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A Fixed Point Theorem for Six Mappings in Metric Space

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Abstract

In present paper we prove a common fixed point theorem for six mappings using compatibility, weak compatibility and commutativity. Our results improve one of the result of Imdad and Khan [3], fisher [2].

Keywords: *Fixed Points, Metric Space, Weak Compatibility and Commutativity, Coincidence Point.*

2 Preliminaries

Before starting the result first we discuss some definitions.

Definition 2.1 [15]. *A pair of self-mapping* (A,B) *on a metric space* (X,d) *is said to be weakly commuting if* $d(ABx,BAx) \le d(Bx,Ax)$. For all $x \in X$.

Definition 2.2. *A pair of self mapping* (A,B) *of a metric space* (X,d) *is said to be compatible if* $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$

Whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t,$ for all $t \in X$.

Theorem 2.1 [2]. *Let S and T be two self-mapping of a complete metric space* (X,d) *such that for all x, y in X either*

(i) $$d(Sx,Ty) \le \frac{b[d(x,Ty)]^2 + c[d(y,Sx)]^2}{d(x,Ty)+d(y,Sx)}$$

if $d(x,Ty)+d(y,Sx) \neq 0, \quad 0 \le b,c,\ b+c<1$

(ii) $d(Sx,Ty)=0$ if $d(x,Ty)+d(y,Sx)=0$

if one of S or T is continuous then S and T have a unique common fixed point.

3 Main Result

Theorem 3.1. *Let* F,G,H,R,S and T *be self mappings of a complete metric space* (X,d) *satisfying the conditions:*

$FG(X) \subset T(X), HR(X) \subset S(X)$ *and for each* $x,y \in X$ *either*

$$(3.1.1)\ d(FGx,HRy) \le \alpha \frac{\left[d(FGx,Ty)^2 + d(HRy,Sx)^2\right]}{\left[d(FGx,Ty)+d(HRy,Sx)\right]}$$

$$+\beta \frac{\left[d(FGx,Ty)+d(HRy,Sx)+d(FGx,Sx)+d(HRy,Ty)\right]}{1+\left[d(FGx,Ty).d(HRy,Sx).d(FGx,Sx).d(HRy,Ty)\right]}$$

$$+\gamma\left[d(FGx,Sx)+d(HRy,Ty)\right]+\delta d(Sx,Ty)$$

if $d(FGx,Ty)+d(HRy,Sx) \neq 0,\ \alpha,\beta,\gamma,\delta \ge 0$ and $2\alpha+4\beta+2\gamma+\delta<1$

or

$(3.1.2)$ $d(FGx, HRy) = 0$ if $d(FGx, Ty) + d(HRy, Sx) = 0$ *if either*

(i) (FG, S) *are compatible, S or FG is continuous and* (HR, T) *are weakly compatible or*

(ii) (HR, T) *are compatible, T or HR is continuous and* (FG, S) *are weakly compatible, then* FG, HR, S and T *have a unique common fixed point. Further more if the pairs* $(F, G), (F, S), (G, S), (H, R), (H, T)$ and (R, T) *are commuting mappings then* F, G, H, R, S and T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $GF(X) \subset T(X)$, we can find a point x_1 in X such that $FGx_0 = Tx_1$. Also since $HR(X) \subset S(X)$ we can choose a point x_2 such that $HRx_1 = Sx_2$. Using this argument repeatedly one can constract a sequence $\{p_n\}$ such that

$$p_{2n} = FGx_{2n} = Tx_{2n+1},\ p_{2n+1} = HRx_{2n+1} = Sx_{2n+2},\ \text{for } n = 0, 1, 2, 3, ...$$

$$d(p_{2n+1}, p_{2n+2}) = d(HRx_{2n+1}, GFx_{2n+2})$$

$$\leq \alpha \frac{\left[d(FGx_{2n+2}, Tx_{2n+1})^2 + d(HRx_{2n+1}, Sx_{2n+2})^2\right]}{\left[d(FGx_{2n+2}, Tx_{2n+1}) + d(HRx_{2n+1}, Sx_{2n+2})\right]}$$

