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A Common Fixed Point Theorem Under Certain Conditions

R. Umamaheshwar Rao[1] and V. Srinivas[2]

[1, 2] Department of Mathematics, Sreenidhi Institue of Science and Technology,
Ghatkesar, R.R. Dist-501 301, Andhra Pradesh, India.
[1]E-mail: maheshwarrao_r@yahoo.co.in
[2] E-mail: srinivasveladi@yahoo.co.in

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Abstract

The aim of this paper is to present a common fixed point theorem in a metric space which extends the result of P.C.Lohani & V.H.Bhadshah using the weaker conditions such as Reciprocally continuous, Compatible mappings, Weakly compatible and Associated sequence.

Keywords: *Fixed point, Self maps, reciprocally continuous, compatible mappings, weakly compatible, associated sequence.*

1 Introduction

G.Jungck gave a common fixed point theorem for commuting mapping maps, which generalizes the Banach's fixed point theorem. This result was further generalized and extended in various ways by many authors. S.Sessa[5] defined weak commutativity and proved common fixed point theorems for weakly commuting maps. Further G.Jungck [1] initiated the concept of compatible maps which is weaker than weakly commuting maps. Afterwards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps.

On the other hand, R.P.Pant [2] introduced a new notion of continuity namely reciprocal continuity for a pair of self maps and proved some common fixed point theorems.
The purpose of this paper is to prove a common fixed point theorem for four self maps in which one pair is reciprocally continuous and compatible and other pair is weakly compatible.

2 Definitions and Preliminaries

Definition 2.1 *If S and T are mappings from a metric space (X,d) into itself, are called weakly commuting mappings on X, if $d(STx,TSx) \le d(Sx,Tx)$ for all x in X.*

Definition 2.2. *Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= t \text{ for some } t\in X.$$

Clearly commuting mappings are weakly commuting, but the converse is not necessarily true.

Definition 2.3. *Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e if $Su=Tu$ for some $u\in X$ then $STu=TSu$.*
It is clear that every compatible pair is weakly compatible but its converse need not be true.
Definition 2.4. *Two self maps S and T of a metric space (X,d) are said to be reciprocally continuous if $\lim_{n\to\infty} TSx_n=Tt$ and $\lim_{n\to\infty} STx_n= St$ when ever $<x_n>$ is a sequence such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=t$ for some $t\in X$.*

If S and T are both continuous then they are obviously reciprocally continuous. But the converse is not true. More over, in the setting of common fixed point theorems for compatible maps satisfying contractive conditions, continuity of one of the mappings S or T implies their reciprocal continuity but not conversely.

P.C.Lohani and V.H.Badshah [6] proved the following theorem.

Theorem 2.5. *Let P,Q,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions*

$$S(X)\subset Q(X) \text{ and } T(X)\subset P(X) \quad \ldots\ldots\text{(2.5.1)}$$

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta\, d(Px,Qy) \ldots\ldots\ldots\text{(2.5.2)}$$

for all x,y in X where α,β≥0, α+β<1.

One of P,Q,S and T is continuous …….(2.5.3)
Pairs (S,P) and (T,Q) are compatible on X …….(2.5.4)

then P,Q,S and T have a unique common fixed point in X.

Associated sequence 2.6. Suppose P,Q,S and T are self maps of a metric space (X,d) satisfying the condition (2.5.1). Then for an arbitrary $x_0 \in X$ such that $Sx_0 = Qx_1$ and for this point x_1, there exists a point x_2 in X such that $Tx_1 = Px_2$ and so on. Proceeding in the similar manner, we can define a sequence $<y_n>$ in X such that $y_{2n}=Sx_{2n}= Qx_{2n+1}$ and $y_{2n+1}=Px_{2n+2} = Tx_{2n+1}$ for $n \geq 0$. We shall call this sequence as an "*Associated sequence of x_0* "relative to the four self maps P,Q,S and T.

