot{\theta}\right). \tag{10}$$

Proof. By taking $P\left(x_p,y_p\right)$ in equation (1), we have the pole points

$$\bar{P}(\bar{x}_p,\bar{y}_p)=\left(-\frac{h^2\dot{b}\dot{\theta}+\dot{h}h\dot{a}}{\dot{h}^2+h^2\dot{\theta}^2}+a\quad,\quad\frac{h^2\dot{a}\dot{\theta}-\dot{h}h\dot{b}}{\dot{h}^2+h^2\dot{\theta}^2}+b\right) \tag{11}$$

and the equation of the pole points (10) is obtained. The pole points of homothetic motions M_I obtained from homothetic motions M_{II} on a fixed plane at the position of $(\lambda,\mu)=(0,0)$ give the following results.

Corollary 2.7 *On the fixed plane $\theta\left(\lambda,\mu\right)=const$, the pole points lie on the line are*

$$\left(h_\mu b_\lambda-h_\lambda b_\mu\right)\bar{x}_p+\left(h_\lambda a_\mu-h_\mu a_\lambda\right)\bar{y}_p=h\left(a_\lambda b_\mu-a_\mu b_\lambda\right). \tag{12}$$

Corollary 2.8 *As a special case in Corollary 4 if $h\left(\lambda,\mu\right)=1$, the pole points of the fixed and moving planes are congurent.*

Corollary 2.9 *If $h\left(\lambda,\mu\right)\neq 0$ is constant, the pole points of fixed planes lie on the line equation (9) [5].*

Corollary 2.10 *As a special case in Corollary 2, if $h\left(\lambda,\mu\right)=1$, the pole points of moving planes are congruent to pole lines of fixed plane in Corollary 6.*

If the pole points of homothetic motions M_I obtained from homothetic motions M_{II} are chosen y axis, then, $x_p=0$ at the position of $\lambda=\mu=0$. Hence, we have

$$y_p=\frac{\dot{a}}{h\dot{\theta}}.$$

Therefore, there is a relation between the pole lines of the fixed plane and the pole lines of a moving plane as follows:

$$\bar{y}_p=hy_p.$$

Now, we investigate the sliding velocity $\overrightarrow{V_f}=\left(\dot{y}_1,\dot{y}_2\right)$ of any $B\left(x,y\right)$ points at the position of $\lambda=\mu=0$. Equation (2) is derived with respect to t and with the position of $\lambda=\mu=0$, we have

$$\begin{aligned}\dot{y}_1&=\dot{h}x-h\dot{\theta}y+\dot{a},\\ \dot{y}_2&=\dot{h}y+h\dot{\theta}x+\dot{b}.\end{aligned} \tag{13}$$

Thus, the sliding velocity is obtained as follows:

$$\overrightarrow{V_f}=\left(\dot{h}x-h\dot{\theta}y+\dot{a}\,,\,\dot{h}y+h\dot{\theta}x+\dot{b}\right). \tag{14}$$

Theorem 2.11 *In homothetic motions M_I obtained from homothetic motions M_{II}, let y-axis be the pole axis at the position of $\lambda = \mu = 0$. Then, the relation between the pole ray going from the pole point $P(x_p, y_p)$ to the point $B(x, y)$ and the sliding velocity $\overrightarrow{V_f}$ of the point $B(x, y)$ is*

$$\left\langle \overrightarrow{V_f}, \overrightarrow{PB} \right\rangle = \dot{h}\left(x^2 + y^2\right) + 2\dot{b}y - \frac{\dot{a}\dot{b}}{h\dot{\theta}}. \tag{15}$$

Proof. By reason of the fact that the pole axis is y-axis, we have $(x_p, y_p) = \left(0, \frac{\dot{a}}{h\dot{\theta}}\right)$ and $\overrightarrow{PB} = \left(x, y - \frac{\dot{a}}{h\dot{\theta}}\right)$ from equation (5). Then it is seen that

$$\begin{aligned} \left\langle \overrightarrow{V_f}, \overrightarrow{PB} \right\rangle &= \left\langle \left(\dot{h}x - h\dot{\theta}y + \dot{a}\,,\, \dot{h}y + h\dot{\theta}x + \dot{b}\right), \left(x\,,\, y - \tfrac{\dot{a}}{h\dot{\theta}}\right) \right\rangle \\ &= \dot{h}\left(x^2 + y^2\right) + 2\dot{b}y - \tfrac{\dot{a}\dot{b}}{h\dot{\theta}} \end{aligned}$$

