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Subdivisions of the Spectra for the Triple Band Matrix over Certain Sequence Spaces

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Abstract

In a series of papers, B. Altay, F. Başar and A.M. Akhmedov recently investigated the spectra and fine spectra for difference operator, considered as bounded operator over various sequence spaces. In the present paper approximation point spectrum, defect spectrum and compression spectrum of an operator represented by the triple band matrix $B(r,s,t)$ over the sequence spaces c_0, c, ℓ_p and bv_p are determined, where bv_p denotes the space of all sequences (x_k) such that $(x_k - x_{k-1})$ belongs to the sequence space ℓ_p with $1 \leq p < \infty$ and was studied by Başar and Altay [Ukrainian Math. J. ***55****(1)(2003), 136–147].*

Keywords: *Spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum and triple band matrix $B(r,s,t)$.*

1 Preliminaries, Background and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, \ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $\mathcal{D}(T)$ is dense in X, or that T has closed graph $\{(x, Tx) : x \in \mathcal{D}(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number λ is perturbed operator

$$T_\lambda = \lambda I - T,$$

defined on the same domain $\mathcal{D}(T)$ as T. The *spectrum* $\sigma(T, X)$ consists of those $\lambda \in \mathbb{C}$ for which T_λ is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\lambda \mapsto T_\lambda^{-1}$.

2 Subdivisions of the Spectrum

In this section, we mention from the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1 The point spectrum, continuous spectrum and residual spectrum

The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided that T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and is bounded and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [16, pp. 370-371]):

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and is bounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectras form a disjoint subdivisions

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X). \tag{1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite dimensional case.

2.2 The approximate point spectrum, defect spectrum and compression spectrum

In this subsection, following Appell et al. [5], we give the definitions of the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T, X) := \{\lambda : \text{ exists a Weyl sequence for } \lambda I - T\} \tag{2}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\lambda : \lambda I - T \text{ is not surjective}\} \tag{3}$$

is called *defect spectrum* of T.

The two subspectra given by (2) and (3) form a (not necessarily disjoint) subdivisions

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T, X) = \{\lambda : \overline{R(\lambda I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X)$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (1) we note that

$$\sigma_r(T, X) = \sigma_{co}(T, X) \backslash \sigma_p(T, X)$$

and

$$\sigma_c(T, X) = \sigma(T, X) \backslash [\sigma_p(T, X) \cup \sigma_{co}(T, X)].$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 2.1 [5, Proposition 1.3, p. 28] *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum is dual to the compression spectrum.

The last equation (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [5]).

2.3 Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$.

(II) $R(T) \neq \overline{R(T)} = X$.

(III) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.

(2) T^{-1} exists but is discontinuous.

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [13]).

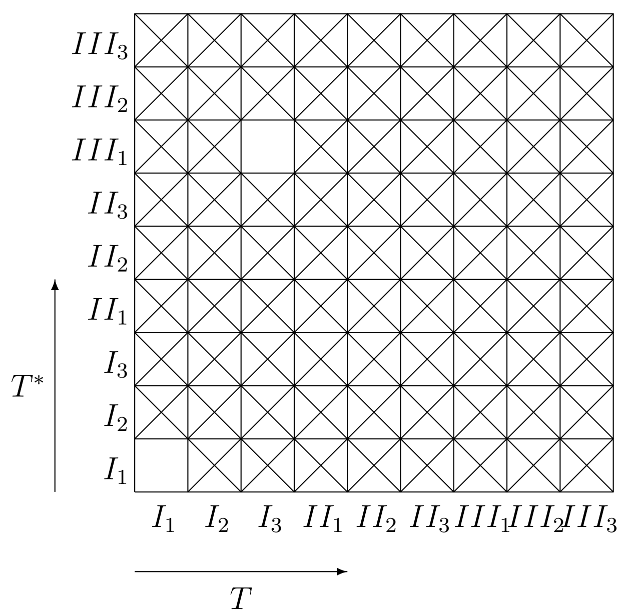

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $T_\lambda = \lambda I - T \in I_1$ or $T_\lambda = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T,X)$ gives rise to the fine spectrum of T. That is, $\sigma(T,X)$ can be divided into the subsets $I_2\sigma(T,X)=\emptyset$, $I_3\sigma(T,X)$, $II_2\sigma(T,X)$, $II_3\sigma(T,X)$, $III_1\sigma(T,X)$, $III_2\sigma(T,X)$, $III_3\sigma(T,X)$. For example, if $T_\lambda=\lambda I-T$ is in a given state, III_2 (say), then we write $\lambda\in III_2\sigma(T,X)$.

