Gen. Math. Notes, Vol. 3, No. 1, March 2011, pp. 26-31
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Recurrence Relations for Single and Product Moments of Lower Record Values from Modified-Inverse Weibull Distribution

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(Received: 26-11-10/ Accepted: 25-1-11)

Abstract

In this study we give recurrence relations of single and product moments of lower record values from modified-inverse Weibull distribution.

Keywords: *Record, single moments, product moments, recurrence relations, modified-inverse Weibull distribution.*

1 Introduction

The model of record statistics defined by Chandler [9] as a model for successive extremes in a sequence of independent and identically distributed (iid) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These recorded voltages are the lower record value sequence.

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with cumulative density function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = \max$ $(\min)\{X_1, X_2, \ldots, X_n\}$, $n = 1,2,\ldots$. We say X_j is an lower record value of this sequence if $Y_j > (<)\ Y_{j-1}$, $j \geq 2$. By definition, X_1 is an upper as well as a lower record value The indices at which the lower record values occur are given by record times $\{L_{(n)}, n \geq 1\}$, when $L_{(n)} = \min\{j \mid j > L_{(n-1)}, X_j > L_{(n-1)}\}$, $n \geq 2$ with $L_{(1)} = 1$ (Ahsanullah, [1]).

The theory of record values arising from a sequence of *iid* continuous random variables and has now spread in various directions. Interested readers may refer to the works Glick [10], Nevzorov ([12], [13]), Resnick [16], Arnold and Balakrishnan [2] and Arnold *et al.* ([3], [4]).

We shall denote

$$\mu_{L(n)}^{(r)} = E(X_{L(n)}^{r}), \quad r, n = 1, 2, \ldots,$$

$$\mu_{L(m,n)}^{(r,s)} = E(X_{L(m)}^{r} X_{L(n)}^{s}), \quad 1 \leq m \leq n-1 \text{ and } r, s = 1, 2, \ldots,$$

$$\mu_{L(m,n)}^{(r,0)} = E(X_{L(m)}^{r}) = \mu_m^{(r)}, \quad 1 \leq m \leq n-1 \text{ and } r = 1, 2, \ldots,$$

$$\mu_{L(m,n)}^{(0,s)} = E(X_{L(n)}^{s}) = \mu_n^{(s)}, \quad 1 \leq m \leq n-1 \text{ and } s = 1, 2, \ldots.$$

Let $\{X_n, n \geq 1\}$ be the sequence of lower record values from (1.1). Then the pdf of $X_{L(n)}$, $n \geq 1$ is given by

$$f_n(x) = \frac{1}{(n-1)!}[-\ln(F(x))]^{n-1} f(x) \tag{1.1}$$

and the joint pdf of X_m and X_n, $1 \leq m < n$, $n > 2$ is given by

$$f_{m,n}(x, y) = \frac{1}{(m-1)!(n-m-1)!}[-\ln(F(x))]^{m-1}$$

$$\times[-\ln(F(y)) + \ln(F(x))]^{n-m-1} \frac{f(x)}{F(x)} f(y), \quad x < y. \tag{1.2}$$

Recurrence relations for single and product moments of record values from generalized Pareto, lomax, exponential and generalized extreme value distribution are derived by Balakrishnan and Ahsanullah ([5], [6], and [7]) and Balakrishnan *et al.* [9] respectively. Pawlas and Szynal ([14], [15]) and Saran and Singh [17] have established recurrence relations for single and product moments of $k-$th record values from Weibull, Gumbel and linear exponential distribution.

A random variable X is said to have modified-inverse weibull distribution if its pdf is of the form

$$f(x) = \alpha(\beta + \lambda x)\, x^{-(\beta+1)}\, e^{-\lambda x}\, e^{-\alpha x^{-\beta} e^{-\lambda x}}, \quad x \geq 0,\ \alpha, \beta, \lambda > 0 \tag{1.3}$$

and the corresponding df is

$$F(x) = e^{-\alpha x^{-\beta} e^{-\lambda x}}, \quad x \geq 0,\ \alpha, \beta, \lambda > 0. \tag{1.4}$$

The inverse Weibull distribution plays an important role in many applications, including the dynamic components of diesel engines and several dataset such as the times to breakdown of an insulating fluid subject to the action of constant tension. More details on the inverse Weibull distribution see Murty *et al.* [11].

In the present study, we established some recurrence relations satisfied by the single and product moments of lower record values from modified-inverse Weibull distribution.

2 Recurrence Relations for Single Moments

Note that for modified-inverse Weibull distribution defined in (1.3)

$$f(x) = \left(\frac{\beta + \lambda x}{x}\right)[-\ln(F(x))]F(x). \tag{2.1}$$

The relation in (2.1) will be exploited in this paper to derive recurrence relations for the moments of record values from the modified-inverse Weibull distribution.

Recurrence relations for single moments of lower record values from df (1.4) can be derived in the following theorem.