Corollary 2.12 *If $h(\lambda, \mu)$ is a constant never vanishing and the pole axis is the y-axis, then the pole ray and the sliding velocity $\overrightarrow{V_f}$ are perpendicular [5].*

Theorem 2.13 *The length of the sliding velocity vector of homothetic motions M_I obtained from homothetic motion M_{II} is*

$$\left\| \overrightarrow{V_f} \right\| = \sqrt{\dot{h}^2 + h^2\dot{\theta}^2} \left\| \overrightarrow{PB} \right\| \tag{16}$$

at the position of each (λ, μ).

Proof. Substituting the differentiation of C given in equation (1) into $\overrightarrow{V_f}$, we get

$$\overrightarrow{V_f} = \begin{pmatrix} \left(\dot{h}\cos\theta - h\dot{\theta}\sin\theta\right)(x - x_p) - \left(\dot{h}\sin\theta + h\dot{\theta}\cos\theta\right)(y - y_p)\,, \\ \left(\dot{h}\sin\theta + h\dot{\theta}\cos\theta\right)(x - x_p) + \left(\dot{h}\cos\theta - h\dot{\theta}\sin\theta\right)(y - y_p) \end{pmatrix}$$

Then, the length of the sliding velocity vector $\overrightarrow{V_f}$ is obtained.

Corollary 2.14 *If $h(\lambda, \mu) = 1$, then we obtain $\left\| \overrightarrow{V_f} \right\| = \left|\dot{\theta}\right| \left\| \overrightarrow{PB} \right\|$ [5].*

Theorem 2.15 *For all homothetic motions M_I obtained from homothetic motions M_{II}, let ψ be angle between the pole ray going from the pole point P to the point B and the sliding velocity vector $\overrightarrow{V_f}$. Then, we have the relation*

$$\cos\psi(\lambda, \mu) = \frac{\dot{h}\cos\theta - h\dot{\theta}\sin\theta}{\sqrt{\dot{h}^2 + h^2\dot{\theta}^2}} \tag{17}$$

at the position of each (λ, μ).

Proof. There is the following relation between the pole ray $\overrightarrow{PB} = (x - x_p, y - y_p)$ and sliding velocity vector $\overrightarrow{V_f}$:

$$\left\langle \overrightarrow{PB}, \overrightarrow{V_f} \right\rangle = \left(\dot{h}\cos\theta - h\dot{\theta}\sin\theta\right) \left\| \overrightarrow{PB} \right\|^2 .$$

On the other hand, it is know that

$$\left\langle \overrightarrow{PB}, \overrightarrow{V_f} \right\rangle = \left\| \overrightarrow{V_f} \right\| \left\| \overrightarrow{PB} \right\| \cos\psi\left(\lambda, \mu\right) .$$

From the equality of the last two equations, we obtain equation (17).

Corollary 2.16 *If $h\left(\lambda, \mu\right) \neq 0$ is constant, then we obtain*

$$\psi\left(\lambda, \mu\right) = \frac{\pi}{2} + \theta\left(\lambda, \mu\right) \quad , \quad \theta = 2k\pi \quad (k = 0, 1, ...) \text{ [1]}.$$

Definition 2.17 *When the sliding velocity vectors of a fixed point are carried to the initial point, without changing the directions, then the locus of the end points of these vectors is a curve called a hodograph.*

Now we investigate any (x, y) points of the locus of the hodographs in all homothetic motions M_I obtained from homothetic motion M_{II}, according to the position of $\dot{\lambda}$ and $\dot{\mu}$. For this let $\dot{\lambda}^2 + \dot{\mu}^2 = 1$. By taking the derivatives with respect to t of the equation (2), we have

$$\begin{aligned} \dot{y}_1 &= \left(h_\lambda x\cos\theta - h_\lambda y\sin\theta - h\theta_\lambda x\sin\theta - h\theta_\lambda y\cos\theta + a_\lambda\right)\dot{\lambda} \\ &\quad + \left(h_\mu x\cos\theta - h_\mu y\sin\theta - h\theta_\mu x\sin\theta - h\theta_\mu y\cos\theta + a_\mu\right)\dot{\mu} \\ \dot{y}_2 &= \left(h_\lambda x\sin\theta + h_\lambda y\cos\theta + h\theta_\lambda x\cos\theta - h\theta_\lambda y\sin\theta + b_\lambda\right)\dot{\lambda} \\ &\quad + \left(h_\mu x\sin\theta + h_\mu y\cos\theta + h\theta_\mu x\cos\theta - h\theta_\mu y\sin\theta + b_\mu\right)\dot{\mu} . \end{aligned} \tag{18}$$

