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Suzuki Type n-Tupled Fixed Point Theorems in Ordered Metric Spaces

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Abstract

In this paper we prove a Suzuki type unique n-tupled common fixed point theorem in a partially ordered metric space.

Keywords: *Partial order, Metric space, n-tupled fixed point, W-compatible maps.*

1 Introduction and Preliminaries

Bhaskar and Lakshmikantham [13] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems in partially ordered complete metric spaces under certain conditions. Later Lakshmikantham and Ciric [17] extended these results by defining the mixed g-monotone property to generalize the corresponding fixed point theorems contained in [13]. After that, Berinde and Borcut [16] introduced the concept of tripled fixed point and proved some related theorems. In this continuation, Karapinar [4] introduced the quadruple fixed point and proved some results on the existence and uniqueness of quadruple fixed points.

Recently Imdad et al.[8] introduced the concept of n-tupled coincidence and n-tupled common fixed point theorems for nonlinear ϕ-contraction mappings. For more details see [9, 10].

In 2008, Suzuki [14, 15] introduced generalized versions of both Banach's and Edelstain's basic results. Many other works in this direction have been considered, for example refer [1, 2, 3, 5, 6, 12] and the references threin.

Combining the concepts of n-tupled fixed point theorems and Suzuki type theorems ,in this paper, we prove n-tupled coincidence and n-tupled common fixed point theorems of Suzuki-type in a partially ordered metric space.

Now we give some known definitions.

Let $(X, \preceq)$ be a partially ordered set and we denote $X \times X \times X \cdots \times X$(n times) by X^n. X^n is cquipped with the following partial ordering: for $x, y \in X^n$ where $x = (x^1, x^2, \cdots, x^n)$ and $y = (y^1, y^2, \cdots, y^n)$, $x \preceq y \Leftrightarrow x^i \preceq y^i$ if i is odd and $x^i \succeq y^i$ if i is even.

Definition 1.1 ([8]) *Let $(X, \preceq)$ be a partially ordred set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then the mapping F is said to have the mixed g-monotone property if F is g-non decreasing in its odd position arguments and g-non increasing in its even position arguments, that is, for all $x_1^i, x_2^i \in X$,*

$$gx_1^i \preceq gx_2^i \Rightarrow \begin{cases} F(x^1, x^2, \cdots, x_1^i, \cdots, x^n) \preceq F(x^1, x^2, \cdots, x_2^i, \cdots, x^n) & \text{if } i \text{ is odd,} \\ F(x^1, x^2, \cdots, x_1^i, \cdots, x^n) \succeq F(x^1, x^2, \cdots, x_2^i, \cdots, x^n) & \text{if } i \text{ is even} \end{cases}$$

Definition 1.2 ([8]) *An element $(x^1, x^2, \cdots x^n) \in X$ is called a n-tupled coincidence point of $F : X^n \to X$ and $g : X \to X$ if*

$$\begin{aligned} F(x^1, x^2, \cdots, x^n) &= gx^1, \\ F(x^2, x^3, \cdots, x^n) &= gx^2, \\ &\vdots \\ F(x^n, x^1, x^2, \cdots, x^{n-1}) &= gx^n. \end{aligned}$$

Definition 1.3 ([8]) *An element $(x^1, x^2, \cdots x^n) \in X$ is called a n-tupled common fixed point of $F : X^n \to X$ and $g : X \to X$ if*

$$\begin{aligned} F(x^1, x^2, \cdots, x^n) &= gx^1 = x^1, \\ F(x^2, x^3, \cdots, x^n) &= gx^2 = x^2, \\ &. \\ &. \\ &. \\ F(x^n, x^1, x^2, \cdots, x^{n-1}) &= gx^n = x^n. \end{aligned}$$

Definition 1.4 ([7]) *The mappings $F : X \times X \to X$ and $f : X \to X$ are called W-compatible if $f(F(x,y)) = F(fx, fy)$ and $f(F(y,x)) = F(fy, fx)$ whenever $fx = F(x,y)$ and $fy = F(y,x)$.*

Lemma 1.5 ([11]) *Let X be a non-empty set and $g : X \to X$ be a mapping. Then there exists a subset E of X such that $g(E) = g(X)$ and the mapping $g : E \to X$ is one-one.*

Now we prove our main results.

