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Unique Fixed Point Theorem for Weakly S-Contractive Mappings

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Abstract

In this paper, we have unique fixed point theorem using S-contractive mappings in complete metric space. We supported our result by some examples.

Keywords: *Complete metric space, Fixed point, Weak S-contraction.*

1 Introduction

It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping $T:X\rightarrow X$ where (X,d) is a metric space, is said to be a contraction if there exist $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq k\, d(x, y) \text{ for all } x, y, \in X \qquad (1.l)$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point which established by Banach (1922). The contractive definition (1.1) implies that. T is uniformly continuous. It is natural to ask if there is contractive definition which do not force T to be continuous. It was answered in affirmative by Kannan [5] who establish a fixed point theorem for mapping satisfying.

$$d(Tx, Ty) \leq k\,[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

for all x, y ∈ x and $0 \leq k < ½$

The mapping satisfying (1.2) are called Kannan type mapping. It is clear that contractions are always continuous and Kannan mapping are not necessarily continuous.

There is a large literature dealing with Kannan type mapping and generalization some of which are noted in [2, 4, 6, 7]

A similar contractive condition has been introduced by Shukla's we call this contraction a S-contraction.

Definition 1.1. S-contraction

Let $T : X \to X$ where (X, d) is a complete metric space is called a S-contraction if there exist $0 \leq k < 1/3$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq k\,[d(x, Ty) + d(Tx, y) + d(x, y)] \tag{1.3}$$

A weaker contraction has been introduced in Hilbert spaces in [1].

Definition 1.2. Weakly contractive mapping

A mapping $T: X \to X$ where (X,d) is a complete metric space is said to be weakly contractive [3] if

$$d(Tx, Ty) \leq d(x, y) - \psi\,[d(x, y)] \tag{1.4}$$

where $x, y \in X$, $\psi : [0, \infty) \to [0, \infty)$ is continuous and non decreasing

$$\psi(x) = 0 \text{ iff } x = 0 \text{ and } \lim_{x\to\infty} \psi(x) = \infty$$

If we take $\psi(x) = kx$ where $0 \leq k < 1$ then (1.4) reduces to (1.1)

Definition1.3. Weak S-contraction

A mapping $T : X \to X$ where (X, d) is a complete metric space is said to be weakly S-contractive or a weak S-Contraction if for all $x, y \in X$ such that

$$d(Tx, Ty) \leq 1/3[d(x, Ty)+d(Tx, y)+d(x,y)] - \psi\,[d(x, Ty),d(Tx, y),d(x,y)] \tag{1.5}$$

where $\psi : [0, \infty)^3 \to [0, \infty)$ is a continuous mapping such that

$\psi(x, y, z) = 0$ *iff* $x = y = z = 0$ *and* $\lim_{x\to\infty}\psi(x) = \infty$

If we take $\psi(x, y, z) = k(x + y + z)$ *where* $0 \le k < 1/3$ *then (1.5) reduces to (1.3). i.e. weak S-contractions are generalization of S-Contraction. The next section we established that in a complete metric space a weak S-contraction has a unique fixed point. At the end of the next section we supported some examples.*

2 Main Results

Theorem 2.1. *Let* $T : X \to X$*, where* (X, d) *is a complete metric space be a weak S-contraction. Then T has a unique fixed point.*

Proof. Let $x_o \in X$ and $n \ge 1$, $x_{n+1} = Tx_n$. (2.1)
If $x_n = x_{n+l} = Tx_n$
then x_n is a fixed point of T.
So we assume $x_n \ne x_{n+1}$
Putting $x = x_{n-l}$ and $y = x_n$ in (1.5) we have for all n = 0, 1,2, ...