Let us investigate the solution of the last equation system by taking $(\lambda, \mu) = (0, 0)$ for simplicity. From equation (18), we find

$$\begin{aligned} \det\Delta = {} & h(x^2 + y^2)\left(h_\lambda\theta_\mu - h_\mu\theta_\lambda\right) + hx\left(a_\lambda\theta_\mu - a_\mu\theta_\lambda\right) + hy\left(b_\lambda\theta_\mu - b_\mu\theta_\lambda\right) \\ & + x\left(h_\lambda b_\mu - h_\mu b_\lambda\right) - y\left(h_\lambda a_\mu - h_\mu a_\lambda\right) + a_\lambda b_\mu - a_\mu b_\lambda , \end{aligned}$$

that is,

$$\begin{aligned} & \left[h^2x^2\left(\theta_\lambda^2 + \theta_\mu^2\right) + y^2\left(h_\lambda^2 + h_\mu^2\right) + 2hxy\left(h_\lambda\theta_\lambda + h_\mu\theta_\mu\right) + 2hx\left(b_\lambda\theta_\lambda + b_\mu\theta_\mu\right)\right. \\ & \left. + 2y\left(h_\lambda b_\lambda + h_\mu b_\mu\right) + \left(b_\lambda^2 + b_\mu^2\right)\right]\dot{y}_1^2 + \left[x^2\left(h_\lambda^2 + h_\mu^2\right) + h^2y^2\left(\theta_\lambda^2 + \theta_\mu^2\right)\right. \\ & \left. - 2hxy\left(h_\lambda\theta_\lambda + h_\mu\theta_\mu\right) + 2x\left(h_\lambda a_\lambda + h_\mu a_\mu\right) - 2hy\left(a_\lambda\theta_\lambda + a_\mu\theta_\mu\right) + \left(a_\lambda^2 + a_\mu^2\right)\right]\dot{y}_2^2 \\ & - 2\left[h\left(x^2 - y^2\right)\left(h_\lambda\theta_\lambda + h_\mu\theta_\mu\right) + xy\left(h_\lambda^2 + h_\mu^2 - h^2\left(\theta_\lambda^2 + \theta_\mu^2\right)\right) + hx\left(a_\lambda\theta_\lambda + a_\mu\theta_\mu\right)\right. \\ & \left. - hy\left(b_\lambda\theta_\lambda + b_\mu\theta_\mu\right) + x\left(h_\lambda b_\lambda + h_\mu b_\mu\right) + y\left(h_\lambda a_\lambda + h_\mu a_\mu\right) + a_\lambda b_\lambda + a_\mu b_\mu\right]\dot{y}_1\dot{y}_2 \\ & = \left(\det\Delta\right)^2 . \end{aligned} \tag{19}$$

Finally, if we find the values of $\dot{\lambda}$ and $\dot{\mu}$ and substitute these values into the equation $\dot{\lambda}^2 + \dot{\mu}^2 = 1$, and the following theorem is found.

Theorem 2.18 *In all homothetic motions M_I obtained from homothetic motions M_{II}, the locus of the hodograph is a ellipse at the position of $\lambda = \mu = 0$*

Proof. Setting $\lambda = \mu = 0$ in equation (19) and taking into consideration the general conic form, we can say that

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

and

$$\det \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{pmatrix} h(x^2+y^2)\left(h_\lambda \theta_\mu - h_\mu \theta_\lambda\right) + hx\left(a_\lambda \theta_\mu - a_\mu \theta_\lambda\right) + hy\left(b_\lambda \theta_\mu - b_\mu \theta_\lambda\right) \\ +x\left(h_\lambda b_\mu - h_\mu b_\lambda\right) - y\left(h_\lambda a_\mu - h_\mu a_\lambda\right) + a_\lambda b_\mu - a_\mu b_\lambda \end{pmatrix}^2 > 0.$$

That is, the locus of the hodograph is a ellipse.