By the definitions given above, we can illustrate the subdivisions (1) in the following table:

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exist
I	$R(\lambda I-T)=X$	$\lambda\in\rho(T,X)$	–	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$
II	$\overline{R(\lambda I-T)}=X$	$\lambda\in\rho(T,X)$	$\lambda\in\sigma_c(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$
III	$\overline{R(\lambda I-T)}\neq X$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$

Table 1.2: Subdivisions of spectrum of a linear operator

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I-T$.

By a *sequence space*, we understand a linear subspace of the space $\omega=\mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty=\sup_{k\in N}|x_k|$ and $\|x\|_{bv}=\sum_{k=0}^\infty|x_k-x_{k+1}|$ while ϕ is not a Banach space with respect to any norm, respectively. Also by ℓ_p, we denote the space of all p–absolutely summable sequences which is a Banach space with the norm $\|x\|_p=(\sum_{k=0}^\infty|x_k|^p)^{1/p}$, where $1\leq p<\infty$.

In this paper, our main focus is the operator $B(r,s,t)$ represented by the following triple band matrix

$$B(r,s,t) = \begin{pmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ t & s & r & 0 & \cdots \\ 0 & t & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We assume here and after that s and t are complex parameters which do not simultaneously vanish.

We summarize the knowledge in the existing literature concerning with the spectrum and the fine spectrum of the linear operators defined by some particular limitation matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p has been studied by Gonzàlez [14], where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p have been investigated by Cartlidge [9]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv have also been investigated by Okutoyi [17, 18]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [3]. Same authors have studied the fine spectrum of the generalized difference operator $B(r,s)$ over c_0 and c, in [4]. The fine spectra of Δ over ℓ_1 and bv has been studied by Kayaduman and Furkan [15]. Recently, the fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p has been studied by Akhmedov and Başar [1, 2], where $1 \leq p < \infty$. Also, the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv has been studied by Furkan et al. [12]. Recently, the fine spectrum of $B(r,s,t)$ over the sequence spaces c_0 and c has been studied by Furkan et al. [10]. Also the fine spectrum of the same operator over ℓ_1 and bv has been studied by Bilgiç and Furkan [8]. More recently the fine spectrum of the operator $B(r,s)$ over ℓ_p and bv_p has been studied by Bilgiç and Furkan [7]. In 2007; Furkan et al. [10] determined the spectra and the fine spectra of the operator $B(r,s,t)$ on the spaces c_0 and c. In the same year, Bilgiç and Furkan [8] determined the spectra and the fine spectra of the operator $B(r,s,t)$ on the spaces ℓ_1 and bv. In 2010, Srivastava and Kumar [19] determined the spectra and the fine spectra of generalized difference operator Δ_ν on the space ℓ_1. Recently, Furkan et al. [11] studied the spectra and the fine spectra with respect to the Goldberg's classification of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p, where bv_p denotes the space of all sequences (x_k) such that $(x_k - x_{k-1})$ in ℓ_p with $1 < p < \infty$ which is studied by Başar and Altay in [6]. In this paper, we have determined the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(r,s,t)$ over the sequence spaces c_0, c, ℓ_p and bv_p, where $1 \leq p < \infty$.

3 The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum of $B(r,s,t)$

In this section, we deal with the approximate point spectrum, defect spectrum and compression spectrum of $B(r,s,t)$ over the sequence spaces c_0, c, ℓ_p and bv_p, where $1 \leq p < \infty$.