2 Main Results

Theorem 2.1 . *Let $(X, \preceq, d)$ be a partially ordered metric space and $F : X^n \to X$ and $f : X \to X$ be two mappings such that F has the mixed g-monotone property on X and satisfying the following :*
(2.1.1) *$F(X^n) \subseteq g(X)$ and $g(X)$ is complete,*
(2.1.2) *If there exists a constant $\theta \in [0,1)$ such that*

$$\eta(\theta) \min \left\{ \begin{array}{c} d\left(gx^1, F(x^1, x^2, \cdots, x^n)\right), \\ d\left(gx^2, F(x^2, x^3, \cdots, x^n, x^1)\right), \\ \vdots \\ d\left(gx^n, F(x^n, x^1, \cdots, x^{n-1})\right) \end{array} \right\} \leq \max \left\{ \begin{array}{c} d(gx^1, gy^1), \\ d(gx^2, gy^2), \\ \vdots \\ d(gx^n, gy^n) \end{array} \right\}$$

implies

$$d(F(x^1, x^2, \cdots, x^n), F(y^1, y^2, \cdots, y^n))$$

$$\leq \ \theta \ \max \left\{ \begin{array}{c} d(gx^1, gy^1), d(gx^2, gy^2), \cdots, d(gx^n, gy^n), \\ d\left(gx^1, F(x^1, x^2, \cdots, x^n)\right), \cdots, d\left(gx^n, F(x^n, x^1, \cdots, x^{n-1})\right), \\ d\left(gy^1, F(x^1, x^2, \cdots, x^n)\right), \cdots, d\left(gy^n, F(x^n, x^1, \cdots, x^{n-1})\right) \end{array} \right\}$$

for all $x^1, x^2, \cdots, x^n, y^1, y^2, \cdots, y^n \in X$ for which gx^i and gy^i $(i = 1, 2, \cdots, n)$ are comparable, where $\eta : [0,1) \to (\frac{1}{2}, 1]$ defined by $\eta(\theta) = \frac{1}{1+\theta}$ is a strictly

decreasing function,
(2.1.3) *There exist elements* $x_0^1, x_0^2 \cdots, x_0^n \in X$ *such that*

$$gx_0^i \preceq F(x_0^i, x_0^{i+1}, \cdots, x_0^n, x_0^1, x_0^2, \cdots, x_0^{i-1}) \text{ if } i \text{ is odd and}$$
$$gx_0^i \succeq F(x_0^i, x_0^{i+1}, \cdots, x_0^n, x_0^1, x_0^2, \cdots, x_0^{i-1}) \text{ if } i \text{ is even.}$$

(2.1.4) *(a) Suppose* F *and* g *are continuous*

or

(b) X *has the following properties :*
(i) If a non-decreasing sequence $\{x_m\} \to x$*, then* $x_m \preceq x$*, for all* m*,*
(ii) If a non-increasing sequence $\{y_m\} \to y$*, then* $y \preceq y_m$*, for all* m*.*
Then F *and* g *have a n-tupled coincidence point in* X*.*

Proof. Let $x_0^1, x_0^2 \cdots, x_0^n \in X$ be satisfying (2.1.3).
In view of (2.1.1), we construct sequences $\{x_m^1\}, \{x_m^2\}, \cdots, \{x_m^n\}$ in X as follows:

$$\begin{array}{c} gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, \cdots, x_{m-1}^n), \\ gx_m^2 = F(x_{m-1}^2, x_{m-1}^3, \cdots, x_{m-1}^n, x_{m-1}^1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ gx_m^n = F(x_{m-1}^n, x_{m-1}^1, \cdots, x_{m-1}^{n-1},) \end{array} \tag{1}$$

for all $m \geq 1$.
We claim for all $m \geq 0$, that

$$gx_m^i \preceq gx_{m+1}^i \text{ if i is odd and } gx_m^i \succeq gx_{m+1}^i \text{ if i is even} \tag{2}$$

Relations (2.1.3) and (1) implies that (2) holds for $m = 0$.
Suppose (2) holds for $m = k > 0$.
For odd i, consider x_{k+1}^i and using mixed g-monotone property of F, we get

$$\begin{aligned} gx_{k+1}^i &= F(x_k^i, x_k^{i+1}, \cdots, x_k^n, x_k^1, \cdots, x_k^{i-1}) \\ &\preceq F(x_{k+1}^i, x_k^{i+1}, \cdots, x_k^n, x_k^1, \cdots, x_k^{i-1}) \\ &\preceq F(x_{k+1}^i, x_{k+1}^{i+1}, \cdots, x_k^n, x_k^1, \cdots, x_k^{i-1}) \\ &\vdots \\ &\preceq F(x_{k+1}^i, x_{k+1}^{i+1}, \cdots, x_{k+1}^n, x_{k+1}^1, \cdots, x_{k+1}^{i-1}) \\ &= gx_{k+2}^i \end{aligned}$$