3 The Acceleration Pole of the Homothetic Motions

Now we will investigate the locus of the points which have zero sliding acceleration. So, we need to solve the equation $\left(\ddot{h}A + h\ddot{A} + 2\dot{h}\dot{A}\right)X + \ddot{C} = 0$. The solution of this equation gives the coordinates of the acceleration pole points. From this we get

$$\begin{aligned} x_{ip} &= \frac{\ddot{a}\left(-\ddot{h}\cos\theta + h\dot{\theta}^2\cos\theta + h\ddot{\theta}\sin\theta + 2\dot{h}\dot{\theta}\sin\theta\right) - \ddot{b}\left(\ddot{h}\sin\theta - h\dot{\theta}^2\sin\theta + h\ddot{\theta}\cos\theta + 2\dot{h}\dot{\theta}\cos\theta\right)}{\left(\ddot{h} - h\dot{\theta}^2\right)^2 + \left(2\dot{h}\dot{\theta} + h\ddot{\theta}\right)^2}, \\ y_{ip} &= \frac{\ddot{a}\left(\ddot{h}\sin\theta - h\dot{\theta}^2\sin\theta + h\ddot{\theta}\cos\theta + 2\dot{h}\dot{\theta}\cos\theta\right) + \ddot{b}\left(-\ddot{h}\cos\theta + h\dot{\theta}^2\cos\theta + h\ddot{\theta}\sin\theta + 2\dot{h}\dot{\theta}\sin\theta\right)}{\left(\ddot{h} - h\dot{\theta}^2\right)^2 + \left(2\dot{h}\dot{\theta} + h\ddot{\theta}\right)^2}. \end{aligned} \tag{20}$$

Thus, for $\lambda = \mu = 0$, the acceleration pole points are given by

$$P\left(x_{ip}, y_{ip}\right) = \left(\frac{-\ddot{h}\ddot{a} - 2\dot{h}\ddot{b}\dot{\theta} + h\left(\ddot{a}\dot{\theta}^2 - \ddot{b}\ddot{\theta}\right)}{\left(\ddot{h} - h\dot{\theta}^2\right)^2 + \left(2\dot{h}\dot{\theta} + h\ddot{\theta}\right)^2}, \frac{-\ddot{h}\ddot{b} + 2\dot{h}\ddot{a}\dot{\theta} + h\left(\ddot{b}\dot{\theta}^2 + \ddot{a}\ddot{\theta}\right)}{\left(\ddot{h} - h\dot{\theta}^2\right)^2 + \left(2\dot{h}\dot{\theta} + h\ddot{\theta}\right)^2} \right). \tag{21}$$

Theorem 3.1 *The equation of the acceleration poles of the homothetic motions M_I obtained from homothetic motions M_{II} on the moving plane is*

$$(h\ddot{a}\ddot{\theta} - \ddot{h}\ddot{b})x_{ip} + (\ddot{h}\ddot{a} + h\ddot{b}\ddot{\theta})y_{ip} = 0 \tag{22}$$

at position $\lambda = \mu = \dot{\lambda} = \dot{\mu} = 0$.

Proof. Setting $\lambda = \mu = \dot{\lambda} = \dot{\mu} = 0$ in equation (1) gives us the desired equation. Therefore, we can give following corollaries at the position of $(\lambda, \mu) = (0, 0)$.

Corollary 3.2 *The acceleration pole points on the moving plane lie on the line given by equation (7) if* $\theta(\lambda,\mu)$ *is constant.*

Corollary 3.3 *The acceleration pole points on the moving plane lie on the line given by equation (8) if* $h(\lambda,\mu)\neq 0$ *is constant.*

Corollary 3.4 *The acceleration pole points on the moving plane lie on the line given by equation (9) if* $h(\lambda,\mu)=1$ *[5].*

Corollary 3.5 *If* $h(\lambda,\mu)\neq 0$ *is constant, the pole line on the moving planes obtained from Corollary 2 and the acceleration pole line on the moving planes obtained from Corollary 12 are congruent [5].*

Theorem 3.6 *The equation of the acceleration pole points of the homothetic motions* M_I *obtained from homothetic motions* M_{II} *on the fixed plane is*

$$(h\ddot{a}\ddot{\theta}-\ddot{h}\ddot{b})\bar{x}_{ip}+(\ddot{h}\ddot{a}+h\ddot{b}\ddot{\theta})\bar{y}_{ip}=0 \tag{23}$$

at position $\lambda=\mu=\dot{\lambda}=\dot{\mu}=0$.