Lemma 2.7. *Let P,Q,S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions (2.5.1) and (2.5.2). Then the associated sequence $\{y_n\}$relative to four self maps is a Cauchy sequence in X.*

Proof: From the definition of associated sequence (2.6), we have

$$d(y_{2n},y_{2n+1}) = d(Sx_{2n},Tx_{2n+1}) \leq \alpha \frac{d(Qx_{2n+1},Tx_{2n+1})[1+d(Px_{2n},Sx_{2n})}{[1+d(Px_{2n},Qy_{2n+1})]} + \beta d(Px_{2n},Qy_{2n+1})$$

$$= \alpha \frac{d(y_{2n},y_{2n+1})[1+d(y_{2n-1},y_{2n})}{[1+d(y_{2n-1},y_{2n})]} + \beta\, d(y_{2n-1},y_{2n})$$

$$= \alpha\, d(y_{2n},y_{2n+1}) + \beta\, d(y_{2n-1},y_{2n})$$

$$(1-\alpha)\, d(y_{2n},y_{2n+1}) \leq \beta\, d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \leq \frac{\beta}{(1-\alpha)}\, d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \leq h\, d(y_{2n-1},y_{2n}) \quad where\ \ h = \frac{\beta}{(1-\alpha)}$$

Now

$$d(y_n,y_{n+1}) \leq h\, d(y_{n-1},y_n) \leq h^2\, d(y_{n-2},y_{n-1}) \leq ....h^n\ d(y_0,y_1)$$

For every integer $p > 0$, we get

$$d(y_n,y_{n+p}) \leq\ d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + ...+\ d(y_{n+p-1},y_{n+p})$$

$$\leq h^n d(y_0,y_1) + h^{n+1} d(y_0,y_1) + ...+ h^{n+p-1} d(y_0,y_1)$$

$$\leq \left(h^n + h^{n+1} + ...+ h^{n+p-1}\right) d(y_0,y_1)$$

$$\leq h^n \left(1 + h + h^2 + ...+ h^{p-1}\right) d(y_0,y_1)$$

Since $h<1$, , $h^n \to 0$ as $n\to \infty$, so that $d(y_n,y_{n+p})\to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, it converges to a limit, say $z\in X$.
The converse of the Lemma is not true, that is P,Q,S and T are self maps of a metric space (X,d) satisfying (2.5.1) and (2.5.2), even if for $x_0\in X$ and for associated sequence of x_0 converges, the metric space (X,d) need not be complete. The following example establishes this.

Example 2.8. Let $X=(-1,1)$ with $d(x,y)=|x-y|$

$$Sx=Tx=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{1}{6} & if\ \frac{1}{6}\le x<1\end{cases}\qquad Px=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{6x+5}{36} & if\ \frac{1}{6}\le x<1\end{cases}\qquad Qx=\begin{cases}\frac{1}{5} & if\ -1<x<\frac{1}{6}\\ \frac{1}{3}-x & if\ \frac{1}{6}\le x<1\end{cases}$$

Then $S(X)=T(X)=\left\{\frac{1}{5},\frac{1}{6}\right\}$ while $P(X)=\left\{\frac{1}{5}\cup\left[\frac{1}{6},\frac{11}{36}\right)\right\}$, $Q(X)=\left\{\frac{1}{5}\cup\left[\frac{1}{6},\frac{-2}{3}\right)\right\}$
so that $S(X)\subset Q(X)$ and $T(X)\subset P(X)$ proving the condition (2.5.1). Clearly (X,d) is not a complete metric space. It is easy to prove that the associated sequence $Sx_0,Tx_1,Sx_2,Tx_3,..,Sx_{2n},Tx_{2n+1}..$, converges to $\frac{1}{5}$ if $-1<x<\frac{1}{6}$; and converges to $\frac{1}{6}$ if $\frac{1}{6}\le x<1$.
Now we prove our theorem.

3 Main Result

Theorem 3.1. *Let P,Q, S and T are self maps of a metric space (X,d) satisfying (2.5.1), (2.5.2) and the conditions*
The pair (S,P) is Reciprocally continuous and compatible and the pair (T,Q) is weakly compatible ….(3.1.1)
Also
The associated sequence relative to four self maps P,Q, S and T such that the sequence $Sx_0,Tx_1,Sx_2,Tx_3,\ldots,Sx_{2n},Tx_{2n+1},\ldots$ *converges to* $z\in X$. *as* $n\to\infty$ ……..(3.1.2)

then P,Q,S and T have a unique common fixed point z in X.