For even i, consider

$$\begin{aligned} gx_{k+2}^i &= F(x_{k+1}^i, x_{k+1}^{i+1}, \cdots, x_{k+1}^n, x_{k+1}^1, \cdots, x_{k+1}^{i-1}) \\ &\preceq F(x_k^i, x_{k+1}^{i+1}, \cdots, x_{k+1}^n, x_{k+1}^1, \cdots, x_{k+1}^{i-1}) \\ &\preceq F(x_k^i, x_k^{i+1}, \cdots, x_{k+1}^n, x_{k+1}^1, \cdots, x_{k+1}^{i-1}) \\ &\vdots \\ &\preceq F(x_k^i, x_k^{i+1}, \cdots, x_k^n, x_k^1, \cdots, x_k^{i-1}) \\ &= gx_{k+1}^i. \end{aligned}$$

Hence by mathematical induction, (2) holds for all $m \geq 0$.
Suppose $gx^1_{m+1} = gx^1_m$, $gx^2_{m+1} = gx^2_m$, $\cdots$, $gx^n_{m+1} = gx^n_m$ for some m.
Then $(x^1_m, x^2_m, \cdots, x^n_m)$ is a n-tupled coincidence point of F and g.
Assume that $gx^1_{m+1} \neq gx^1_m$ or $gx^2_{m+1} \neq gx^2_m$, or $\cdots$ or $gx^n_{m+1} \neq gx^n_m$ for all m.
Since

$$\eta(\theta)\min\left\{\begin{array}{c} d(gx^1_0, F(x^1_0, x^2_0, \cdots, x^n_0), \\ \vdots \\ d(gx^n_0, F(x^n_0, x^1_0, \cdots, x^{n-1}_0), \end{array}\right\} \leq \min\left\{\begin{array}{c} d(gx^1_0, gx^1_1), \cdots, \\ d(gx^n_0, gx^n_1) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(gx^1_0, gx^1_1), \cdots, \\ d(gx^n_0, gx^n_1) \end{array}\right\},$$

by (2.1.2) we have

$$\begin{aligned} d(gx^1_1, gx^1_2) &= d(F(x^1_0, x^2_0, \cdots, x^n_0), F(x^1_1, x^2_1, \cdots, x^n_1)) \\ &\leq \theta \ \max\left\{\begin{array}{c} d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \\ d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \\ d(gx^1_1, gx^1_1), \cdots, d(gx^n_1, gx^n_1) \end{array}\right\} \\ &= \theta \ \max\left\{ d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \right\}. \end{aligned}$$

Similarly

$$\begin{aligned} d(gx^2_1, gx^2_2) &\leq \theta \ \max\left\{ d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \right\}. \\ &\vdots \\ d(gx^n_1, gx^n_2) &\leq \theta \ \max\left\{ d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \right\}. \end{aligned}$$

Thus

$$\max\left\{ d(gx^1_1, gx^1_2), \cdots, d(gx^n_1, gx^n_2) \right\} \leq \theta \ \max\left\{ d(gx^1_0, gx^1_1), \cdots, d(gx^n_0, gx^n_1) \right\}.$$

Continuing in this way, we obtain

$$\begin{aligned} \max\left\{\begin{array}{c} d(gx^1_m, gx^1_{m+1}), \cdots, \\ d(gx^n_m, gx^n_{m+1}) \end{array}\right\} &\leq \theta \max\left\{\begin{array}{c} d(gx^1_{m-1}, gx^1_m), \cdots, \\ d(gx^n_{m-1}, gx^n_m) \end{array}\right\} \\ &\leq \theta^2 \max\left\{\begin{array}{c} d(gx^1_{m-2}, gx^1_{m-1}), \cdots, \\ d(gx^n_{m-2}, gx^n_{m-1}) \end{array}\right\} \\ &\vdots \\ &\leq \theta^m \max\left\{\begin{array}{c} d(gx^1_0, gx^1_1), \cdots, \\ d(gx^n_0, gx^n_1) \end{array}\right\}. \end{aligned} \tag{3}$$

For $m > l$, consider

$$\begin{aligned} d(gx_l^1, gx_m^1) &\leq d(gx_l^1, gx_{l+1}^1) + d(gx_{l+1}^1, gx_{l+2}^1) + \cdots + d(gx_{m-1}^1, gx_m^1) \\ &\leq (\theta^l + \theta^{l+1} + ... + \theta^{m-1}) \max \left\{ \begin{array}{c} d(gx_0^1, gx_1^1), \cdots, \\ d(gx_0^n, gx_1^n) \end{array} \right\} \quad from\ (3) \\ &\leq \tfrac{\theta^l}{1-\theta} \max \left\{ \begin{array}{c} d(gx_0^1, gx_1^1), \cdots, \\ d(gx_0^n, gx_1^n) \end{array} \right\} \\ &\to 0 \quad \text{as} \quad l \to \infty. \end{aligned}$$

