Gen. Math. Notes, Vol. 20, No. 1, January 2014, pp.27-33
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Note on the Stability of System of Differential Equations $\dot{x} = f(t, x(t))$

Qusuay H. Alqifiary[1,2] and J. Knezević-Miljanović[2]

[1]University of Al-Qadisiyah
Aldiwaniyah, Iraq
E-mail: qhaq2010@gmail.com
[2]University of Belgrade, Belgrade, Serbia
E-mail: knezevic@matf.bg.ac.rs

(Received: 26-10-13 / Accepted: 9-12-13)

Abstract

In this paper, we examine the relation between practical stability and Hyers-Ulam-stability and Hyers-Ulam-Rassias stability as well. In addition,by practical stability we gave a sufficient condition in order that the first order nonlinear systems of differential equations has local generalized Hyers-Ulam stability and local generalized Hyers-Ulam-Rassias stability.

Keywords: *Practical stability;Local Hyers-Ulam stability;Local Hyers-Ulam-Rassias stability;System of differential equations;Lyapunov function*

1 Introduction

There are many definitions provided for stability of differential equations, including Lyaopunov stabiity, practical stability, Hyers-Ulam stability and Hyers-Ulam-Rassias stability. In 1961, the notion of practical stability was discussed in the monograph by Lasalle and Lefschetz [7]. In which they point out that stability investigations may not assure practical stability and vice versa. For example an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Motivated by this, Weiss and Infante introduced the concept of finite time stability[11].

In 1940, Hyers-Ulam stability was introduced by S.M.Ulam [14] to answer the question:suppose one has a function $y(t)$ which is close to solve an equation.

Is there an exact solution $x(t)$ of the equation which is close to $y(t)$? [5],[12]. In 1941, D.H.Hyers [5] gave an affirmative answer to the equation of Ulam for additive Cauchy equation in Banach spaces. A generalized solution to Ulam's problem for approximately linear mappings was proved by Th.M.Rassias in 1978.Th.M.Rassias [18] considered a mapping $f : E_1 \to E_2$ such that $t \to f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) for\ any\ x, y \in E_1.$$

Definitions both of Hyers-Ulam stability and Hyers-Ulam-Rassias stability have applicable significance since it means that if one is studying an Hyers-Ulam stable or Hyers-Ulam-Rassias stable system then one does not have to reach the exact solution.(which usually is quite difficult or time consuming).This is quite useful in many applications e.g. numerical analysis, optimization, biology and economics etc., where finding the exact solution is quite difficult.

The basic statements of data dependence in the theory of ordinary differential equations are the following (see for example [1],[3],[4],[6],[19]):monotony with respect to data, continuity with respect to data, differentiability with respect to parameters, Lyapunov stability, structural stability. There are some results for some differential and integral equations ([15],[8],[9],[10],[17]).

There are many studies about the relation between types of stability, Lyapunov stability and practical stability (see [7],[15],[2]). With these results in mind we shall present, in this paper, the relation between practical stability and local generalized Hyers-Ulam stability and also local generalized Hyers-Ulam-Rassias stability. Because of practical stability of system with initial condition we study this kind of system. In addition, will give a sufficient condition in order that the first order nonlinear system of differential equations has local generalized Hyers-Ulam stability and local generalized Hyers-Ulam-Rassias stability.

2 Preliminaries

Let $(\mathbb{B}, \|.\|)$ be a Banach space (real or complex), and let $J = [t_0, t_0 + T)$ for some $T > 0$, $t_0 \geq 0$. We consider two systems : a system

$$\dot{x} = f(t, x)\ , \forall t \in J, \tag{1}$$

where f is defined and continuous on $J \times \mathbb{B}$. The equilibrium state is at the origin : $f(t, 0) = 0, \forall t \in J$.

A system that depends on parameter $\epsilon \in (0, \epsilon_0]$,$(\epsilon_0 \in (0, \infty))$ which is said to be perturbed system

$$\dot{x} = f(t, x) + p(t, x). \tag{2}$$

Let P be the set of all perturbations p satisfying $\|p(t,x)\| = \epsilon \leq \epsilon_0$ for all $t \in J$ and all x,let Q be a closed and bounded set of $\mathbb{B}$ containing the origin and let Q_0 be a subset of Q .

Definition 1 *Practical stability[7]*
Let $x^*(t, x_0, t_0)$ be the solution of (2) satisfying $x^*(t_0, x_0, t_0) = x_0$. If for each $p \in P$,i.e. $\epsilon \in (0, \epsilon_0]$, x_0 in Q_0 and each $t_0 \geq 0$, $x^*(t, x_0, t_0)$ in Q for all $t \in J$, then the origin is said to be (Q_0, Q, ϵ_0)-practically stable .
The slutions which start initially in Q_0 remain thereafter in Q .