$$+\beta \frac{\left[d(FGx_{2n+2}, Tx_{2n+1}) + d(HRx_{2n+1}, Sx_{2n+2}) + d(FGx_{2n+2}, Sx_{2n+2}) + d(HRx_{2n+1}, Tx_{2n+1})\right]}{1 + \left[d(FGx_{2n+2}, Tx_{2n+1}).d(HRx_{2n+1}, Sx_{2n+2}).d(FGx_{2n+2}, Sx_{2n+2}).d(HRx_{2n+1}, Tx_{2n+1})\right]}$$

$$+\gamma\left[d(FGx_{2n+2}, Sx_{2n+2}) + d(HRx_{2n+1}, Tx_{2n+1})\right] + \delta d(Sx_{2n+2}, Tx_{2n+1})$$

$$\leq \alpha\left[d(p_{2n+2}, p_{2n+1}) + d(p_{2n+1}, p_{2n})\right]$$

$$+2\beta\left[d(p_{2n+2}, p_{2n+1}) + d(p_{2n+1}, p_{2n})\right]$$

$$+\gamma\left[d(p_{2n+2}, p_{2n+1}) + d(p_{2n+1}, p_{2n})\right] + \delta d(p_{2n+1}, p_{2n})$$

$$\leq (\alpha + 2\beta + \gamma) d(p_{2n+2}, p_{2n+1}) + (\alpha + 2\beta + \gamma + \delta) d(p_{2n+1}, p_{2n})$$

$$d\left(p_{2n+2}, p_{2n+1}\right) \le \left\{\frac{\alpha+2\beta+\gamma+\delta}{1-(\alpha+2\beta+\gamma)}\right\} d\left(p_{2n+1}, p_{2n}\right)$$

$$d\left(p_{2n+2}, p_{2n+1}\right) \le k\, d\left(p_{2n+1}, p_{2n}\right), \qquad \text{where } k=\frac{\alpha+2\beta+\gamma+\delta}{1-(\alpha+2\beta+\gamma)}<1$$

Similarly, $d\left(p_{2n}, p_{2n+1}\right) \le k\, d\left(p_{2n-1}, p_{2n}\right)$

Thus for every *n*, we have

(3.1.3) $d\left(p_n, p_{n+1}\right) \le k\, d\left(p_{n-1}, p_n\right)$

Which shows that $\{p_n\}$ is a Cauchy Sequence in the complete metric space (X,d). Hence the sequence $FGx_n = Tx_{2n+1}$ and $HRx_{2n+1} = Sx_{2n+2}$ which are subsequences also converges to the point *p*.

Let us now assume that *S* is continuous so that the sequences $\{S^2x_{2n}\}$ and $\{SFGx_{2n}\}$ converge to *Sp*. Also the compatibility of $\{S, FG\}, \{FGSx_{2n}\}$ converges to *Sp*.

Now

$$d\left(FGSx_{2n}, HRx_{2n+1}\right) \le \alpha \frac{\left[d\left(FGSx_{2n}, Tx_{2n+1}\right)^2 + d\left(HRx_{2n+1}, S^2x_{2n}\right)^2\right]}{\left[d\left(FGSx_{2n}, Tx_{2n+1}\right) + d\left(HRx_{2n+1}, S^2x_{2n}\right)\right]}$$

$$+\beta \frac{\left[\begin{array}{l} d\left(FGSx_{2n}, Tx_{2n+1}\right) + d\left(HRx_{2n+1}, S^2x_{2n}\right) + d\left(FGSx_{2n}, S^2x_{2n}\right) \\ + d\left(HRx_{2n+1}, Tx_{2n+1}\right)\end{array}\right]}{1+\left[\begin{array}{l} d\left(FGSx_{2n}, Tx_{2n+1}\right).d\left(HRx_{2n+1}, S^2x_{2n}\right).d\left(FGSx_{2n}, S^2x_{2n}\right) \\ .d\left(HRx_{2n+1}, Tx_{2n+1}\right)\end{array}\right]}$$

