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Oscillation of Second Order Neutral Difference Inequalities with Oscillating Coefficients

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Abstract

In this paper, we established some sufficient conditions for the oscillation of second order neutral difference inequalities

$$(-1)^{\delta}x(n)\left\{\Delta^2 z(n)+(-1)^{\delta}q(n)f(x(\sigma(n)))\right\}\le 0,\quad n\ge n_0 \qquad (*)$$

where $\delta=0$ *or* $\delta=1$, $z(n)=x(n)+p(n)x(n-\tau)$, τ *is a positive integer,* $\{p(n)\}$, $\{q(n)\}$ *are sequences of real numbers,* $\{\sigma(n)\}$ *is a sequence of nonnegative integers and* $f:R\to R$ *where* R *is the set of real numbers. There are proved sufficient conditions under which every bounded solution of* $(*)$ *is either oscillatory or* $\liminf_{n\to\infty}|x(n)|=0$.

Keywords: *Neutral difference equation, oscillation, oscillating coefficients.*

1 Introduction

Consider the second order neutral difference inequalities

$$(-1)^{\delta}x(n)\left\{\Delta^2 z(n)+(-1)^{\delta}q(n)f(x(\sigma(n)))\right\}\le 0,\quad n\ge n_0, \qquad (E_\delta)$$

where $\delta=0$ or $\delta=1$, $z(n)=x(n)+p(n)x(n-\tau)$, τ is a positive integer, $\{p(n)\}$, $\{q(n)\}$ are sequences of real numbers, $\{\sigma(n)\}$ is a sequence of nonnegative integers and $f:R\to R$ where R is the set of real numbers.

The symbol Δ denotes the forward difference operator defined by
$\Delta x(n) = x(n+1) - x(n)$, $\Delta^i x(n) = \Delta\left(\Delta^{i-1} x(n)\right)$, $i = 1, 2, 3, ...$ and $\Delta^0 = 1$.

Recently several authors have been studying the oscillatory properties of solutions of neutral delay and advanced difference equations and inequalities of the first and higher order. In the oscillation theory of difference equations and inequalities, one of the important problems is to find sufficient conditions that every (bounded) solution of (E_δ) is either oscillatory or tends to zero as $n \to \infty$.

Let $m = \min\left\{\inf_{n \geq n_0} \sigma(n), n_0 - \tau\right\}$. By a solution of (E_δ), we mean a real sequence $\{x(n)\}, n \in N(m) = \{m, m+1, m+2, ...\}$ satisfy (E_δ). We consider only such solutions which are non trivial for all large n. A solution of (E_δ) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

In this paper, we give some new aspects in the study of the oscillatory properties of solutions of the inequalities (E_δ) with oscillating coefficients $q(n)$. With respect to the oscillation of delay difference equation with oscillating coefficients, reader can refer to $[6, 7]$. For the several background on difference equation, one can refer to $[1 - 5]$.

Throughout this paper, we define

$$N(a) = \{a, a+1, a+2, ...\}$$

and

$$N(a, b) = \{a, a+1, a+2, ..., b\}$$

where a and b are integers with $a \leq b$.

The following conditions are assumed to be hold throughout the paper.

(c_1) $\lim_{n\to\infty} \sigma(n) = \infty$.

(c_2) $\{q(n)\}$ is allowed to oscillate on $N(n_0)$.

(c_3) $\{p(n)\}$ and $\{q(n)\}$ are not identically zero.

(c_4) $uf(u) > 0$ for $u \neq 0$.

As a starting point, we introduce the following lemmas that are required for the proof of our main results.

Lemma 1.1 *Let* $\{x(n)\}$ *be a bounded solution of* (E_δ) *and* $\{p(n)\}$ *be a bounded sequence. Set*

$$z(n) = x(n) + p(n)x(n-\tau). \tag{1}$$

Then the sequence $\{z(n)\}$ is bounded.

Proof. The proof of Lemma is evident.

Lemma 1.2 *Let* $\{f(n)\}$, $\{g(n)\}$ *be sequences of real numbers on* $N(n_0)$ *and* τ *be an integer such that*

$$f(n) = g(n) + p(n)g(n-\tau), \quad n \geq n_0 + \max\{0, \tau\}. \tag{2}$$

Assume that $p(n)$ is one of the following ranges:

(i) $p_1 \leq p(n) \leq 0$,

(ii) $0 \leq p(n) \leq p_2 < 1$,

(iii) $1 < p_3 \leq p(n) \leq p_4$.