$d(x_n, x_{n+l}) = d(Tx_{n-l}, Tx_n)$
$\le 1/3\,[d(x_{n-l}, Tx_n) + d(Tx_{n-1}, x_n) + d(x_{n-l}, x_n)]$
$-\ \psi\,[d(x_{n-l}, Tx_n), d(Tx_{n-1}, x_n), d(x_{n-l}, x_n)]$
$= 1/3\,[d(x_{n-1}, x_{n+1}) + d(x_n, x_n) + d(x_{n-l}, x_n)]$
$-\ \psi\,[d(x_{n-l}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_n)]$
$= 1/3\,[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)] - \psi\,[d(x_{n-l}, x_{n+1}), 0, d(x_{n-1}, x_n)]$
$\le 1/3\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-l}, x_n)]$
$-\ \psi\,[d(x_{n-l}, x_n) + d(x_n, x_{n+1}), 0, d(x_{n-1}, x_n)]$ (2.2)
$2/3\, d(x_n, x_{n+1}) \le 2/3\, d(x_{n-1}, x_n) - \psi\,[d(x_{n-1}, x_n) + d(x_n, x_{n+1}), d(x_{n-1}, x_n)]$
$\le 2/3\, d(x_{n-1}, x_n)$
$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)$ (2.3)

i.e. $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of (2.3) decreasing sequence of non-negative real numbers and hence is convergent.
i.e. $\lim_{n\to\infty} d(x_n, x_{n+1})$ is exist.
let $d(x_n, x_{n+1}) \to r$ as $n \to \infty$ (2.4)
We next prove that r = 0.

$d(x_n, x_{n+l}) = d(Tx_{n-l}, Tx_n)$
$\le 1/3\,[d(x_{n-l}, Tx_n) + d(Tx_{n-l}, x_n) + d(x_{n-l}, x_n)]$
$-\ \psi\,[d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n), d(x_{n-1}, x_n)]$
$= 1/3\,[d(x_{n-l}, x_{n+1}) + d(x_n, x_n) + d(x_{n-l}, x_n)]$
$-\ \psi\,[d(x_{n-1}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_n)]$
$\le 1/3\,[d(x_{n-l}, x_{n+1}) + d(x_{n-1}, x_n)]$ (2.5)

taking $n \to \infty$ in (2.5) we have by (2.4).

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \leq 1/3 \lim_{n\to\infty} [d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]$$
$$r \leq 1/3 [\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) + r]$$
$$2r \leq \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \qquad (2.6)$$

Since $d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$
taking limit as $n \to \infty$ in above we have by (2.4)

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) \leq 2r \qquad (2.7)$$

from (2.6) and (2.7)

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r$$

Again taking $n \to \infty$ in (2.2)

$$\lim_{n\to\infty} d(x_n, x_{n-1}) \leq 1/3 [\lim_{n\to\infty} d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$$
$$- \psi [\lim_{n\to\infty} \{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, \lim_{n\to\infty} d(x_{n-1}, x_n)]$$
$$r \leq 1/3 [r + r + r] - \psi (2r, r, 0)$$
$$r \leq r - \psi (2r, r, 0)$$

or $\psi (2r, r, 0) \leq 0$ which is contraction unless $r = 0$
Thus we have established that

$$d(x_n, x_{n+l}) \to 0 \text{ as } n \to \infty \qquad (2.9)$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, then there exist $\in > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers 'k',

$$n(k) > m(k) > k,$$
$$d(x_{m(k)}, x_{n(k)}) \geq \in \qquad (2.10)$$

and $d(x_{m(k)}, x_{n(k)-1}) < \in$ (2.11)

Then,

$$\in \leq d(x_{m(k)}, x_{n(k)}) = d(Tx_{m(k)-l}\ Tx_{n(k)-l})$$
$$\leq 1/3 [d(x_{m(k)-l}, Tx_{n(k)-l}) + d (Tx_{m(k)-l}, x_{n(k)-l}) +(d(x_{m(k)-l}, x_{n(k)-l})]$$
$$- \psi [d(x_{m(k)-l}, Tx_{n(k)-l}) + d(Tx_{m(k)-l}, x_{n(k)-l}), d(x_{m(k)-l}, x_{n(k)-l})]$$
$$= 1/3 [d(x_{m(k)-l}, x_{n(k)}) + d (x_{m(k)}, x_{n(k)-l}) +(d(x_{m(k)-l}, x_{n(k)-l})]$$

$$- \psi\, [d(x_{m(k)-l}, x_{n(k)}), d(x_{m(k)}, x_{n(k)-l}), d(x_{m(k)-l}, x_{n(k)-l})] \quad (2.12)$$