$$+\gamma\left[d\left(FGSx_{2n}, S^2x_{2n}\right) + d\left(HRx_{2n+1}, Tx_{2n+1}\right)\right] + \delta\, d\left(S^2x_{2n}, Tx_{2n+1}\right)$$

which on letting $n \to \infty$, we have

$$d(SP,P) \le \alpha \frac{\left[d(Sp,p)^2 + d(p,Sp)^2\right]}{\left[d(SP,P)+d(p,SP)\right]} + \beta \frac{\left[d(SP,P)+d(P,SP)+d(SP,SP)+d(P,p)\right]}{1+\left[d(SP,P).d(P,SP).d(SP,SP).d(P,p)\right]}$$

$$+\gamma\left[d(SP,SP)+d(P,P)\right]+\delta\, d(SP,P)$$

$$\le \alpha\, d(SP,P) + 2\beta\, d(SP,P) + \delta\, d(SP,P)$$

$$\le (\alpha+2\beta+\delta)\, d(SP,P)$$

Which is possible only when $SP = P$, since $\alpha+2\beta+\delta<1$

Now

$$d\left(FGP,HRx_{2n+1}\right)\le\alpha\frac{\left[d\left(FGP,Tx_{2n+1}\right)^2+d\left(HRx_{2n+1},SP\right)^2\right]}{\left[d\left(FGP,Tx_{2n+1}\right)+d\left(HRx_{2n+1},SP\right)\right]}$$

$$+\beta\frac{\left[d\left(FGP,Tx_{2n+1}\right)+d\left(HRx_{2n+1},SP\right)+d\left(FGP,SP\right)+d\left(HRx_{2n+1},Tx_{2n+1}\right)\right]}{1+\left[d\left(FGP,Tx_{2n+1}\right).d\left(HRx_{2n+1},SP\right).d\left(FGP,SP\right).d\left(HRx_{2n+1},Tx_{2n+1}\right)\right]}$$

$$+\gamma\left[d\left(FGP,SP\right)+d\left(HRx_{2n+1},Tx_{2n+1}\right)\right]+\delta d\left(SP,Tx_{2n+1}\right)$$

On letting $n\to\infty$ and using $SP=P$, we obtain

$$d\left(FGP,P\right)\le\alpha\frac{\left[d\left(FGP,P\right)^2+d\left(P,P\right)^2\right]}{\left[d\left(FGP,P\right)+d\left(P,P\right)\right]}$$

$$+\beta\frac{\left[d\left(FGP,P\right)+d\left(P,P\right)+d\left(FGP,P\right)+d\left(P,P\right)\right]}{1+\left[d\left(FGP,P\right).d\left(P,P\right).d\left(FGP,P\right).d\left(P,P\right)\right]}$$

$$+\gamma\left[d\left(FGP,P\right)+d\left(P,P\right)\right]+\delta d\left(P,P\right)$$

$$\le\alpha d\left(FGP,P\right)+2\beta d\left(FGP,P\right)+\gamma d\left(FGP,P\right)$$

$$\le\left(\alpha+2\beta+\gamma\right)d\left(FGP,P\right)$$

which is possible only when $FGP=P$, since $\alpha+2\beta+\gamma<1$
Since $FG\left(X\right)\subset T\left(X\right)$, a point q Then $Tq=P$ so $HRP=HR\left(Tq\right)$ Now

$$d\left(P,HRq\right)=d\left(FGP,HRq\right)$$

$$\le\alpha\frac{\left[d\left(FGP,Tq\right)^2+d\left(HRq,SP\right)^2\right]}{\left[d\left(FGP,Tq\right)+d\left(HRq,SP\right)\right]}$$

$$+\beta\frac{\left[d\left(FGP,Tq\right)+d\left(HRq,SP\right)+d\left(FGP,SP\right)+d\left(HRq,Tq\right)\right]}{1+\left[d\left(FGP,Tq\right).d\left(HRq,SP\right).d\left(FGP,SP\right).d\left(HRq,Tq\right)\right]}$$

$$+\gamma\left[d\left(FGP,SP\right)+d\left(HRq,Tq\right)\right]+\delta\ d\left(SP,Tq\right)$$