Proof. If we substitute the acceleration pole points into equation (1), we find

$$\bar{P}(\bar{x}_{ip},\bar{y}_{ip})=\left(h\frac{-\ddot{h}\ddot{a}-2\dot{h}\ddot{b}\dot{\theta}+h(\ddot{a}\dot{\theta}^2-\ddot{b}\ddot{\theta})}{(\ddot{h}-h\dot{\theta}^2)^2+(2\dot{h}\dot{\theta}+h\ddot{\theta})^2}+a\,,\ h\frac{-\ddot{h}\ddot{b}+2\dot{h}\ddot{a}\dot{\theta}+h(\ddot{b}\dot{\theta}^2+\ddot{a}\ddot{\theta})}{(\ddot{h}-h\dot{\theta}^2)^2+(2\dot{h}\dot{\theta}+h\ddot{\theta})^2}+b\right). \tag{24}$$

If we take $\lambda=\mu=\dot{\lambda}=\dot{\mu}=0$ in the last equation, we have equation (23).
So, we can give the following corollaries at the position of $\lambda=\mu=\dot{\lambda}=\dot{\mu}=0$.

Corollary 3.7 *The acceleration pole points on the fixed plane lie on the line given by equation (12) if* $\theta(\lambda,\mu)$ *is constant.*

Corollary 3.8 *As a special case, if* $h(\lambda,\mu)=1$ *and* $\theta(\lambda,\mu)$ *is constant, the acceleration pole points on the moving plane and the acceleration pole points on the fixed plane are congruent.*

Corollary 3.9 *The acceleration pole points on the fixed plane lie on the line given by equation (9) if* $h(\lambda,\mu)\neq 0$ *is constant.*

Corollary 3.10 *If* $h(\lambda,\mu)=1$ *the acceleration pole line of a moving plane obtained from Corollary 12 and the acceleration pole line of a fixed plane obtained from Corollary 15 are congruent.*

Corollary 3.11 *As is seen from Corollaries 4-15 and 6-17 , the pole line of a fixed plane and the acceleration pole line of a fixed plane are congruent.*

4 Two-Parameter Homothetic Motion Along a Curve in Euclidean Space

In this section, we define two-parameter homothetic motion along a curve in a Euclidean Space and obtain characterization of the same trajectory surface.

Let E^3 and E'^3 are moving and fixed Euclidean Space, respectively, then motion of E^3 with respect to E'^3 depend on 6 independent variable. The first three of them are three component of orthogonal matrix represents rotation and the other three of them are three component of represents translation. Let s and t denotes the parameters of two-parameter motion of E^3 with respect to E'^3. Then generally the locus of a point is a surface.
Two-parameter motion of E^3 with respect to E'^3 is represented by

$$\varphi(s,t) = \mathrm{A}(s,t)\vec{p} + \vec{d}(s,t). \tag{25}$$

In this section, some parametrizations of orbit surface are given in special case of $\mathrm{A}(s,t)$ is orthogonal matrix and $\vec{d}(s,t)$ is translation vector. $\mathrm{O}(3)$ denotes the set of all 3×3 orthogonal matrices and $\Omega(3)$ denotes a vector space, given by

$$\Omega(3) = \left\{ \Omega = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix} \ , \ w_i \in \mathbb{R} \right\}$$

Let, P is column matrix corresponding to $\vec{p}$ for $\vec{p} \in E^3$ and Ω is an anti-symmetric matrix corresponding to $\vec{w}$, then

$$\Omega\mathrm{P} = \vec{w} \wedge \vec{p}$$

In the other words, cross product of two vectors are equal to matrix product of corresponding to these vectors.
Let, $\vec{w}(s) = (w_1(s), w_2(s), w_3(s))$, which is a differentiable function with respect to $s \in \mathbb{R}$, a vector-valued function. Accordingly, there is a unique Ω anti-symmetric matrix

$$\Omega(s) = \begin{bmatrix} 0 & w_3(s) & -w_2(s) \\ -w_3(s) & 0 & w_1(s) \\ w_2(s) & -w_1(s) & 0 \end{bmatrix}$$

for all $\forall\, s \in I \subset \mathbb{R}$ satisfying the following equality:

$$\Omega(s)\mathrm{P}(s) = \vec{w}(s) \wedge \vec{p}(s) \tag{26}$$

for $\vec{w}(s)$ and $\vec{p} \in E^3$.