Hence $\{gx_m^1\}$ is a Cauchy sequence in $g(X)$. Similarly we can show that $\{gx_m^2\}$, $\cdots$, $\{gx_m^n\}$ are Cauchy sequences in $g(X)$.
Since $g(X)$ is complete, there exist $p^1, p^2, \cdots, p^n, z^1, z^2, \cdots, z^n \in X$ such that

$$gx_m^1 \to p^1 = gz^1, gx_m^2 \to p^2 = gz^2, \cdots, gx_m^n \to p^n = gz^n. \tag{4}$$

Suppose (2.1.4)(a) holds, i.e F and g are continuous.
From Lemma 1.5, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and the mapping $g : E \to X$ is one - one. Let us define $G : [g(E)]^n \to X$ by $G(gx^1, gx^2, \cdots gx^n) = F(x^1, x^2, \cdots x^n)$ for all $gx^1, gx^2, \cdots gx^n \in g(E)$.
Since F and g are continuous, it follows that G is continuous.
Now, we have

$$\begin{aligned} F(z^1, z^2, \cdots, z^n) &= G(gz^1, gz^2, \cdots, gz^n) \\ &= \lim_{n\to\infty} G(gx_m^1, gx_m^2, \cdots, gx_m^n) \\ &= \lim_{n\to\infty} F(x_m^1, x_m^2, \cdots, x_m^n) \\ &= \lim_{n\to\infty} gx_{m+1}^1 = gz^1. \end{aligned}$$

Similarly we have

$$gz^2 = F(z^2, \cdots, z^n, z^1), \cdots, gz^n = F(z^n, z^1, \cdots, z^{n-1}).$$

Thus $(z^1, z^2, \cdots z^n)$ is a n-tupled coincidence point of F and g.
Suppose (2.1.4)(b) holds.
Since $gx_{m+1}^1 \neq gx_m^1$ or $gx_{m+1}^2 \neq gx_m^2$ or $\cdots$ or $gx_{m+1}^n \neq gx_m^n$ for all m and $gx_m^1 \to gz^1$, $gx_m^2 \to gz^2$, $\cdots$, $gx_m^n \to gz^n$ it follows that
$\max\{d(gx_m^1, gz^1), d(gx_m^2, gz^2), \cdots, d(gx_m^n, gz^n)\} > 0$ for infinitely many m.

$$Claim : \max \left\{ \begin{array}{c} d(gz^1, F(x^1, x^2, \cdots, x^n)), \cdots, \\ d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} d(gz^1, gx^1), \cdots, \\ d(gz^n, gx^n) \end{array} \right\}$$

for all $x^1, x^2, \cdots, x^n \in X$ with $gz^i \preceq gx^i$ for i is odd and $gz^i \succeq gx^i$ for i is even and $\max\{d(gz^1, gx^1), \cdots, d(gz^n, gx^n)\} > 0$.
Let $x^1, x^2, \cdots, x^n \in X$ with $gz^i \preceq gx^i$ for i is odd and $gz^i \succeq gx^i$ for i is even and $\max\{d(gz^1, gx^1), \cdots, d(gz^n, gx^n)\} > 0$.

Since $gx^i_m \to gz^i$, for $i = 1, 2, \cdots, n$ there exists a positive integer m_0 such that for $m \geq m_0$ we have

$$\max\left\{ \begin{array}{c} d(gx^1_m, gz^1), \cdots, \\ d(gx^n_m, gz^n) \end{array} \right\} \leq \frac{1}{6} \max\left\{ \begin{array}{c} d(gz^1, gx^1), \cdots, \\ d(gz^n, gx^n) \end{array} \right\} \tag{5}$$

Now for $m \geq m_0$, consider

$$\eta(\theta) \min\left\{ \begin{array}{c} d(gx^1_m, F(x^1_m, x^2_m, \cdots, x^n_m)), \\ \vdots \\ d(gx^n_m, F(x^n_m, x^1_m, \cdots, x^{n-1}_m)) \end{array} \right\} \leq \max\left\{ \begin{array}{c} d(gx^1_m, gx^1_{m+1}), \\ \vdots \\ d(gx^n_m, gx^n_{m+1}) \end{array} \right\}$$

$$\leq \max\left\{ \begin{array}{c} d(gx^1_m, gz^1) + d(gz^1, gx^1_{m+1}), \cdots, \\ d(gx^n_m, gz^n) + d(gz^n, gx^n_{m+1}) \end{array} \right\}$$

$$\leq \max\left\{ d(gx^1_m, gz^1) + \cdots + d(gx^n_m, gz^n) \right\} + \max\left\{ d(gz^1, gx^1_{m+1}) + \cdots + d(gz^n, gx^n_{m+1}) \right\}$$

$$\leq \tfrac{2}{6} \max\left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\} \; from (5)$$