For simplicity in notation, here and in what follows, we define the sets S and $\widetilde{S}$ by

$$S = \left\{ \lambda : \left| \frac{2(r-\lambda)}{-s+\sqrt{s^2-4t(r-\lambda)}} \right| \leq 1 \right\}$$

and

$$\widetilde{S} = \begin{cases} S \backslash \{r\} & , \quad |t| < |s|, \\ S & , \quad |t| \geq |s|, \end{cases}$$

respectively, and denote the set of all interior points of the set S by $Int\ S$, as usual; where s is a complex number such that $\sqrt{s^2} = -s$.

3.1 Subdivisions of the spectrum of $B(r,s,t)$ on c_0

In this subsection, we give the subdivisions of the spectrum of the operator $B(r,s,t)$ over the sequence space c_0.

Theorem 3.1 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t),c_0] = \widetilde{S}$.

(b) $\sigma_{\delta}[B(r,s,t),c_0] = S$.

(c) $\sigma_{co}[B(r,s,t),c_0] = Int\ S$.

Proof. (a) Since $\sigma_{ap}[B(r,s,t),c_0] = \sigma[B(r,s,t),c_0] \backslash III_1\sigma[B(r,s,t),c_0]$, one can derive by Theorems 2.1 and 2.7 of Furkan et al. [10] that $\sigma_{ap}[B(r,s,t),c_0] = \widetilde{S}$.

(b) Since we have

$$\sigma_{\delta}[B(r,s,t),c_0] = \sigma[B(r,s,t),c_0] \backslash I_3\sigma[B(r,s,t),c_0]$$

from Table 1.2 and

$$I_3\sigma[B(r,s,t),c_0] = II_3\sigma[B(r,s,t),c_0] = III_3\sigma[B(r,s,t),c_0] = \emptyset$$

is obtained by Theorem 2.2 of Furkan et al. [10], we obtain from Theorem 2.1 of Furkan et al. [10] that $\sigma_{\delta}[B(r,s,t),c_0] = \sigma[B(r,s,t),c_0]$.

(c) Since the equality

$$\sigma_{co}[B(r,s,t),c_0] = III_1\sigma[B(r,s,t),c_0] \cup III_2\sigma[B(r,s,t),c_0] \cup III_3\sigma[B(r,s,t),c_0]$$

holds from Table 1.2, the desired result is immediate by Theorems 2.1–2.8 of Furkan et al. [10].

The next corollary can be obtained from Proposition 2.1:

Corollary 3.2 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t)^*,\ell_1] = S$.

(b) $\sigma_{\delta}[B(r,s,t)^*,\ell_1] = \widetilde{S}$.

(c) [10, Theorem 2.3] $\sigma_p[B(r,s,t)^*,\ell_1] = Int\ S$.

3.2 Subdivisions of the spectrum of $B(r,s,t)$ on c

In the present subsection, we give the subdivisions of the spectrum of the operator $B(r,s,t)$ over the sequence space c.

Theorem 3.3 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t),c] = \widetilde{S}$.

(b) $\sigma_{\delta}[B(r,s,t),c] = S$.

(c) $\sigma_{co}[B(r,s,t),c] = Int\ S \cup \{r+s+t\}$.

Proof. (a) Table 1.2 gives that $\sigma_{ap}[B(r,s,t),c] = \sigma[B(r,s,t),c]\backslash III_1\sigma[B(r,s,t),c]$. Then, by Theorem 2.10 of Furkan et al. [10] that $\sigma_{ap}[B(r,s,t),c] = \widetilde{S}$.

(b) $\sigma_{\delta}[B(r,s,t),c] = \sigma[B(r,s,t),c]\backslash I_3\sigma[B(r,s,t),c]$ is obtained from Table 1.2. Moreover, since

$$\sigma_p[B(r,s,t),c] = I_3\sigma[B(r,s,t),c] \cup II_3\sigma[B(r,s,t),c] \cup III_3\sigma[B(r,s,t),c] = \emptyset,$$

$I_3\sigma[B(r,s,t),c] = \emptyset$ by the part (ii) of Theorem 2.10 of Furkan et al. [10]. Hence, we have $\sigma_{\delta}[B(r,s,t),c] = \sigma[B(r,s,t),c]$.