Suppose that $g(n) > 0$ for $n \geq n_0$, $\liminf_{n\to\infty} g(n) = 0$ and that $\lim_{n\to\infty} f(n) = L \in R$ exists. Than $L = 0$.

Proof. From (2), we see that

$$f(n+\tau) - f(n) = g(n+\tau) + [p(n+\tau) - 1]g(n) - p(n)g(n-\tau). \tag{3}$$

Let $\{n_k\}$ be a sequence of integers such that

$$\lim_{k\to\infty} n_k = \infty \quad \textit{and} \quad \lim_{k\to\infty} g(n_k) = 0. \tag{4}$$

We should prove the lemma when (i) holds. The cases where (ii) or (iii) holds are similar and will be omitted. By replacing n by $n_k$in (3) and by using (4) and the fact that $\{p(n)\}$ is bounded, we obtain

$$\lim_{k\to\infty} \Big[g(n_k+\tau) - p(n_k)g(n_k-\tau) \Big] = 0.$$

As $g(n_k+\tau) > 0$ and $p(n_k)g(n_k-\tau) \leq 0$, it follows that $\lim_{k\to\infty} p(n_k)g(n_k-\tau) = 0$ and so

$$L = \lim_{k\to\infty} f(n_k) = \lim_{k\to} \Big[g(n_k) - p(n_k)g(n_k-\tau) \Big] = 0.$$

The proof is complete.

Lemma 1.3 *Let $\{f(n)\}$, $\{g(n)\}$ and $\{p(n)\}$ be sequences of real numbers and τ be a positive integer such that*

$$f(n) = g(n) + p(n)g(n-\tau) \quad for \quad n \geq n_0 + \tau.$$

Assume that $0 < g(n) \leq g_0 < \infty$ and $\lim_{n\to\infty} f(n) = 0$. In addition, we suppose that there exists constants p_1, p_2 such that either

$$-1 < p_1 \leq p(n) \leq 0 \quad or \quad 0 \leq |p(n)| \leq |p_1| < 1, \tag{5}$$

or

$$p(n) \leq p_2 < -1. \tag{6}$$

Then

$$\lim_{n\to\infty} g(n) = 0.$$

Proof. (i) Let (5) holds. Then

$$g(n) = f(n) - p(n)g(n-\tau) \leq f(n) + |p_1|\, g(n-\tau), \quad n \geq n_0 + \tau.$$

By iteration, for sufficiently large n, we have

$$g(n) \leq f(n)+|p_1|\, f(n-\tau)+|p_1|^2 f(n-2\tau)+...+|p_1|^{k-1} f(n-(k-1)\tau)+|p_1|^k g(n-k\tau).$$

The last relation we can written in the form

$$0 < g(n+k\tau) \leq f(n+k\tau) + |p_1|\, f(n+(k-1)\tau) + |p_1|^2 f(n+(k-2)\tau) + ...+$$
$$+ |p_1|^{k-1} f(n+\tau) + |p_1|^k g(n)$$

for sufficiently large n.

In view of $\lim_{n\to\infty} f(n) = 0$, for any $\epsilon_1 > 0$ there exists sufficiently large N such that

$$|f(n)| < \epsilon_1 \quad for \quad n \geq N.$$

Then

$$|g(n+k\tau)| < \epsilon_1 \frac{1}{1-|p_1|} + |p_1|^k g_0, \quad n \geq N. \tag{7}$$

Therefore for any $\epsilon > 0$ there exists ϵ_1 and $k = k_0$ such that

$$\frac{\epsilon_1}{1+p_1} + |p_1|^{k_0} q_0 < \epsilon.$$

Then from (7) in view of the last relation, we have

$$\lim_{n\to\infty} g(n) = 0.$$

(*ii*) Let (6) hold. Then from $p(n)g(n-\tau) = f(n) - g(n)$ with regard to (6), we get

$$g(n) \leq \frac{1}{p_2}\left(f(n+\tau) - g(n+\tau)\right), \quad n \geq n_0 + 2\tau.$$

By iteration for sufficiently large n, we have

$$g(n) \leq \frac{1}{p_2}f(n+\tau) - \frac{1}{p_2^2}f(n+2\tau) + \ldots + (-1)^{k-1}\frac{1}{p_2^k}f(n+k\tau) + (-1)^k\frac{1}{p_2^k}g(n+k\tau).$$

In view of $\lim_{n\to\infty} f(n) = 0$, for any $\epsilon_1 > 0$, there exists sufficiently large N such that $|f(n)| \leq \epsilon_1$, for $n \geq N$. Then

$$|g(n)| \leq \frac{\epsilon_1}{|p_2| - 1} + \frac{g_0}{|p_2|^k}.$$

Then analogously as in the case (*i*) we obtain $\lim_{n\to\infty} g(n) = 0$.