Proof. From the condition (3.1.2), $Sx_0,Tx_1,Sx_2,Tx_3,\ldots,Sx_{2n},Tx_{2n+1},\ldots$ converges to $z\in X$. as $n\to\infty$

First suppose that the pair (S,P) is reciprocally continuous and compatible, then from the definition of reciprocally continuity of (S,P) if $Sx_{2n} \to z$, $Px_{2n} \to z$ as $n \to \infty$ then

$SPx_{2n} \to Sz$, $PSx_{2n} \to Pz$ ……...(3.1.3)

From the compatibility of the pair (S,P) we get $\lim_{n\to\infty} d(SPx_{2n}, PSx_{2n}) = 0$ or $\lim_{n\to\infty} SPx_{2n} = \lim_{n\to\infty} PSx_{2n}$ Using (3.1.3) this gives that Sz=Pz.

Since $S(X) \subset Q(X)$ there exists $u \in X$ such that Sz=Qu.

we consider

$$d(Sz,z) = \lim_{n\to\infty} d(Sz, Tx_{2n+1}) \le \lim_{n\to\infty}\left\{\alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1+d(Pz,Sz)]}{[1+d(Pz,Qx_{2n+1})]} + \beta\, d(Pz, Qx_{2n+1})\right\}$$
$$= \beta\, d(Sz,z)$$

this gives $d(Sz,z) \le \beta\, d(Sz,z)$, since $\beta \ge 0, \alpha+\beta<1$ giving that d(Sz,z)=0. Thus Sz=z.

Hence Sz=Pz=z=Qu. This shows that 'z' is a common fixed point of P and S.

Now we prove Qu=Tu.

Consider

$$d(z,Tu) = d(Sz,Tu) \le \left\{\alpha \frac{d(Qu,Tu)[1+d(Pz,Sz)]}{[1+d(Pz,Qu)]} + \beta\, d(Pz,Qu)\right\}$$
$$= \alpha\ d(z,Tu)$$

this gives $d(z,Tu) \le \alpha$ d(z,Tu), since $\alpha \ge 0, \alpha+\beta<1$ giving that d(z,Tu)=0. Thus Tu=z. Hence Tu=Qu=z.

Also since the pair (T,Q) is weakly compatible and since Tu=Qu=z, we get TQu=QTu or Tz=Qz.

Again we consider

$$d(z,Tz) = d(Sz,Tz) \le \left\{\alpha \frac{d(Qz,Tz)[1+d(Pz,Sz)]}{[1+d(Pz,Qz)]} + \beta\, d(Pz,Qz)\right\}$$
$$= \beta\, d(z,Tz)$$

this gives $d(z,Tz) \le \beta\, d(z,Tz)$, since $\beta \ge 0, \alpha+\beta<1$ giving that d(z,Tz)=0. Thus Tz=z.

Hence Qz=Tz=z. Therefore Pz=Qz=Sz=Tz= z, showing that 'z' is a common fixed point of P,Q,S and T.

The uniqueness of the fixed point can be easily proved.

Remark 3.2. From the Example 2.8, clearly the pair (S,P) is reciprocally continuous since if $x_n = \left(\frac{1}{6} + \frac{1}{6^n}\right)$ for $n \geq 1$, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Px_n = \frac{1}{6}$ then $\lim_{n\to\infty} SPx_n = \frac{1}{6} = S(t)$ and $\lim_{n\to\infty} PSx_n = \frac{1}{6} = P(t)$. But none of S and P is continuous. Since $\lim_{n\to\infty} d(SPx_n, PSx_n) = 0$, the pair (S,P) is compatible. Also the pair (T,Q) is weakly compatible as they commute at coincident points $\frac{1}{5}$ and $\frac{1}{6}$. The rational inequality holds for the values of $\alpha, \beta \geq 0$, $\alpha + \beta < 1$. Moreover $\frac{1}{6}$ is the unique common fixed point of P,Q,S and T.

Remark 3.3. Theorem 3.1 is a generalization of Theorem 2.5 by virtue of the weaker conditions such as the reciprocal continuity and compatibility of the pair (S,P) in place continuity of one of the mappings; weakly compatibility of the pair (T,Q) in place of compatibility; and associated sequence relative to four self maps P,Q,S and T in place of the complete metric space.

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