$$\mathrm{A}(s,t) = I + (\sin t)\Omega + (1-\cos t)\Omega^2 \tag{27}$$

is the orthogonal matrix defined via the anti-symmetric matrix $\Omega(s)$ corresponding to the vector $\vec{w}(s) = (w_1(s), w_2(s), w_3(s))$ [8].
Let $\vec{p}(s)$ and P indicates position vector and matrix form of $\vec{p} \in E'^3$, respectively. Then, from equation (27) we get

$$\mathrm{A}(s,t)\vec{p} = \mathrm{A}(s,t)\mathrm{P} = \left[I + (\sin t)\Omega + (1-\cos t)\Omega^2\right]\mathrm{P}. \tag{28}$$

Also, since $\Omega\mathrm{P} = \vec{w} \wedge \vec{p}$ and $\vec{w} \wedge (\vec{w} \wedge \vec{p}) = \langle \vec{w}, \vec{p} \rangle \vec{w} - \langle \vec{w}, \vec{w} \rangle \vec{p}$ by using the equations (27) and (28), we obtain

$$\mathrm{A}(s,t)\vec{p} = \vec{p}\cos t + \langle \vec{w}, \vec{p} \rangle \vec{w}(1-\cos t) + (\vec{w} \wedge \vec{p})\sin t. \tag{29}$$

Definition 4.1 *Two-parameter homothetic motion in a Euclidean space along the curve $\alpha(s)$ is defined by*

$$\varphi(s,t) = h(s,t)A(s,t)\vec{p} + \alpha(s).$$

Let $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ be the Frenet frame of the curve α of the point p. The trajectory $\varphi(s,t)(p)$ of the point p is a surface. From equations (27) and (29), we obtain the parametrization of this surface as follows.

$$\varphi(s,t)(p) = \vec{p}\cos t + \left\langle \vec{T}, \vec{p} \right\rangle \vec{T}(1-\cos t) + \left(\vec{T} \wedge \vec{p}\right)\sin t + \alpha(s).$$

Then, two-parameter homothetic motion in Euclidean space along the curve $\alpha(s)$ can be deduced to

$$\varphi(s,t)(p) = h(s,t)\left[\vec{p}\cos t + \left\langle \vec{T}, \vec{p} \right\rangle \vec{T}(1-\cos t) + \left(\vec{T} \wedge \vec{p}\right)\sin t\right] + \alpha(s). \tag{30}$$

Now, we obtain the normal of the surface drawn by the trajectory of the points p. Since the Frenet formulas are

$$\vec{T}' = k_1\vec{N} \quad , \quad \vec{N}' = -k_1\vec{T} + k_2\vec{B} \quad , \quad \vec{B}' = -k_2\vec{N}$$

then from the equation (30) we get

$$\begin{aligned}\varphi_t = {} & h_t(s,t)\left[\vec{p}\cos t + \left\langle \vec{T}, \vec{p} \right\rangle \vec{T}(1-\cos t) + \left(\vec{T} \wedge \vec{p}\right)\sin t\right] \\ & + h(s,t)\left[-\vec{p}\sin t - \left\langle \vec{T}, \vec{p} \right\rangle \vec{T}\sin t + \left(\vec{T} \wedge \vec{p}\right)\cos t\right]\end{aligned}$$

and

$$\begin{aligned}\varphi_s = {} & h_s(s,t)\left[\vec{p}\cos t + \left\langle \vec{T}, \vec{p} \right\rangle \vec{T}(1-\cos t) + \left(\vec{T} \wedge \vec{p}\right)\sin t\right] \\ & + h(s,t)\left[k_1\sin t\left(\vec{N} \wedge \vec{p}\right) + (1-\cos t)k_1\left\langle \vec{N}, \vec{p} \right\rangle \vec{T} + (1-\cos t)\left\langle \vec{T}, \vec{p} \right\rangle k_1\vec{N}\right] + \vec{T}.\end{aligned}$$

If we take $\vec{p}=\lambda\vec{N}$, then

$$\varphi_t=(h_t\lambda\cos t-h\lambda\sin t)\,\vec{N}+(h_t\lambda\sin t+h\lambda\cos t)\,\vec{B},$$

$$\varphi_s=[hk_1\lambda\,(1-\cos t)+1]\,\vec{T}+h_s\lambda\cos t\,\vec{N}+h_s\lambda\sin t\,\vec{B}.$$