$$= \tfrac{2}{5} \left[\max\left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\} - \tfrac{1}{6} \max\left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\}\right]$$

$$\leq \tfrac{2}{5} \left[\max\left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\} - \max\left\{ d(gx^1_m, gz^1), \cdots, d(gx^n_m, gz^n) \right\}\right] \; from (5)$$

$$\leq \max\left\{ \begin{array}{c} d(gz^1, gx^1) - d(gx^1_m, gz^1), \cdots, \\ d(gz^n, gx^n) - d(gx^n_m, gz^n) \end{array} \right\}$$

$$\leq \max\left\{ d(gx^1_m, gx^1), \cdots, d(gx^n_m, gx^n) \right\}.$$

From (2), (4) and (2.1.4)(b), we have $gx^i_m \preceq gz^i$ if i is odd and $gz^i \preceq gx^i_m$ if i is even for all m. Hence for all m,we have

$$gx^i_m \preceq gz^i \preceq gx^i \quad \text{for i is odd and} \quad gx^i \preceq gz^i \preceq gx^i_m \quad \text{for i is even.} \tag{6}$$

Hence by (2.1.2), we get

$$d(F(x^1_m, x^2_m, \cdots, x^n_m), F(x^1, x^2, \cdots, x^n))$$

$$\leq \; \theta \; \max\left\{ \begin{array}{c} d(gx^1_m, gx^1), \cdots, d(gx^n_m, gx^n), \\ d(gx^1_m, gx^1_{m+1}), \cdots, d(gx^n_m, gx^n_{m+1}), \\ d(gx^1, gx^1_{m+1}), \cdots, d(gx^n, gx^n_{m+1}) \end{array} \right\}.$$

Letting $m \to \infty$, we get

$$d(gz^1, F(x^1, x^2, \cdots, x^n)) \leq \theta \max\left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\}.$$

Analogously we can prove that

$$\begin{aligned} d(gz^2, F(x^2, x^3, \cdots, x^n, x^1)) &\leq \theta \max\left\{\ d(gz^1, gx^1), \cdots, d(gz^n, gx^n)\ \right\}. \\ &\vdots \\ d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) &\leq \theta \max\left\{\ d(gz^1, gx^1), \cdots, d(gz^n, gx^n)\ \right\}. \end{aligned}$$

Thus

$$\max\left\{\begin{array}{c} d(gz^1, F(x^1, x^2, \cdots, x^n)), \\ \cdots, \\ d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} d(gz^1, gx^1), \\ \cdots, \\ d(gz^n, gx^n) \end{array}\right\} \tag{7}$$

Hence the claim.
Now consider

$$\begin{aligned} d(gx^1, F(x^1, x^2, \cdots, x^n)) &\leq d(gx^1, gz^1) + d(gz^1, F(x^1, x^2, \cdots, x^n)) \\ &\leq d(gx^1, gz^1) + \theta \max\left\{\begin{array}{c} d(gz^1, gx^1), \\ \cdots, \\ d(gz^n, gx^n) \end{array}\right\} \ from \ \ (7) \\ &\leq (1+\theta) \max\left\{\begin{array}{c} d(gx^i, gz^i), \\ \cdots, \\ d(gz^n, gx^n) \end{array}\right\} \end{aligned}$$

Thus

$$\eta(\theta) d(gx^1, F(x^1, x^2, \cdots, x^n)) \leq \max\left\{\begin{array}{c} d(gx^1, gz^1), \\ \cdots, \\ d(gz^n, gx^n) \end{array}\right\}.$$

Hence

$$\eta(\theta) \min\left\{\begin{array}{c} d(gx^1, F(x^1, x^2, \cdots, x^n)), \cdots, \\ d(gx^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array}\right\} \leq \max\left\{\begin{array}{c} d(gz^1, gx^1), \\ \cdots, \\ d(gz^n, gx^n) \end{array}\right\}.$$

Now from (2.1.2), we have

$$\begin{aligned} &d(F(x^1, x^2, \cdots, x^n), F(z^1, z^2, \cdots, z^n)) \\ &\leq \theta \max\left\{\begin{array}{c} d(gx^1, gz^1), \cdots, d(gx^n, gz^n), \\ d(gx^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gx^n, F(x^n, x^1, \cdots, x^{n-1})), \\ d(gz^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array}\right\} \end{aligned} \tag{8}$$

Now from(8), we obtain

$$\begin{aligned} &d(F(x^1_m, x^2_m, \cdots, x^n_m), F(z^1, z^2, \cdots, z^n)) \\ &\leq \theta \max\left\{\begin{array}{c} d(gx^1_m, gz^1), \cdots, d(gx^n_m, gz^n), \\ d(gx^1_m, gx^1_{m+1}), \cdots, d(gx^n_m, gx^n_{m+1}), \\ d(gz^1, gx^1_{m+1}), \cdots, d(gz^n, gx^n_{m+1}) \end{array}\right\}. \end{aligned}$$