Lemma 1.4 *Let $\{w(n)\}_{n=n_0}^{\infty}$ and $\{v(n)\}_{n=n_0}^{\infty}$ be two sequences of real numbers. If the limit $\lim_{n\to\infty}[w(n)v(n) + v(n+1)]$ exists in the extended real line R^*, then the limit $\lim_{n\to\infty} v(n)$ exists in R^**

Proof. If the conclusion is false, then there are numbers ξ and η such that

$$\liminf_{n\to\infty} v(n) < \xi < \eta < \limsup_{n\to\infty} v(n).$$

We are able to select an increasing sequence $\{n_k\}_{k=1}^{\infty}$ with the following properties:

$$\lim_{k\to\infty} n_k = \infty, \quad \lim_{k\to\infty} \Delta v(n_k) = 0, \tag{8}$$

$$v(n_{2k-1}) < \xi, \quad v(n_{2k}) > \eta, \quad k = 1, 2, 3, ... \tag{9}$$

In view of (8) we see that the limit

$$\lim_{k\to\infty}[w(n_k)\Delta v(n_k) + v(n_k+1)] = \lim_{k\to\infty} v(n_k+1)$$

exists in R^*. However, this is a contradiction, since (9) implies that the sequence $\{v(n_k)\}_{k=1}^{\infty}$ cannot have a limit in R^*.

We are now in a position to state and prove our main results.

2 Main Results

In addition we suppose that

(C_1) There exists two sequences $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ of nonnegative integers such that

$$\bigcup_{j=1}^{\infty} N(a_j, b_j) \subset N(n_0), \quad \lim_{j\to\infty} a_j = \infty,$$

and for any $j = 1, 2, 3, ...,$

$$a_j + \tau < b_j < a_{j+1}, \quad a_{j+1} - a_j \leq M < \infty.$$

(C_2) $q(n) \geq 0$ for $n \in \bigcup_{j=1}^{\infty} N(a_j, b_j)$ and $\liminf_{n\to\infty} q(n) = 0$.

(C_3) Let there exists constants p_1 and p_2 such that the following holds:

$$p_1 \leq p(n) \leq p_2, \quad n \geq n_0.$$

Denote

$$A_k = \bigcup_{j=k}^{\infty} N(a_j, b_j)$$

Theorem 2.1 *Let (C_1), (C_2), (C_3) hold. If*

$$\lim_{j\to\infty} \sum_{n=a_j}^{b_j-1} q(n) = \infty, \tag{10}$$

then every bounded solution of (E_0) is either oscillatory or $\lim_{n\to\infty} |x(n)| = 0$.

Proof. Let $\{x(n)\}$ be a nonoscillatory bounded solution of (E_0). Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive and bounded solution of (E_0). Then there exist an integer $n_1 \geq n_0$ such that $\{x(n)\}$ is bounded, $x(n-\tau) > 0$ and $x(\sigma(n)) > 0$, for all $n \geq n_1$.

Then from (E_0), we get that $\{\Delta z(n)\}$ is decreasing and $\{z(n)\}$ is monotone on $A_1 \cap N(n_1)$.

In view of that $\{x(n)\}$ is bounded and $x(n) > 0$ on $N(n_1)$, there exists a constant K and $n_2 \geq n_1$ such that $|f(x(\sigma(n)))| \leq K$ for all $n \geq n_2$. With regard to (C_2) for any there exists a $n_3 \geq n_2$ such that

$$q(n) \geq -\delta/KM \quad for \quad n \geq n_3. \tag{11}$$

Then from (E_0) with regard to (11), we have $\Delta^2 z(n) \leq \delta/M$ for $n \geq n_3$. Summing the last inequality from b_j to $a_{j+1} - 1$ $(b_j \geq n_3, j \in N)$ we have

$$\Delta z(a_{j+1}) \le \Delta z(b_j) + \delta, \quad b_j \ge n_3, \quad j \in N. \tag{12}$$