$$\leq \alpha \frac{\left[d(P,P)^2 + d(HRq,P)^2\right]}{\left[d(P,P) + d(HRq,P)\right]}$$

$$+ \beta \frac{\left[d(P,P) + d(HRq,P) + d(P,P) + d(HRq,P)\right]}{1 + \left[d(P,P).d(HRq,P).d(P,P).d(HRq,P)\right]}$$

$$+ \gamma\left[d(P,P) + d(HRq,P)\right] + \delta\ d(P,P)$$

$$\leq \alpha d(P,HRq) + 2\beta\ d(P,HRq) + \gamma d(P,HRq)$$

$$\leq (\alpha + 2\beta + \gamma)\ d(P,HRq)$$

Which is possible only when $HRq = P = Tq$, since $\alpha + 2\beta + \gamma < 1$.
Which shows that q is a point of HR and T.
Now by weak compatibility of (HR,T), we have

$$HRP = HR(Tq) = T(HRq) = TP$$

Which shows that P is also a point of (HR,T).
Now
$d(P,HRP) = d(FGP,HRP)$

$$\leq \alpha \frac{\left[d(FGP,TP)^2 + d(HRP,SP)^2\right]}{\left[d(FGP,TP) + d(HRP,SP)\right]}$$

$$+ \beta \frac{\left[d(FGP,TP) + d(HRP,SP) + d(FGP,SP) + d(HRP,TP)\right]}{1 + \left[d(FGP,TP).d(HRP,SP).d(FGP,SP).d(HRP,TP)\right]}$$

$$+ \gamma\left[d(FGP,SP) + d(HRP,TP)\right] + \delta\ d(SP,TP)$$

$$\leq \alpha\ d(P,HRq) + 2\beta\ d(P,HRq) + \gamma\ d(P,HRq)$$

$$\leq (\alpha + 2\beta + \gamma)\ d(P,HRq)$$

Which is possible only when $HRq = P = Tq$, since $\alpha + 2\beta + \gamma < 1$
Which shows that q is a point of HR and T.
Now by weak compatibility of (HR,T), we have

$$HRP = HR(Tq) = T(HRq) = TP$$

Which shows that P is also a point of (HR,T).

Now

$$d(P,HRP)=d(FGP,HRP)$$

$$\leq \alpha\frac{\left[d(FGP,TP)^2+d(HRP,SP)^2\right]}{\left[d(FGP,TP)+d(HRP,SP)\right]}$$

$$+\beta\frac{\left[d(FGP,TP)+d(HRP,SP)+d(FGP,SP)+d(HRP,TP)\right]}{1+\left[d(FGP,TP).d(HRP,SP).d(FGP,SP).d(HRP,TP)\right]}$$

$$+\gamma\left[d(FGP,SP)+d(HRP,TP)\right]+\delta\ d(SP,TP)$$

$$\leq \alpha\frac{\left[d(P,P)^2+d(HRP,P)^2\right]}{\left[d(P,P)+d(HRP,P)\right]}$$

$$+\beta\frac{\left[d(P,P)+d(HRP,P)+d(P,P)+d(HRP,P)\right]}{1+\left[d(P,P).d(HRP,P).d(P,P).d(HRP,P)\right]}$$

$$+\gamma\left[d(P,P)+d(HRP,P)\right]+\delta\ d(P,P)$$

$$\leq(\alpha+2\beta+\gamma)\ d(HRP,P)$$

Which is possible only when $HRP=P=TP$, since $\alpha+2\beta+\gamma<1$.
Hence P is a common fixed point of FG,S,HR and T.
Now suppose that FG is continuous so that the sequence $\{FG^2x_{2n}\}$ and $\{FGSx_{2n}\}$ converges to FGP. Since (FG,S) are compatible it follows that $\{SFGx_{2n}\}$ also converges to FGP. Thus