Letting $m \to \infty$, we get
$d(gz^1, F(z^1, z^2, \cdots, z^n)) \leq 0$ so that $gz^1 = F(z^1, z^2, \cdots, z^n)$.
Analogously, we can show that $gz^2 = F(z^2, z^3, \cdots, z^n, z^1), \cdots$,
$gz^n = F(z^n, z^1, \cdots, z^{n-1})$.
Thus $(z^1, z^2, \cdots, z^n)$ is a n-tupled coincidence point of F and g.

Theorem 2.2 *In addition to the hypotheses of Theorem 2.1, suppose that for any* $(x^1, x^2, \cdots, x^n)$, $(y^1, y^2, \cdots, y^n) \in X^n$, *there exists* $(u^1, u^2, \cdots, u^n) \in X^n$ *such that* $(F(u^1, u^2, \cdots, u^n), F(u^2, u^3, \cdots, u^n, u^1), \cdots, F(u^n, u^1, \cdots, u^{n-1}))$ *is comparable with* $(F(x^1, x^2, \cdots, x^n), F(x^2, x^3, \cdots, x^n, x^1), \cdots, F(x^n, x^1, \cdots, x^{n-1}))$ *and*
$(F(y^1, y^2, \cdots, y^n), F(y^2, y^3, \cdots, y^n, y^1), \cdots, F(y^n, y^1, \cdots, y^{n-1}))$. *Further more assume that* F *and* g *are* W*-compatible, then* F *and* g *have a unique* n*-tupled common fixed point.*

Proof. From Theorem 2.1, the set of n-tupled coincidence points of F and g is non-empty.
Let $(x^1, x^2, \cdots, x^n)$ and $(y^1, y^2, \cdots, y^n)$ be two n-tupled coincidence points of F and g, That is

$$\begin{aligned} F(x^1, x^2, \cdots, x^n) &= gx^1, F(y^1, y^2, \cdots, y^n) = gy^1, \\ F(x^2, x^3, \cdots, x^n, x^1) &= gx^2, F(y^2, y^3, \cdots, y^n, y^1) = gy^2, \\ &\vdots \\ F(x^n, x^1, \cdots, x^{n-1})) &= gx^n, F(y^n, y^1, \cdots, y^{n-1})) = gy^n. \end{aligned}$$

Now we shall show that

$$gx^1 = gy^1, gx^2 = gy^2, \cdots, gx^n = gy^n. \tag{9}$$

By the assumption, there exists $(u^1, u^2, \cdots, u^n) \in X \times X$ such that
$(F(u^1, u^2, \cdots, u^n), F(u^2, u^3, \cdots, u^n, u^1), \cdots, F(u^n, u^1, \cdots, u^{n-1}))$ is comparable with $(F(x^1, x^2, \cdots, x^n), F(x^2, x^3, \cdots, x^n, x^1), \cdots, F(x^n, x^1, \cdots, x^{n-1}))$ and
$(F(y^1, y^2, \cdots, y^n), F(y^2, y^3, , \cdots, y^n, y^1), \cdots, F(y^n, y^1, \cdots, y^{n-1}))$.
Put $u_0^1 = u^1, u_0^2 = u^2, \cdots, u_0^n = u^n$ and choose $u_1^1, u_1^2, \cdots, u_1^n \in X$ such that

$$\begin{aligned} gu_1^1 &= F(u_0^1, u_0^2, \cdots, u_0^n) \\ gu_1^2 &= F(u_0^2, u_0^3, \cdots, u_0^n, u_0^1) \\ &\vdots \\ gu_1^n &= F(u_0^n, u_0^1, \cdots, u_0^{n-1}) \end{aligned}$$

As in in the proof of Theorem 2.1, we can define the sequences $\{u_m^1\}$, $\{u_m^2\}$, $\cdots$, $\{u_m^n\}$ such that

$$\begin{aligned} gu_m^1 &= F(u_{m-1}^1, u_{m-1}^2, \cdots, u_{m-1}^n) \\ gu_m^2 &= F(u_{m-1}^2, u_{m-1}^3, \cdots, u_{m-1}^n, u_{m-1}^1) \\ &\vdots \\ gu_m^n &= F(u_{m-1}^n, u_{m-1}^1, \cdots, u_{m-1}^{n-1}) \quad \text{for} \quad m \geq 1. \end{aligned}$$