(c) From Table 1.2

$$\sigma_{co}[B(r,s,t),c] = III_1\sigma[B(r,s,t),c] \cup III_2\sigma[B(r,s,t),c] \cup III_3\sigma[B(r,s,t),c],$$

$$III_1\sigma[B(r,s,t),c] \cup III_2\sigma[B(r,s,t),c] = \sigma_r[B(r,s,t),c]$$

and $III_3\sigma[B(r,s,t),c] = \emptyset$ from the parts (ii) and (iii) of Theorem 2.10 of Furkan et al. [10], we have

$$\sigma_{co}[B(r,s,t),c] = Int\ S \cup \{r+s+t\}.$$

The next corollary is an immediate consequence of Proposition 2.1:

Corollary 3.4 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t)^*,\ell_1] = S$.

(b) $\sigma_{\delta}[B(r,s,t)^*,\ell_1] = \widetilde{S}$.

(c) [10, Theorem 2.9] $\sigma_{p}[B(r,s,t)^*,\ell_1] = Int\ S \cup \{r+s+t\}$.

3.3 Subdivisions of the spectrum of $B(r,s,t)$ on ℓ_p, $(1 \leq p < \infty)$.

In this subsection, we give the subdivisions of the spectrum of the operator $B(r,s,t)$ over the sequence space ℓ_p, where $1 \leq p < \infty$.

Theorem 3.5 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t),\ell_1] = \widetilde{S}$.

(b) $\sigma_{\delta}[B(r,s,t),\ell_1] = S$.

(c) $\sigma_{co}[B(r,s,t),\ell_1] = S$.

Proof. (a) Since $\sigma_{ap}[B(r,s,t),\ell_1] = \sigma[B(r,s,t),\ell_1]\backslash III_1\sigma[B(r,s,t),\ell_1]$, $\sigma_{ap}[B(r,s,t),\ell_1] = \widetilde{S}$ is obtained by Theorems 2.1 and 2.4 of Bilgiç and Furkan [8].

(b) Since $\sigma_{\delta}[B(r,s,t),\ell_1] = \sigma[B(r,s,t),\ell_1]\backslash I_3\sigma[B(r,s,t),\ell_1]$ from Table 1.2 and

$$I_3\sigma[B(r,s,t),\ell_1] = II_3\sigma[B(r,s,t),\ell_1] = III_3\sigma[B(r,s,t),\ell_1] = \emptyset$$

by Theorem 2.2 of Bilgiç and Furkan [8], we obtain from Theorem 2.1 of Bilgiç and Furkan [8] that $\sigma_{\delta}[B(r,s,t),\ell_1] = \sigma[B(r,s,t),\ell_1]$.

(c) Since the equality

$$\sigma_{co}[B(r,s,t),\ell_1] = III_1\sigma[B(r,s,t),\ell_1] \cup III_2\sigma[B(r,s,t),\ell_1] \cup III_3\sigma[B(r,s,t),\ell_1]$$

holds from Table 1.2, Theorems 2.2–2.4 of Bilgiç and Furkan [8] show that the compression spectrum of $B(r,s,t)$ over the sequence space ℓ_1 is the set S.

The following corollary is an easy consequence of Proposition 2.1:

Corollary 3.6 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t)^*,\ell_\infty] = S$.

(b) $\sigma_{\delta}[B(r,s,t)^*,\ell_\infty] = \widetilde{S}$.

(c) [8, Theorem 2.5] $\sigma_{p}[B(r,s,t)^*,\ell_\infty] = S$.

Theorem 3.7 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t),\ell_p] = \widetilde{S}$.

(b) $\sigma_{\delta}[B(r,s,t),\ell_p] = S$.

(c) $\sigma_{co}[B(r,s,t),\ell_p] = Int\ S$.

Proof. (a) Since $\sigma_{ap}[B(r,s,t),\ell_p] = \sigma[B(r,s,t),\ell_p]\backslash III_1\sigma[B(r,s,t),\ell_p]$, $\sigma_{ap}[B(r,s,t),\ell_p] = \widetilde{S}$ is obtained by Theorems 2.2 and 2.7 of Furkan et al. [11].