$$d(FG^2x_{2n},HRx_{2n+1})\leq\alpha\frac{\left[d(FG^2x_{2n},Tx_{2n+1})^2+d(HRx_{2n+1},SFGx_{2n})^2\right]}{\left[d(FG^2x_{2n},Tx_{2n+1})+d(HRx_{2n+1},SFGx_{2n})\right]}$$

$$+\beta\frac{\begin{bmatrix}d(FG^2x_{2n},Tx_{2n+1})+d(HRx_{2n+1},SFGx_{2n})\\+d(FG^2x_{2n},SFGx_{2n})+d(HRx_{2n+1},Tx_{2n+1})\end{bmatrix}}{1+\begin{bmatrix}d(FG^2x_{2n},Tx_{2n+1}).d(HRx_{2n+1},SFGx_{2n})\\.d(FG^2x_{2n},SFGx_{2n}).d(HRx_{2n+1},Tx_{2n+1})\end{bmatrix}}$$

$$+\gamma\left[d(FG^2x_{2n},SFGx_{2n})+d(HRx_{2n+1},Tx_{2n+1})\right]+\delta\ d(SFGx_{2n},Tx_{2n+1})$$

On making $n \to \infty$, we obtain

$$d(FGP,P) \le \alpha \frac{\left[d(FGP,P)^2 + d(P,FGP)^2\right]}{\left[d(FGP,P) + d(P,FGP)\right]}$$

$$+ \beta \frac{\left[d(FGP,P) + d(P,FGP) + d(FGP,FGP) + d(P,P)\right]}{1 + \left[d(FGP,P).d(P,FGP).d(FGP,FGP).d(P,P)\right]}$$

$$+ \gamma\left[d(FGP,FGP) + d(P,P)\right] + \delta\ d(FGP,P)$$

$$\le (\alpha + 2\beta + \delta)\ d(FGP,P)$$

Which is possible only when $FGP = P$, since $\alpha + 2\beta + \delta < 1$.
As earlier there exists q in X such that $FGP = P = Tq.$ Then

$$d\left(FG^2x_{2n},HRq\right) \le \alpha \frac{\left[d\left(FG^2x_{2n},Tq\right)^2 + d\left(HRq,SFGx_{2n}\right)^2\right]}{\left[d\left(FG^2x_{2n},Tq\right) + d\left(HRq,SFGx_{2n}\right)\right]}$$

$$+\beta \frac{\left[d\left(FG^2x_{2n},Tq\right) + d\left(HRq,SFGx_{2n}\right) + d\left(FG^2x_{2n},SFGx_{2n}\right) + d\left(HRq,Tq\right)\right]}{1+\left[d\left(FG^2x_{2n},Tq\right).d\left(HRq,SFGx_{2n}\right).d\left(FG^2x_{2n},SFGx_{2n}\right).d\left(HRq,Tq\right)\right]}$$

$$+\gamma\left[d\left(FG^2x_{2n},SFGx_{2n}\right) + d\left(HRq,Tq\right)\right] + \delta\, d\left(SFGx_{2n},Tq\right)$$

which on letting $n \to \infty$, we obtain

$$d(P,HRq) \le \alpha \frac{\left[d(P,P)^2 + d(HRq,P)^2\right]}{\left[d(P,P) + d(HRq,P)\right]} + \beta \frac{\left[d(P,P) + d(HRq,P) + d(P,P) + d(HRq,P)\right]}{1+\left[d(P,P).d(HRq,P).d(P,P).d(HRq,P)\right]}$$

$$+\gamma\left[d(P,P) + d(HRq,P)\right] + \delta\, d(P,P)$$

$$\le (\alpha + 2\beta + \gamma)\ d(HRq,P)$$

which given $HRq = P = Tq$, since $\alpha + 2\beta + \gamma < 1$.
Thus q is a coincidence point of (HR,T). Since the pair (HR,T) is weakly compatible one has $HRP = HR(Tq) = T(HRq) = TP$ which shows that $HRP = TP$. Further

$$d\left(FGx_{2n},HRP\right)\le\alpha\frac{\left[d\left(FGx_{2n},TP\right)^2+d\left(HRP,Sx_{2n}\right)^2\right]}{\left[d\left(FGx_{2n},TP\right)+d\left(HRP,Sx_{2n}\right)\right]}$$