Further, set $x_0^1 = x^1, x_0^2 = x^2, \cdots, x_0^n = x^n$ and $y_0^1 = y^1, y_0^2 = y^2, \cdots,$ $y_0^n = y^n$ in the same way, we define the sequences $\{gx_m^1\}$,$\{gx_m^2\}$, $\cdots$, $\{gx_m^n\}$ and $\{gy_m^1\}$,$\{gy_m^2\}$, $\cdots$, $\{gy_m^n\}$ by

$$\begin{aligned} gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, \cdots, x_{m-1}^n), gy_m^1 = F(y_{m-1}^1, y_{m-1}^2, \cdots, y_{m-1}^n), \\ gx_m^2 = F(x_{m-1}^2, x_{m-1}^3, \cdots, x_{m-1}^n, x_{m-1}^1), gy_m^2 = F(y_{m-1}^2, y_{m-1}^3, \cdots, y_{m-1}^n, y_{m-1}^1), \\ \vdots \\ gx_m^n = F(x_{m-1}^n, x_{m-1}^1, \cdots, x_{m-1}^{n-1})), gy_m^n = F(y_{m-1}^n, y_{m-1}^1, \cdots, y_{m-1}^{n-1})). \end{aligned}$$

Without loss of generality assume that
$(F(x^1, x^2, \cdots, x^n), F(x^2, x^3, \cdots, x^n, x^1), \cdots, F(x^n, x^1, \cdots, x^{n-1})) \preceq$
$(F(u^1, u^2, \cdots, u^n), F(u^2, u^3, \cdots, u^n, u^1), \cdots, F(u^n, u^1, \cdots, u^{n-1}))$ and
$(F(y^1, y^2, \cdots, y^n), F(y^2, y^3, \cdots, y^n, y^1), \cdots, F(y^n, y^1, \cdots, y^{n-1})) \preceq$
$(F(u^1, u^2, \cdots, u^n), F(u^2, u^3, \cdots, u^n, u^1), \cdots, F(u^n, u^1, \cdots, u^{n-1}))$.
Then we have $gx^i \preceq gu_1^i$ for i is odd and $gx^i \succeq gu_1^i$ for i is even.
As in Theorem 2.1, we have $gu_m^i \preceq gu_{m+1}^i$ for i is odd and $gu_m^i \succeq gu_{m+1}^i$ for i is even for all m.
Hence $gx^i \preceq gu_m^i$ for i is odd and $gx^i \succeq gu_m^i$ for i is even for all m.
Since

$$\eta(\theta) \min \left\{ \begin{array}{c} d(gx^1, F(x^1, x^2, \cdots, x^n)), \\ \vdots \\ d(gx^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array} \right\} = 0 \leq \max \left\{ \begin{array}{c} d(gx^1, gu_m^1), \\ \vdots \\ d(gx^n, gu_m^n) \end{array} \right\}.$$

We have by (2.1.2) that

$$d(F(x^1, x^2, \cdots, x^n), F(u_m^1, u_m^2, \cdots, u_m^n))$$

$$\leq \theta \max \left\{ \begin{array}{c} d(gx^1, gu_m^1), \cdots, d(gx^n, gu_m^n), \\ d(gx^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gx^n, F(x^n, x^1, \cdots, x^{n-1})) \\ d(gu_m^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gu_m^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array} \right\}$$

which implies that

$$\begin{aligned} d(gx^1, gu_{m+1}^1) & \leq \theta \max \left\{ \begin{array}{c} d(gx^1, gu_m^1), \cdots, d(gx^n, gu_m^n), \\ 0, \cdots, 0 \\ d(gu_m^1, gx^1), \cdots, d(gu_m^n, gx^n) \end{array} \right\} \\ & = \theta \max \{ d(gx^1, gu_m^1), \cdots, d(gx^n, gu_m^n) \}. \end{aligned} \quad (10)$$

Similarly, for $i = 2, 3, \cdots, n$ we can we show that

$$d(gx^i, gu_{m+1}^i) \leq \theta \max \left\{ d(gx^1, gu_m^1), \cdots, d(gx^n, gu_m^n) \right\}.$$

Thus

$$\max\left\{\begin{array}{c} d(gx^1, gu^1_{m+1}), \cdots, \\ d(gx^n, gu^n_{m+1}) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} d(gx^1, gu^1_m), \cdots, \\ d(gx^n, gu^n_m) \end{array}\right\}. \tag{11}$$

Let $r_m = \max\{d(gx^1, gu^1_m), \cdots, d(gx^n, gu^n_m)\}$.
Then from (11), we have $r_{m+1} \leq \theta r_m$.
Hence $r_{m+1} \leq \theta r_m \leq \theta^2 r_{m-1} \leq ... \leq \theta^m r_0 \to 0$ as $m \to \infty$.
Hence

$$\lim_{m\to\infty} d(gx^i, gu^i_m) = 0 \text{ for } i = 1, 2, \cdots, n. \tag{12}$$