Definition 2 *Local generalized Hyers- Ulam stability [9],[16]*
Let ϵ be a positive real number. We consider the system (1) *with following differential inequality*

$$\|\dot{y}(t) - f(t, y(t))\| \leq \epsilon \ , \forall t \in J. \tag{3}$$

The equation (1) is local generalized Hyers-Ulam stability (LGHUs) if for each $\epsilon \in (0, \epsilon_0]$ and for each solution $y(t, t_0, x_0) \in C^1(J, \mathbb{B})$ of (3) there exists a solution $x \in C^1(J, \mathbb{B})$ of (1) with

$$|y(t) - x(t)| \leq K(\epsilon),$$

where $K(\epsilon)$ is an expression of ϵ with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Definition 3 *Local generalized Hyers-Ulam-Rassias stability [13],[18]*
We consider the system (1) *with following differential inequality*

$$\|\dot{y}(t) - f(t, y(t))\| \leq \varphi(t) \ \forall t \in J, \tag{4}$$

where $\varphi : J \to [0, \infty)$ is a continuous function. The equation (1) is local generalized Hyers-Ulam-Rassias stability (LGHURs) with respect to φ if there exists $K > 0$ such that for each solution $y(t, t_0, x_0) \in C^1(J, \mathbb{B})$ of (4) there exists a solution $x \in C^1(J, \mathbb{B})$ of (1) with

$$|y(t) - x(t)| \leq K\varphi(t), \forall t \in J$$

Definition 4 *[15] We say that $V : J \times \mathbb{B} \to \mathbb{R}$ is a Lyapunov function if $V(t, x)$ is continuous in (t, x) , bounded on bounded subset of $\mathbb{B}$.*

3 Main Results

Lemma 5 *Consider the following differential equation*

$$\dot{x} = f(t, x(t)) \ , t \in J \tag{5}$$

with initial condition

$$x_0 = x(t_0) \in Q_0, \tag{6}$$

where f is defined and continuous on $J \times \mathbb{B}$*,and equilibrium state is at the origin :*$f(t,0) = 0$ *,*$\forall t \in J$*. The system (5),(6) to be* (Q_0, Q, ϵ_0)*-practically stable it is sufficient that there exists a continuous non increasing on the system (5) solutions Lyapunov function* $V(t,x)$ *such that*

$$\{x \in \mathbb{B} : V(t,x) \leq 1\} \subseteq Q \ , t \in J \tag{7}$$

$$Q_0 \subseteq \{x \in \mathbb{B} : V(t_0, x) \leq 1\} \tag{8}$$

Proof. We will prove by contradiction. Suppose that conditions (7) ,(8) are satisfied but there are $\tau \in J$ and $x_0 \in Q_0$ such that the solution $x(t) = x(t, x_0, t_0)$ of (5) leaves the set Q. From (7) follows inequality $V(\tau, x(\tau)) > 1$which contradicts the condition (8). Our assumption is false and the equilibrium of system (5),(6) is (Q_0, Q, ϵ_0)-practically stable . ■

Theorem 6 *Consider two systems : the system of differential equation* (5),(6) *and the system* (2) *. If equilibrium of* (5) *(at the origin) is* (Q_0, Q, ϵ_0)*-practically stable then the system* (5)*,* (6) *is local generalized Hyers-Ulam stability.*

Proof. Since Q is closed and bounded set then there exists real number $M > 0$ such that $Q = \{x : \|x\| \leq M\}$.

Now , let $x^* = f(t, x_0, t_0)$ satisfying (3) for arbitrary $\epsilon \in (0, \epsilon_0]$, then x^* satisfies (2). Since the equilibrium of (5) is (Q_0, Q, ϵ_0)-practically stable then x^* in Q ,that means that $\|x^*\| \leq M$. Since $M > 0$, $\epsilon > 0$ then there exists $s > 0$ such that $M = s\epsilon$.

Hence, $\|x^*\| \leq s\epsilon$ for all $t \in J$,$\lim\limits_{\varepsilon \to 0} K(\varepsilon) = \lim\limits_{\varepsilon \to 0} s\varepsilon = 0$. Obviously ,$w(t) = 0$ satisfies the equation (5) and

$$\|x^*(t) - w(t)\| \leq s\epsilon \ , \forall t \in J.$$

Hence, the equation (5) with initial condition (6) has local generalized Hyers-Ulam stability. ■

Corollary 7 *For the system (5) ,(6) to be local generalized Hyers-Ulam stability it sufficient that there exists a continuous non increasing on the system (5) solutions Lyapunov function* $V(t,x)$ *such that satisfies the conditions (7) and (8).*