$$+\beta\frac{\left[d\left(FGx_{2n},TP\right)+d\left(HRP,Sx_{2n}\right)+d\left(FGx_{2n},Sx_{2n}\right)+d\left(HRP,TP\right)\right]}{1+\left[d\left(FGx_{2n},TP\right).d\left(HRP,Sx_{2n}\right).d\left(FGx_{2n},Sx_{2n}\right).d\left(HRP,TP\right)\right]}$$

$$+\gamma\left[d\left(FGx_{2n},Sx_{2n}\right)+d\left(HRP,TP\right)\right]+\delta\ d\left(Sx_{2n},TP\right)$$

making $n\to\infty$. We obtain

$$d\left(P,HRP\right)\le\alpha\frac{\left[d\left(P,P\right)^2+d\left(HRP,P\right)^2\right]}{\left[d\left(P,P\right)+d\left(HRP,P\right)\right]}$$

$$+\beta\frac{\left[d\left(P,P\right)+d\left(HRP,P\right)+d\left(P,P\right)+d\left(HRP,P\right)\right]}{1+\left[d\left(P,P\right).d\left(HRP,P\right).d\left(P,P\right).d\left(HRP,P\right)\right]}$$

$$+\gamma\left[d\left(P,P\right)+d\left(HRP,P\right)\right]+\delta\ d\left(P,P\right)$$

$$\le\left(\alpha+2\beta+\gamma\right)\ d\left(HRP,P\right)$$

It follows that $HRP=P=TP$, since $\alpha+2\beta+\gamma<1$.
The point P therefore is in the range of HR and since $HR\left(X\right)\subset S\left(X\right)$. There exists a point w in X such that $Sw=P$. Thus

$$d\left(FGw,P\right)=d\left(FGw,HRP\right)$$

$$\le\alpha\frac{\left[d\left(FGw,TP\right)^2+d\left(HRP,Sw\right)^2\right]}{\left[d\left(FGw,TP\right)+d\left(HRP,Sw\right)\right]}$$

$$+\beta\frac{\left[d\left(FGw,TP\right)+d\left(HRP,Sw\right)+d\left(FGw,Sw\right)+d\left(HRP,TP\right)\right]}{1+\left[d\left(FGw,TP\right).d\left(HRP,Sw\right).d\left(FGw,Sw\right).d\left(HRP,TP\right)\right]}$$

$$+\gamma\left[d\left(FGw,Sw\right)+d\left(HRP,TP\right)\right]+\delta\ d\left(Sw,TP\right)$$

$$\leq \alpha \frac{\left[d(FGw,P)^2 + d(P,P)^2\right]}{\left[d(FGw,P) + d(P,P)\right]} + \beta \frac{\left[d(FGw,P) + d(P,P) + d(FGw,P) + d(P,P)\right]}{1 + \left[d(FGw,P).d(P,P).d(FGw,P).d(P,P)\right]}$$

$$+ \gamma \left[d(FGw,P) + d(P,P)\right] + \delta\, d(P,P)$$

$$\leq (\alpha + 2\beta + \gamma)\, d(FGw,P)$$

Hence $FGw = P$, since $\alpha + 2\beta + \gamma < 1$.
Also $FGP = SP = P$.
Thus we have proved that P is common fixed point of FG, HR, S and T.
Let v be another fixed point of S, T, FG and HR, then

$$d(P,v) = d(FGP, HRv)$$

$$\leq \alpha \frac{\left[d(FGP,Tv)^2 + d(HRv,SP)^2\right]}{\left[d(FGP,Tv) + d(HRv,SP)\right]}$$

$$+ \beta \frac{\left[d(FGP,Tv) + d(HRv,SP) + d(FGP,SP) + d(HRv,Tv)\right]}{1 + \left[d(FGP,Tv).d(HRv,SP).d(FGP,SP).d(HRv,Tv)\right]}$$