(I) Let there exists a $j_0 \ge 1$ such that $\Delta z(n) < 0$ for all $n \in A_{j_0} \cap N(n_3)$. Summing (E_0) from a_j to $b_j - 1$, $j \ge j_0$ and using that $\Delta z(n) < 0$ we obtain

$$\sum_{n=a_j}^{b_j-1} q(n) f\left(x(\sigma(n))\right) \le \Delta z(a_j) - \Delta z(b_j) \le -\Delta z(b_j). \tag{13}$$

(a) Let $\inf_{j\ge j_0} \{\Delta z(b_j)\} > -\infty$, then from (13) we have

$$\sum_{n=a_j}^{b_j-1} q(n) f\left(x(\sigma(n))\right) < \infty, \quad a_j \ge n_3, \quad j \ge j_0. \tag{14}$$

The last inequality with regard to (10) and the property of the function f and the sequence $\{\sigma(n)\}$ implies $\liminf_{n\to\infty} x(n) = 0$.

(b) Let $\inf_{j\ge j_0} \{\Delta z(b_j)\} = -\infty$. Then in view of (12) and that $\{\Delta z(n)\}$ is eventually negative and decreasing sequence on $A_{j_0} \cap N(n_3)$, we get that $\{z(n)\}$ is unbounded below. Then this, in view of (C_3) and Lemma 1.1, we get that $\{x(n)\}$ is unbounded, which is a contradiction to the assumption that $\{x(n)\}$ is a bounded sequence.

(II) Let there exists a sequence $\{i_r\}_{r=1}^{\infty}$, $i_r \in N$ such that $\Delta z(n) > 0$ and $\{\Delta z(n)\}$ is decreasing for all $n \in A_{i_r} \subset N(n_0)$. Then summing (E_0) from a_{i_r} to $b_{i_r} - 1$, $r \ge 1$, we have

$$\sum_{n=a_{i_r}}^{b_{i_r}-1} q(n) f\left(x(\sigma(n))\right) \le \Delta z(a_{i_r}) - \Delta z(b_{i_r}) \le \Delta z(a_{i_r}). \tag{15}$$

Because $\{x(n)\}$ is bounded on $N(n_1)$, by Lemma 1.1 we get that $\{z(n)\}$ is bounded on $N(n_1)$. Therefore with regard to (12) and the monotonicity of $\{z(n)\}$, $\{\Delta z(n)\}$ we have $\sup_{i_r\ge i_1} \{\Delta z(a_{i_r}\} < \infty$. Thus from (15) we get (14), which implies as in the case (Ia) that $\liminf_{n\to\infty} x(n) = 0$.

The proof of the Theorem is complete.

Theorem 2.2 *Let (C_1), (C_2), (C_3) and* (10) *hold. Then every bounded solution of (E_1) is either oscillatory or* $\liminf_{n\to\infty} |x(n)| = 0$.

Proof. Let $\{x(n)\}$ be a nonoscillatory bounded solution of (E_1). Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive and bounded solution of (E_1). Then there exists an integer $n_1 \ge n_0$ such that $\{x(n)\}$ is bounded, $x(n-\tau) > 0$ and $x(\sigma(n)) > 0$, for all $n \le n_1$. If $q(n) > 0$

for any $n \in A_1 \cap N(n_1)$, then from (E_1) we get that $\{\Delta z(n)\}$ is increasing and $\{z(n)\}$ is monotone on $A_1 \cap N(n_1)$.

Analogously as in the proof of Theorem 2.1 we have (11). Then from (E_1) in view of (11), we have

$$\Delta^2 z(n) \geq -\delta/M \quad for \quad n \geq n_2.$$

Summing the last inequality from b_j to $a_{j+1}-1$, $b_j \geq n_2$, $j \in N$, we obtain

$$\Delta z(a_{j+1}) \geq \Delta z(b_j) + \delta, \quad b_j \geq n_2, \quad j \in N. \tag{16}$$

(I) Let there exists a $j_0 \geq 1$ such that $\Delta z(n) > 0$ for all $n \in A_{j_0}$, $a_{j_0} \geq n_2$. Summing (E_1) from a_j to $b_j - 1$ for any $j \geq j_0$, we obtain

$$\sum_{n=a_j}^{b_j-1} q(n) f\left(x(\sigma(n))\right) \leq \Delta z(b_j) - \Delta z(a_j) \leq \Delta z(b_j). \tag{17}$$