Again

$$\in \le d(x_{m(k)}, x_{n(k)})$$
$$\le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-l}, x_{n(k)}) \text{ by (2.11)},$$
$$\le \in + d(x_{n(k)-1}, x_{n(k)})$$

taking $k \to \infty$ is a above inequality and using (2.9) we obtain

$$\in \le \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \le \in$$

and

$$\in \le \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) + \lim_{k\to\infty} d(x_{n(k)-l}, x_{n(k)}) \le \in$$

we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \in \quad (2.13)$$

And

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \in \quad (2.14)$$

Similarly

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \in \quad (2.15)$$

taking $k \to \infty$ in (2.12) and using (2.9), (2.13), (2.14) and (2.15) we obtain

$$\in \le 1/3\, [\in + \in + \in] - \psi\, (\in, \in, \in)$$
$$\in \le \in - \psi\, (\in, \in, \in)$$
$$\psi\, (\in, \in, \in) \le 0 \text{ which is contraction since } \in > 0$$

Hence $\{x_n\}$ is a Cauchy sequence and therefore is convergent in the complete metric space (X,d)

Let $x_n \to z$ and $n \to \infty$. (2.16)

Then

$$d(z, Tz) \le d\,(z, x_{n+1}) + d\,(x_{n+1}, Tz)$$
$$= d(z, x_{n+1}) + d(Tx_n, Tz).$$
$$\le d\,(z, x_{n+1}) + 1/3\, [d\,(x_n, Tz) + d\,(Tx_n, z) + d(x_n, z)]$$
$$- \psi\, [d\,(x_n, Tz), d\,(Tx_n, z), d(x_n, z)]$$
$$= d\,(z, Tx_n) + 1/3\, [d\,(x_n, Tz) + d\,(Tx_n, z) + d(x_n, z)]$$
$$- \psi\, [d\,(x_n, Tz), d\,(Tx_n, z), d(x_n, z)]$$
$$= d\,(z, Tz) + 1/3\, [d\,(z, Tz) + d\,(Tz, z) + d(z, z)]$$
$$- \psi\, (d\,(z, Tz), d\,(Tz, z), d(z, z)]$$
$$= 2d\,(z, Tz) - \psi\, (d\,(z, Tz), d\,(Tz, z), d(z, z))$$
$$< 2d\,(z, Tz)$$
$$- d\,(z, Tz) < 0$$
$$d(z, Tz) \ge 0$$

Hence $Tz = z$

Next we establish that the fixed point z is unique.
Let z_1 and z_2 be two fixed points of T,
then

$$d(z_1, z_2) = d(Tz_1, Tz_2)$$
$$\leq 1/3\,[d(z_1, Tz_2) + d(Tz_1, z_2) + d(z_1, z_2)] - \psi(d(Tz_1, z_2), d(z_1, Tz_2), d(z_1, z_2))$$

i.e.

$$d(z_1, z_2) \leq d(z_1, z_2) - \psi\, d(z_1, z_2), d(z_1, z_2), d(z_1, z_2))$$

which by property of ψ is a contradiction unless $d(z_l, z_2) = 0$, that is $z_1 = z_2$. Hence fixed point is unique in S-contraction.
consider the following example

Example 2.1. *Let x = {p, q, r,} and d is a metric defined on X as follows.*

(i)	*d(p, q) = 2*	*d(q, r) = 4*	*d(r, p) = 3*
and	*T(p) = q*	*T(q) = q*	*T(r) = p*
(ii)	*d(q, r) = 2*	*d(r, p) = 4*	*d(p,q) = 3*
	T(q) = r	*T(r) = r*	*T(p) = q*
(iii)	*d(r, p) = 2*	*d(p, q) = 4*	*d(q, r) = 3*
	T(r) = p	*T(p) = p*	*T(q) = r*

where T: $X \rightarrow x$ is mapping defined as (i) (ii) and (iii) respectively
Then (X, d) is a complete metric space.
Let $\psi(a, b, c) = 1/3 \min\{a, b, c\}$
Then T is a weak S-contraction and conditions of theorem are satisfied. Hence T must have a unique fixed point.
It is clear that q, r and p are fixed point of T
Corresponding mapping of T.
and if x replace p or q and y replace r then inequality. (1.3) does not holds by definition of T in (i)
Similarly x replace q and r and y replace p then inequality (1.3) does not holds by definition of T in (ii)
and x replace r and p and y replace q then inequality (1.3) does not holds by definition of T in (iii)

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