Proof. Suppose that conditions (7) ,(8) are satisfied,then by lemma 5 the system (5),(6) is (Q_0, Q, ϵ_0)-practically stable .Hence, by theorem 6 the system has local generalized Hyers-Ulam stability. ■

Theorem 8 *Consider the following differential equation*

$$\dot{x} = f(t, x(t)) \ , t \in J \tag{9}$$

with initial condition

$$x_0 = x(t_0) \in Q_0, \tag{10}$$

where f is defined and continuous on $J \times \mathbb{B}$, and equilibrium state is at the origin :$f(t, 0) = 0 \ , \forall t \in J$.
If equilibrium is (Q_0, Q, ϵ_0)-practically stable and there exists $\epsilon_1 > 0$ such that $\epsilon_1 \leq \varphi(t) \leq \epsilon \ \forall t \in J$ then the system (9), (10) *is LGHURs with respect to φ.*

Proof. Since Q is closed and bounded set then there exists real number $M > 0$ such that $Q = \{x : \|x\| \leq M\}$.

Now , let $x^* = f(t, x_0, t_0)$ satisfying (9) ,since $\varphi(t) \leq \epsilon$ then x^* satisfies (2). Since the equilibrium of (9) is (Q_0, Q, ϵ_0)-practically stable then x^* in Q ,that mean that $\|x^*\| \leq M$. Since $M > 0$, $\epsilon_1 > 0$ then there exists $K > 0$ such that $M = K\epsilon_1$.

Then, $\|x^*\| \leq K\epsilon_1$ for all $t \in J$,hence $\|x^*\| \leq K\varphi(t)$ for all $t \in J$. Obviously ,$w(t) = 0$ satisfies the equation (9) and

$$\|x^*(t) - w(t)\| \leq K\varphi(t) \ , \forall t \in J.$$

Hence, the equation (9) with initial condition (10) has local generalized Hyers-Ulam-Rassias stability. ■

Corollary 9 *For the system (9) ,(10) to be local generalized Hyers-Ulam stability it sufficient that there exsist a continuous nonincreasing on the system (9) solutions Lyapunov function $V(t, x)$ such that satisfies the conditions (7) and (8).*

Proof. Suppose that conditions (7) ,(8) are satisfied,then by lemma 5 the system (9),(10) is (Q_0, Q, ϵ_0)-practically stable .Hence, by theorem 8 the system has local generalized Hyers-Ulam-Rassias stability. ■

Theorem 10 *Let $(\mathbb{B}, \|.\|)$ be a Banach space (real or complex) , and let $J = [t_0, t_0 + T)$ for some $T > 0$, $t_0 \geq 0$. Consider two systems : a system*

$$\dot{x} = f(t, x) \ , \forall t \in J, \tag{11}$$

with initial condition

$$x(t_0) = 0 \in Q_0, \tag{12}$$

for a set Q_0, where f is defined, continuous on $J \times \mathbb{B}$ and satisfies Lipschitz condition. The equilibrium state is at the origin : $f(t, 0) = 0, \forall t \in J$.

A system that depends on parameter $\epsilon \in (0, \epsilon_0]$,$(\epsilon_0 \in (0, \infty))$ *which is said to be perturbed system*

$$\dot{x} = f(t, x) + p(t, x). \tag{13}$$

Let P *be the set of all perturbations* p *satisfying* $\|p(t, x)\| = \epsilon \leq \epsilon_0$ *for all* $t \in J$ *and all* x *.*

If the system of differential equation (11),(12) *has Hyers-Ulam stability with Hyers-Ulam constant K then the origin is* (Q_0, Q, ϵ_0)*-practically stable ,where* $Q = \{x : \|x\| \leq K\epsilon_0\}$ *,contains the origin.*

Proof. Let $\epsilon > 0$,$\epsilon \in (0, \epsilon_0]$ and let $x^* = f(t, x_0, t_0)$,$x_0 \in Q_0$ be a solution of (13) ,i.e. $\left\|\dot{x}^* - f(t, x^*)\right\| \leq \epsilon$. Since the system (11),(12) has Hyers-Ulam stability with constant $K > 0$ then there exists y a solution of (11),(12) with $\left\|x^* - y\right\| \leq K\epsilon$ .By uniqueness of solution then y=0 .Hence $\left\|x^*\right\| \leq K\epsilon \leq K\epsilon_0$. Thus the equilibrium of $(11), (12)$ is (Q_0, Q, ϵ_0)-practically stable. ■

Remark 11 *[11] In case* $Q_0 \subset Q$ *then we have expansive stability .If* $Q \subset Q_0$ *then we have contractive stability.*

Remark 12 *If we have a differential equation of n-order in a Banach space* $\mathbb{B}_1$ *then we reduce it to a differential equation of first order in Banach space* $\mathbb{B} = \mathbb{B}_1^n$.

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