Hence, of the surface drawn by the trajectory of the points p is

$$\varphi_t\wedge\varphi_s=\begin{vmatrix}\vec{T} & \vec{N} & \vec{B}\\ 0 & h_t\lambda\cos t-h\lambda\sin t & h_t\lambda\sin t+h\lambda\cos t\\ h\,(1-\cos t)\,k_1\lambda+1 & h_s\lambda\cos t & h_s\lambda\sin t\end{vmatrix}$$

i.e.,

$$\begin{aligned}\varphi_t\wedge\varphi_s=&[-h_sh\lambda^2]\,\vec{T}+[hk_1\lambda^2\,(h_t\sin t+h\cos t)\,(1-\cos t)+h_t\lambda\sin t+h\lambda\cos t]\,\vec{N}\\ &+[hk_1\lambda^2\,(h\sin t-h_t\cos t)\,(1-\cos t)-h_t\lambda\cos t+h\lambda\sin t]\,\vec{B}.\end{aligned}$$

If $h(s,t)$ is a constant that is never vanishing, then the normal of this surface is in a normal plane which is perpendicular to the tangent vector field of the curve $\alpha(s)$.

5 Parametrizations of Trajectory Surfaces

In this section, we find some parametrizations of the trajectory surfaces obtained from two-parameter motions in a Euclidean space.

5.1 Cylinder Surface

Assume that $\alpha(s)=(0,0,s)$ and $p=(p_1,p_2,p_3)\in E^3$. Substituting these into equation (30), we get

$$\varphi(s,t)(p)=(hp_1\cos t-hp_2\sin t\,,\ hp_2\cos t+hp_1\sin t\,,\ hp_3+s)\,.$$

As a special case, if $p=(p_1,p_2,0)$, we obtain

$$\varphi(s,t)(p)=(hp_1\cos t-hp_2\sin t\,,\ hp_2\cos t+hp_1\sin t\,,\ s)\,.$$

For $p_1=r\sin\theta$ and $p_2=r\cos\theta$, we get

$$\varphi(s,t)(p)=(hr\sin\theta\cos t-hr\cos\theta\sin t\,,\ hr\cos\theta\cos t+hr\sin\theta\sin t\,,\ s)\,,$$

that is,

$$\varphi(s,t)(p)=(hr\sin(\theta-t)\ ,\ hr\cos(\theta-t)\ ,\ s)\,. \tag{31}$$

Example 5.1 *Let $-1<s<1$, $0<t,\theta<2\pi$ and $h(s,t)=s+\sin t\cos t$ in equation (31) then we can obtain the homothetic cylinder surface given in Figure 1.*

Example 5.2 *If we take $h(s,t)=1$ in equation (31) the cylinder surface is obtained as given in Figure 2.*

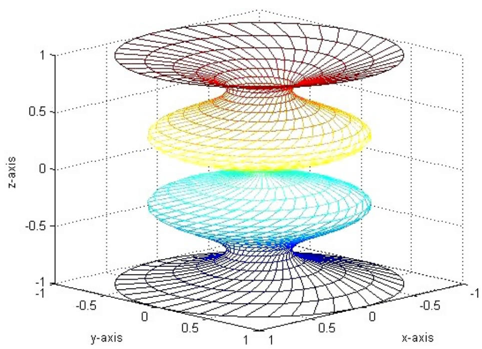

Figure 1: Homothetic cylinder surface

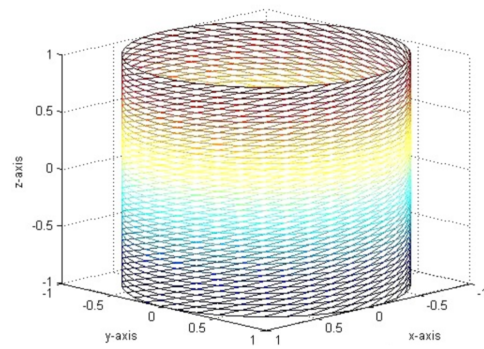

Figure 2: Cylinder surface

5.2 Hyperboloid Surface

Let $\alpha(s) = (0, 0, s)$ and $p(s) = (1, s, 0)$; substituting these into equation (30), we get

$$\varphi(s,t)(p) = (h\cos t - hs\sin t, hs\cos t + h\sin t, s)\,. \tag{32}$$

Example 5.3 *In equation (32) if* $-1 < s < 1$, $0 < t < 2\pi$ *and* $h(s,t) = s + \sin t\cos t$ *are given, then a homothetic hyperboloid surface is obtained as given in Figure 3.*