(b) Since $\sigma_{\delta}[B(r,s,t),\ell_p] = \sigma[B(r,s,t),\ell_p]\backslash I_3\sigma[B(r,s,t),\ell_p]$ from Table 1.2 and

$$I_3\sigma[B(r,s,t),\ell_p] = II_3\sigma[B(r,s,t),\ell_p] = III_3\sigma[B(r,s,t),\ell_p] = \emptyset$$

is observed by Theorem 2.3 of Furkan et al. [11] whose Theorem 2.2 gives that $\sigma_{\delta}[B(r,s,t),\ell_p] = \sigma[B(r,s,t),\ell_p]$.

(c) Since the equality

$$\sigma_{co}[B(r,s,t),\ell_p] = III_1\sigma[B(r,s,t),\ell_p] \cup III_2\sigma[B(r,s,t),\ell_p] \cup III_3\sigma[B(r,s,t),\ell_p]$$

holds from Table 1.2, the desired result can easily be seen by Theorems 2.2–2.7 of Furkan et al. [11].

The following corollary is a consequence of Proposition 2.1:

Corollary 3.8 *Let* $p^{-1} + q^{-1} = 1$, *then we have*

(a) $\sigma_{ap}[B(r,s,t)^*,\ell_q] = S$.

(b) $\sigma_{\delta}[B(r,s,t)^*,\ell_q] = \widetilde{S}$.

(c) [11, Theorem 2.4] $\sigma_p[B(r,s,t)^*,\ell_q] = Int\ S$.

3.4 Subdivisions of the spectrum of $B(r,s,t)$ on bv_p, $(1 \leq p < \infty)$

In the present subsection, we give the subdivisions of the spectrum of the operator $B(r,s,t)$ over the sequence space bv_p. Since the subdivisions of the spectrum of the operator $B(r,s,t)$ on the space bv_p can be derived by analogy to the space ℓ_p, we omit the detail and give the concerning results without proof.

Theorem 3.9 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t),bv] = \widetilde{S}$.

(b) $\sigma_\delta[B(r,s,t), bv] = S$.

(c) $\sigma_{co}[B(r,s,t), bv] = S$.

As a consequence of Proposition 2.1, we also have the following:

Corollary 3.10 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t)^*, bv^*] = S$.

(b) $\sigma_\delta[B(r,s,t)^*, bv^*] = \widetilde{S}$.

(c) [8, Theorem 2.8.(i)] $\sigma_p[B(r,s,t)^*, bv^*] = S$.

Theorem 3.11 *The following results hold:*

(a) $\sigma_{ap}[B(r,s,t), bv_p] = \widetilde{S}$.

(b) $\sigma_\delta[B(r,s,t), bv_p] = S$.

(c) $\sigma_{co}[B(r,s,t), bv_p] = Int\ S$.

The following corollary is derived from Proposition 2.1:

Corollary 3.12 *Let* $p^{-1} + q^{-1} = 1$. *Then, we have:*

(a) $\sigma_{ap}\left(B\left(r,s,t\right)^*, bv_p^*\right) = S$.

(b) $\sigma_\delta[B(r,s,t)^*, bv_p^*] = \widetilde{S}$.

(c) [11, Theorem 3.3] $\sigma_p[B(r,s,t)^*, bv_p^*] = Int\ S$.

Conclusion

There is a wide literature related with the spectrum and fine spectrum of certain linear operators represented by particular limitation matrices over some sequence spaces. Although the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces c_0, c and ℓ_p, bv_p with $1 < p < \infty$ were respectively studied by Furkan et al. [10] and [11], in the present paper, the concepts of the approximate point spectrum, defect spectrum and compression spectrum are introduced, and given the subdivisions of the spectrum of the operator $B(r,s,t)$ over the sequence spaces c_0, c, ℓ_p and bv_p, as the new subdivisions of spectrum, where $1 \leq p < \infty$. This is a new development of the spectrum of an infinite matrix over a sequence space. Following the same way, it is natural that one can derive some new results, on the subdivisions of the spectrum of $B(r,s,t)$ or other particular limitation matrices, for example the double band matrix Δ_ν defined by a strictly decreasing sequence $\nu = (\nu_k)$ of positive real numbers satisfying certain conditions, over the spaces which do not consider here, from the known results via Table 1.2, in the usual sense.

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