Similarly, we can show that

$$\lim_{m\to\infty} d(gy^i, gu^i_m) = 0 \text{ for } i = 1, 2, \cdots, n. \tag{13}$$

Hence $gx^i = gy^i$ for $i = 1, 2, \cdots, n$.
Thus (9) is proved.
Since $gx^1 = F(x^1, x^2, \cdots, x^n)$,$gx^2 = F(x^2, x^3, \cdots, x^n, x^1)$, $\cdots$,
$gx^n = F(x^n, x^1, \cdots, x^{n-1})$, by W-compatibility of F and g, we have

$$\begin{aligned} g(gx^1) = g(F(x^1, x^2, \cdots, x^n)) &= F(gx^1, gx^2, \cdots, gx^n), \\ g(gx^2) = g(F(x^2, x^3, \cdots, x^n, x^1)) &= F(gx^2, gx^3, \cdots, gx^n, gx^1), \\ &\vdots \\ g(gx^n) = g(F(x^n, x^1, \cdots, x^{n-1})) &= F(gx^n, gx^1, \cdots, gx^{n-1}), \end{aligned}$$

Denote $gx^1 = z^1$, $gx^2 = z^2$, $\cdots$,, $gx^n = z^n$ Then

$$\begin{aligned} gz^1 &= F(z^1, z^2, \cdots, z^n), \\ gz^2 &= F(z^2, z^3, \cdots, z^n, z^1), \\ &\vdots \\ gz^n &= F(z^n, z^1, \cdots, z^{n-1}), \end{aligned} \tag{14}$$

Thus $(z^1, z^2, \cdots, z^n)$ is a n-tupled coincidence point of F and g. Then from (9), we have $gx^1 = gz^1, gx^2 = gz^2, \cdots, gx^n = gz^n$
so that

$$z^1 = gz^1, z^2 = gz^2, \cdots, z^n = gz^n. \tag{15}$$

Now by (14) and (15), we conclude that $(z^1, z^2, \cdots, z^n)$ is a n-tupled common fixed point of F and g.
To prove the uniqueness of n-tupled common fixed point of F and g, assume that $(s^1, s^2, \cdots, s^n)$ is another n-tupled common fixed point of F and g.
Then from (9), we have $gz^1 = gs^1, gz^2 = gs^2, \cdots, gz^n = gs^n$ which yields that $z^1 = s^1, z^2 = s^2, \cdots, z^n = s^n$.
Hence $(z^1, z^2, \cdots, z^n)$ is the unique n-tupled common fixed point of F and g.

Now we illustrate Theorem 2.2 with an example when $n = 4$.

Example 2.3 *Let $X = \mathcal{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let us define $\preceq$ by ordering $\leq$.*
Define $F : X^4 \to X$ and $g : X \to X$ by

$$F(x^1, x^2, x^3, x^4) = \frac{x^1 - 2x^2 + 3x^3 - 4x^4}{64}, \quad gx = \frac{x}{4}.$$

Then for (x^1, x^2, x^3, x^4), (y^1, y^2, y^3, y^4) in X^4, we have

$$\begin{aligned} d(F(x^1, x^2, x^3, x^4), F(y^1, y^2, y^3, y^4)) &= \left|\tfrac{x^1-2x^2+3x^3-4x^4}{64} - \tfrac{y^1-2y^2+3y^3-4y^4}{64}\right| \\ &\leq \tfrac{1}{16}\left[\begin{array}{c} \left|\frac{x^1}{4} - \frac{y^1}{4}\right| + 2\left|\frac{x^2}{4} - \frac{y^2}{4}\right| + \\ 3\left|\frac{x^3}{4} - \frac{y^3}{4}\right| + 4\left|\frac{x^4}{4} - \frac{y^4}{4}\right| \end{array}\right] \\ &= \tfrac{1}{16}\left[\begin{array}{c} d(gx^1, gy^1) + 2d(gx^2, gy^2) + \\ 3d(gx^3, gy^3) + 4d(gx^4, gy^4) \end{array}\right] \\ &\leq \tfrac{5}{8}\max\left\{\begin{array}{c} d(gx^1, gy^1), d(gx^2, gy^2), \\ d(gx^3, gy^3), d(gx^4, gy^4) \end{array}\right\} \end{aligned}$$

.

Thus (2.1.2)is satisfied with $\theta = \frac{5}{8}$ and $\eta(\theta) = \frac{8}{13}$. Clearly F and g are W-compatible.One can easily verify the remaining conditions of Theorem 2.2. Clearly $(0, 0, 0, 0)$ is a n-tupled unique common fixed point of F and g.

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