Example 5.4 *In equation (32) if* $h(s,t) = 1$ *is taken, then a hyperboloid surface is obtained as given by Figure 4.*

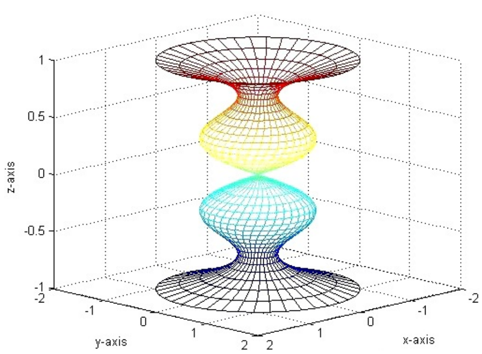

Figure 3: Homothetic hyperboloid surface

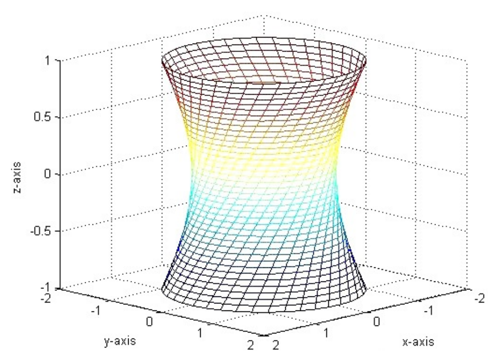

Figure 4: Hyperboloid surface

5.3 Tor Surface

Let the curve $\alpha(s) = (r\sin\theta, r\cos\theta, 0)$ be a circle with radius r on the xy-plane. Then the Frenet frame of the curve $\alpha(s)$ at the point p is

$$\vec{T} = (\cos\theta, -\sin\theta, 0) \quad , \quad \vec{N} = (-\sin\theta, -\cos\theta, 0) \quad , \quad \vec{B} = (0, 0, -1)$$

and for $\vec{p} = \lambda \vec{N}$, substituting these equations into equation (30), we obtain an equation of the tor surface as follows:

$$\varphi(s,t)(p) = (\sin\theta\,[r - h\lambda\cos t]\,, \cos\theta\,[r - h\lambda\cos t]\,, -h\lambda\sin t)\,. \tag{33}$$

Example 5.5 *In equation (33) if* $0 < s < 1$, $0 < t, \theta < 2\pi$ *and* $h(s,t) = s + \sin t \cos t$ *are given, then a homothetic tor surface is obtained as drawn in the Figure 5.*

Example 5.6 *In equation (33) if* $h(s,t) = 1$ *is taken, then a tor surface is obtained as drawn in Figure 6.*

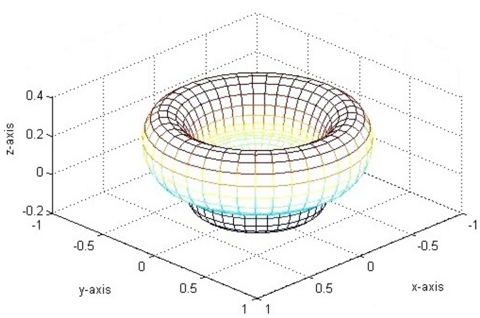

Figure 5: Homothetic tor surface

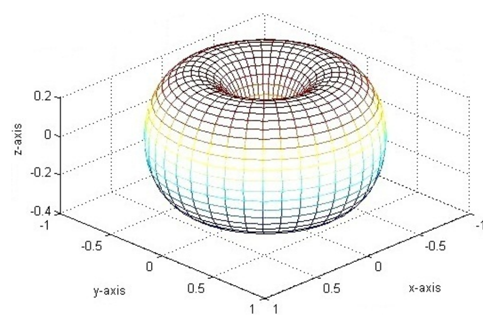

Figure 6: Tor surface

6 Conclusion

The results we have presented deal with Euclidean homothetic motions in which position of the moving object depends on two parameters. The hodographs of two-parameters Euclidean homothetic motions were obtained. A hodograph is the locus of the end points of the velocity of a particle and it is the solution of the first order equation which is Newton's Law. The locus of a the hodograph of Euclidean homothetic motion was found as an ellipse in this study.

Also this paper deals with trajectory surfaces (cylinder, hyperboloid and tor surfaces) generated by a point, the moving body, and figures of these surfaces were drawn by using MATLAB software.

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