(a) Let $\sup_{j \geq j_0} \{\Delta z(b_j)\} < \infty$, then from (17) in view of (10) and the property of the function f and the sequence $\{\sigma(n)\}$, we have

$$\liminf_{n \to \infty} x(n) = 0.$$

(b) Let $\sup_{j \geq j_0} \{\Delta z(b_j)\} = \infty$, then in view of (16) and the fact that $\{\Delta z(n)\}$ is increasing and positive for all $n \in A_{j_0}$, we have that $\{z(n)\}$ is unbounded above. Then in view of (C_3) and of Lemma 1.1 we get that $\{x(n)\}$ is unbounded, which is a contradiction.

(II) Let there exists a sequence $\{i_r\}_{r=1}^{\infty}, i_r \in N$ such that $\Delta z(n) < 0$ and $\{\Delta z(n)\}$ is increasing all $n \in A_{i_r} \subset N(n_0)$. Then summing (E_1) from a_{i_r} to $b_{i_r} - 1$, $r \geq 1$, we obtain

$$\sum_{n=a_{i_r}}^{b_{i_r-1}} q(n) f\left(x(\sigma(n))\right) \leq \Delta z(b_{i_r}) - \Delta z(a_{i_r}) \leq -\Delta z(a_{i_r}). \tag{18}$$

In view of Lemma 1.1 and that $\{x(n)\}$ is bounded and positive on $N(n_1)$, we have that $\{z(n)\}$ is bounded on $N(n_1)$. Then with regard to (16) and the monotonicity of $\{z(n)\}$, $\{\Delta z(n)\}$ we get

$$\sup_{i_r \geq i_1} \{-\Delta z(a_{i_r})\} < \infty.$$

Therefore from (18) we get

$$\sum_{n=a_{i_r}}^{b_{i_r}-1} q(n) f\left(x(\sigma(n))\right) < \infty.$$

The last relation in view of (10) and the property of the function f and the sequence $\{\sigma(n)\}$ we get that

$$\liminf_{n\to\infty} x(n) = 0.$$

The proof of Theorem 2.2 is complete.

Now denote

$$q_+(n) = \max\{0, q(n)\}, \quad q_-(n) = \max\{0, -q(n)\}, \quad n \geq n_0. \tag{19}$$

Then $q(n) = q_+(n) - q_-(n)$.

Theorem 2.3 *Let (C_3) hold. In addition we suppose that*

$$\sum_{n=n_0}^{\infty} q_+(n) = \infty \tag{20}$$

and

$$\sum_{n=n_0}^{\infty} q_-(n) < \infty. \tag{21}$$

Then every bounded solution of (E_0) is oscillatory, or $\liminf_{n\to\infty} |x(n)| = 0$.

Proof. Let $\{x(n)\}$ be a bounded nonoscillatory solution of (E_0). Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive and bounded solution of (E_0). Then there exists an integer $n_1 \geq n_0$ such that $\{x(n)\}$ is bounded, $x(n-\tau) > 0$ and $x(\sigma(n)) > 0$, for $n \geq n_1$. Analogously as in the proof of Theorem 2.1, there exists $K > 0$ and $n_2 \geq n_1$, such that $|f(x(\sigma(n)))| \leq K$ for $n \geq n_2$. Then the inequality (E_0) in view of (19) we can write in the form

$$\Delta^2 z(n) + q_+(n) f\left(x(\sigma(n))\right) - K q_-(n) \leq 0, \quad n \geq n_2. \tag{22}$$

With regard to (21) there exists a $L > 0$ such that $\sum_{n=n_2}^{\infty} q_-(n) = L$. Then (22) via the estimation (19) we have $\Delta z(n) \leq \Delta z(n_2) + KL$, *i.e.*, $\{\Delta z(n)\}$ is bounded above. If $\sum_{n_0}^{\infty} q_+(n) f\left(x(\sigma(n))\right) = \infty$, then the estimation (22) implies that $\lim_{n\to\infty} \Delta z(n) = -\infty$ and therefore $\lim_{n\to\infty} z(n) = -\infty$. Thus in view of Lemma 1.1 and (C_3) contradicts the fact that $\{x(n)\}$ is bounded on $N(n_1)$. Therefore

$$\sum_{n=n_0}^{\infty} q_+(n) f\left(x(\sigma(n))\right) < \infty. \tag{23}$$

Then (23) in view of (20) and the properties of function f and the sequence $\{\sigma(n)\}$ implies that

$$\liminf_{n\to\infty} x(n) = 0 \tag{24}$$

The proof of Theorem 2.3 is complete.