$$+ \gamma \left[d(FGP,SP) + d(HRv,Tv)\right] + \delta\, d(SP,Tv)$$

$$\leq \alpha \frac{\left[d(P,v)^2 + d(v,P)^2\right]}{\left[d(P,v) + d(v,P)\right]} + \beta \frac{\left[d(P,v) + d(v,P) + d(P,P) + d(v,v)\right]}{1 + \left[d(P,v).d(v,P).d(P,P).d(v,v)\right]}$$

$$+ \gamma \left[d(P,P) + d(v,v)\right] + \delta\, d(P,v)$$

$$\leq (\alpha + 2\beta + \delta)\, d(P,v)$$

Yielding thereby $P = v$, since $\alpha + 2\beta + \delta < 1$.
Finally, we prove that P is also a common fixed point of F, G, H, R, S and T.

For this P be the unique common fixed point of both the pairs (FG,S) and (HR,T). Then

$$FP = F(FGP) = F(GFP) = FG(FP),\ FP = F(SP) = S(FP).$$
$$GP = G(FGP) = G(FGP) = GF(GP) = FG(GP),\ GP = G(SP) = S(GP).$$

Which shows that FP and GP is a common fixed point of (FG,S). Yielding thereby $FP = P = GP = SP = FGP$ in the view of uniqueness of the common fixed point of the pair (FG,S).

Similarly using the commutativity of $(H,R),(H,T)$ and (R,T) it can be shown that $HP = P = RP = TP = HRP$.

Now we show that $FP = HP,\ GP = RP$. Also remains a common fixed point of both the pairs (FG,S) and (HR,T). For this

$$d(FP,HP) = d(F(GFP),H(RHP))$$

$$= d(FG(FP),HR(HP))$$

$$\le \alpha \frac{\left[d(FG(FP),T(HP))^2 + d(HR(HP),S(FP))^2\right]}{\left[d(FG(FP),T(HP)) + d(HR(HP),S(FP))\right]}$$

$$+ \beta \frac{\left[\begin{array}{l} d(FG(FP),T(HP)) + d(HR(HP),S(FP)) + d(FG(FP),S(FP)) \\ + d(HR(HP),T(HP)) \end{array}\right]}{1 + \left[\begin{array}{l} d(FG(FP),T(HP)).d(HR(HP),S(FP)).d(FG(FP),S(FP)) \\ .d(HR(HP),T(HP)) \end{array}\right]}$$

$$+ \gamma\left[d(FG(FP),S(FP)) + d(HR(HP),T(HP))\right] + \delta\ d(S(FP),T(HP))$$

$$d(FP,HP) \le 0$$

$$\Rightarrow d(FP,HP) = 0$$

$$\Rightarrow FP = HP$$

Similarly, it can be shown that $GP = RP$.

Thus P is the unique common fixed point of F,G,H,R,S and T.

Now, if $d(FGx,Ty) + d(HRy,Sx) = 0 \Rightarrow d(FGx,HRy) = 0$. Then suppose that there exists n such that $P_n = P_{n+1}$. Then, also $P_{n+1} = P_{n+2}$, suppose not. Then from (3.1.3) we have $0 < d(P_{n+1},P_{n+2}) < k\,d(P_{n+1},P_n)$ yielding thereby $P_{n+1} = P_{n+2}$. Thus

$P_n = P_{n+k}$ for k=1,2,3,… . It then follows that there exists two points z_1 and z_2 such that $v_1 = FGz_1 = Tz_2$ and $v_2 = HRz_2 = Sz_1$. Since $d(FGz_1, Tz_2) + d(HRz_2, Sz_1) = 0$, from (3.1.2) $d(FGz_1, HRz_2) = 0$ *i.e.* $v_1 = FGz_1 = HRz_2 = v_2$. Also $Sv_1 = I(FGz_1) = FG(Iz_1) = FGv_1$. Similarly $HRv_2 = Tv_2$. Define $y_1 = FGv_1, y_2 = HRv_2$. Since $d(FGv_1, Tv_2) + d(HRv_2, Sv_1) = 0$, it follows from (3.1.2) that $d(FGv_1, HRv_2) = 0$ *i.e.* $y_1 = y_2$. Thus $FGv_1 = Sv_1 = HRv_2 = Tv_2$.
But $v_1 = v_2$.
Therefore FG, S, HR and T have a common coincidence point. Define $w = FGv_1$, it then follows that w is also a common coincidence point of FG, S, HR and T. If $FGw \neq FGv_1 = HRv_1$, then $d(FGw, HRv_1) > 0$. But, Since $d(FGw, Tv_1) + d(HRv_1, Sw) = 0$, it follows from (3.1.2) that $d(FGw, HRv_1) = 0$ *i.e.* $FGw = HRv_1$, which is a contradiction.
Hence $FGw = HGv_1 = w$ and w is a common fixed point of FG, HR, S and T. The rest of the proof is identical to the previous calculation, hence it is omitted.
This completes the proof.