Theorem 2.1 *For a* $n \geq 1$ *and* $r = 0, 1, 2, \ldots,$

$$\mu_{L(n)}^{(r+\alpha)} = \frac{n\beta}{r}\left(\mu_{L(n)}^{(r)} - \mu_{L(n+1)}^{(r)}\right) + \frac{n\lambda}{(r+1)}\left(\mu_{L(n)}^{(r+1)} - \mu_{L(n+1)}^{(r+1)}\right). \tag{2.2}$$

Proof. We have from equations (1.1) and (2.1)

$$\mu_{L(n)}^{(r)} = \frac{1}{(n-1)!}[\beta I_1 + \lambda I_2], \tag{2.3}$$

Where

$$I_1 = \int_0^\infty x^{r-1}[-\ln(F(x))]^n F(x)\,dx\,. \tag{2.4}$$

Integrating by parts treating x^{r-1} for integration and the rest of the integrand for differentiation we obtain

$$I_1 = \frac{\Gamma(n+1)}{r}\mu_{L(n)}^{(r)} - \frac{\Gamma(n+1)}{r}\mu_{L(n+1)}^{(r)}, \tag{2.5}$$

Similarly, we write

$$I_2 = \frac{\Gamma(n+1)}{r+1}\mu_{L(n)}^{(r)} - \frac{\Gamma(n+1)}{r+1}\mu_{L(n+1)}^{(r)}\,. \tag{2.6}$$

Now substituting for I_1, I_2 in equation (2.3) and simplifying the resulting expression, we derive the relations in (2.2).

3 Recurrence Relations for Product Moments

Making use of (2.1), we can drive recurrence relations for product moments of lower record values

Theorem 3.1 *For* $m \ge 1$ *and* $r, s = 0,1,2,\ldots,$

$$\mu_{L(m,m+1)}^{(r,s)} = \frac{m\beta}{r}\left(\mu_{L(m,m+1)}^{(r,s)} - \mu_{L(m+1)}^{(r+s)}\right) + \frac{m\lambda}{(r+1)}\left(\mu_{L(m,m+1)}^{(r+1,s)} - \mu_{L(m+1)}^{(r+s+1)}\right) \tag{3.1}$$

and for $1 \le m \le n-2$ *and* $r, s = 0,1,2,\ldots,$

$$\mu_{L(m,n)}^{(r,s)} = \frac{m\beta}{r}\left(\mu_{L(m,n)}^{(r,s)} - \mu_{L(m+1,n)}^{(r,s)}\right) + \frac{m\lambda}{(r+1)}\left(\mu_{L(m,n)}^{(r+1,s)} - \mu_{L(m+1,n)}^{(r+1,s)}\right). \tag{3.2}$$

Proof. From equation (1.2) for $1 \le m \le n-1$, $r, s = 0,1,2,\ldots$ and on using equation (2.1), we get

$$\mu_{L(m,n)}^{(r,s)} = \frac{1}{(m-1)!(n-m-1)!}\int_0^\infty y^s f(y) I(y)\,dy\,, \tag{3.3}$$

where

$$I(y) = \int_y^\infty x^r [-\ln(F(x))]^{m-1}[\ln(F(x)) - \ln(F(y))]^{n-m-1}\frac{f(x)}{F(x)}dx\,. \tag{3.4}$$

For $n = m+1$

$$I(y) = \beta I_3 + \lambda I_4 \tag{3.5}$$

where

$$I_3 = \int_y^\infty x^{r-1}[-\ln(F(x))]^m \, dx .$$

Integrating by parts we get

$$I_3 = -\frac{y^r}{r}[-\ln(F(y))]^m + \frac{m}{r}\int_y^\infty x^r[-\ln(F(x))]^{m-1}\frac{f(x)}{F(x)}dx . \tag{3.6}$$

Similarly, we write

$$I_4 = -\frac{y^{r+1}}{(r+1)}[-\ln(F(y))]^m + \frac{m}{(r+1)}\int_y^\infty x^{r+1}[-\ln(F(x))]^{m-1}\frac{f(x)}{F(x)}dx . \tag{3.7}$$

Now substituting for I_3, I_4 in equation (3.5) and simplifying the resulting expression, we derive the relations in (3.1).

For $1 \le m \le n-1$ and making use of (2.1) we write (3.4) as

$$I(y) = \beta I_5 + \lambda I_6 \ , \tag{3.8}$$

where

$$I_5 = \int_y^\infty x^{r-1}[-\ln\{F(x)\}]^m [\ln(F(x)) - \ln(F(y))]^{n-m-1} dx .$$

Integrating by parts we get,

$$I_5 = \frac{m}{r}\int_y^\infty x^r[-\ln\{F(x)\}]^{m-1}[\ln(F(x)) - \ln(F(y))]^{n-m-1}\frac{f(x)}{F(x)}dx$$
$$-\frac{(n-m-1)}{r}\int_y^\infty x^r[-\ln\{F(x)\}]^m[\ln(F(x)) - \ln(F(y))]^{n-m-2}\frac{f(x)}{F(x)}dx \tag{3.9}$$

and

$$I_6 = \frac{m}{(r+1)}\int_y^\infty x^{r+1}[-\ln\{F(x)\}]^{m-1}[\ln(F(x)) - \ln(F(y))]^{n-m-1}\frac{f(x)}{F(x)}dx$$
$$-\frac{(n-m-1)}{(r+1)}\int_y^\infty x^{r+1}[-\ln\{F(x)\}]^m[\ln(F(x)) - \ln(F(y))]^{n-m-2}\frac{f(x)}{F(x)}dx . \tag{3.10}$$

Now substituting for I_5, I_6 in equation (3.8) and simplifying the resulting expression, we derive the relations in (3.2).

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