Now we consider the equation

$$\Delta^2 z(n) + q(n) f\left(x(\sigma(n))\right) = 0, \quad n \geq n_0 \qquad (E)$$

as a special case of (E_0).

Theorem 2.4 *Let either* (5) *or*

$$-\infty < p_3 \leq p(n) \leq p_2 < -1 \tag{25}$$

hold. In addition we suppose that

$$\sum_{n=n_0}^{\infty} nq_+(n) = \infty \quad \textit{and} \tag{26}$$

$$\sum_{n=n_0}^{\infty} nq_-(n) < \infty. \tag{27}$$

Then every bounded solution of (E) is either oscillatory or

$$\lim_{n\to\infty} x(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \Delta^i z(n) = 0, \quad i = 0, 1.$$

Proof. Let $\{x(n)\}$ be a bounded nonoscillatory solution of (E). Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive and bounded solution of (E). Then there exist an integer $n_1 \geq n_0$ such that $\{x(n)\}$ is bounded, $x(n-\tau) > 0$ and $x(\sigma(n)) > 0$ on $N(n_1)$. Multiplying (E) by n and then summing from n_2 to $n-1$, we have

$$u(n) = \sum_{s=n_2}^{n-1} s\Delta^2 z(s) = \sum_{s=n_2}^{n-1} sq_-(s) f\left(x(\sigma(s))\right) - \sum_{s=n_2}^{n-1} sq_+(s) f\left(x(\sigma(s))\right). \tag{28}$$

If $\sum_{n=n_2}^{\infty} nq_+(n) f\left(x(\sigma(n))\right) = \infty$, then in view of (27) and the boundedness of $\{x(n)\}$ from (28), we get $\lim_{n\to\infty} u(n) = -\infty$. By Lemma 1.4 there exists $\lim_{n\to\infty} z(n) = z_0 \in R^*$. Let $|z_0| < \infty$. Then $\lim_{n\to\infty} u(n) = -\infty$ implies $\lim_{n\to\infty} n\Delta z(n) = -\infty$. From this relations we get $\lim_{n\to\infty} z(n) = -\infty$ which contradicts the fact that $|z_0| < \infty$. Therefore $\lim_{n\to\infty} |z(n)| = \infty$. This in view of Lemma 1.1 gives a contradiction to the fact that $\{x(n)\}$ is bounded. Therefore

$$\sum_{n=n_2}^{\infty} nq_+(n)f\left(x(\sigma(n))\right) < \infty. \tag{29}$$

Then (29) in view of (26) and the property of f and the sequence $\{\sigma(n)\}$ implies that (24) holds.

Now, letting $n \to \infty$ in (28), then using the boundedness of $\{x(n)\}$, (27), (29) and the property of f, we have

$$\lim_{n\to\infty}\Big[n\Delta z(n) - z(n+1)\Big] = L_1, \quad |L_1| < \infty. \tag{30}$$

With regard to Lemma 1.4 and the fact that $\{z(n)\}$ is bounded, we obtain that $\lim_{n\to\infty} z(n) = L$, $|L| < \infty$. Then if we use either (5) or (25), (24) and Lemma 1.2, we obtain that $L = 0$. From (30) in view of $L = 0$, we get that $\lim_{n\to\infty} \Delta z(n) = 0$. We proved that $\lim_{n\to\infty} \Delta^k z(n) = 0$, $k = 0, 1$. Then if we use Lemma 1.3, we have $\lim_{n\to\infty} x(n) = 0$.

The proof of the Theorem 2.4 is complete.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, (1992).

[2] S.N. Elaydi, *An Introduction to Difference Equations*, Springer Verlag, New York, (1996).

[3] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).

[4] W.G. Kelley and A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, (1991).

[5] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Method and Application*, Academic Press, New York, (1988).

[6] X.H. Tang and S.S. Cheng, An oscillation criteria for linear difference equation with oscillating coefficients, *J. Comput. Appl. Math*, 132(2) (2001), 319-329.

[7] W.P. Yan and J.R. Yan, Comparison and oscillation results for delay difference equations with oscillating coefficients, *Int. J. Math. Math. Sci.*, 19(1) (1996), 171-176.