Corollary 3.2. Theorem 3.1 remains true if contraction condition (3.1.1) and (3.1.2) are replaced by any one of the following condition:

(a) By changing $\beta = 0, \delta = 0.$ Then

$$d(FGx, HRy) \leq \alpha \frac{\left[d(FGx, Ty)^2 + d(HRy, Sx)^2\right]}{\left[d(FGx, Ty) + d(HRy, Sx)\right]}$$

$$+\gamma\left[d(FGx, Sx) + d(HRy, Ty)\right]$$

$$if\ d(FGx, Ty) + d(HRy, Sx) \neq 0, \alpha, \gamma > 0, 2\alpha + 2\gamma < 1$$

$$or\ d(FGx, HRy) = 0\ if\ d(FGx, Ty) + d(HRy, Sx) = 0.$$

Which is a results of Fisher [2] and Kannan [9].

(b) By choosing $\beta = 0, \gamma = 0.$ Then

$$d(FGx, HRy) \leq \alpha \frac{\left[d(FGx, Ty)^2 + d(HRy, Sx)^2\right]}{\left[d(FGx, Ty) + d(HRy, Sx)\right]} + \delta d(Sx, Ty)$$

$$if\ d(FGx,Ty)+d(HRy,Sx)\neq 0,\alpha,\delta>0,2\alpha+\delta<1$$

$$or\ d(FGx,HRy)=0\ if\ d(FGx,Ty)+d(HRy,Sx)=0.$$

Which is a results of Fisher [2].

(c) By choosing $\beta=0,\gamma=0,\delta=0.$ Then

$$d(FGx,HRy)\leq\alpha\frac{\left[d(FGx,Ty)^2+d(HRy,Sx)^2\right]}{\left[d(FGx,Ty)+d(HRy,Sx)\right]}$$

$$if\ d(FGx,Ty)+d(HRy,Sx)\neq 0,\alpha>0,\alpha<1/2$$

$$or\ d(FGx,HRy)=0\ if\ d(FGx,Ty)+d(HRy,Sx)=0.$$

Which extends a theorem of Fisher [2].

(d)

$$If\ \frac{\left[d(FGx,Ty)^2+d(HRy,Sx)^2\right]}{\left[d(FGx,Ty)+d(HRy,Sx)\right]}\leq\frac{\left[d(FGx,Ty)+d(HRy,Sx)\right]^2}{\left[d(FGx,Ty)+d(HRy,Sx)\right]}\leq d(FGx,Ty)+d(HRy,Sx)$$

and $\beta=0,$ then condition $(3.1.1)$ be

$$d(FGx,HRy)\leq\alpha\left[d(FGx,Ty)+d(HRy,Sx)\right]+\gamma\left[d(FGx,Sx)+d(HRy,Ty)\right]+\delta d(Sx,Ty)$$

If $2\alpha+2\gamma+\delta<1$.
Which extends a theorem of Hardy-Rogers. [14].

4 Remarks

(i) If we put $F=A,G=B,H=S,R=T,$ $S=I$ and $T=J$ and $\alpha=\alpha_1,\beta=0,\gamma=\alpha_2,\delta=\alpha_3,$ then we get the result of Imdad and Khan.

(ii) Our result is motivated by Fisher and J. Rhoades, Kannan, Hardy-Roger.

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