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# Existence of Nontrivial Solutions for a Nonlocal Elliptic System of $(p, q)$-Kirchhoff Type with Critical Exponent

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**Abstract**

*This paper studies the existence of nontrivial solution for the nonlocal elliptic system of $(p, q)$-Kirchhoff type (1) with critical exponent. Under some conditions on $M_i$ $(i = 1, 2)$ and $F$, the existence of nontrivial solution is established. Our technical approach is based on Bonanno and Molica Bisci's general critical points theorem.*

**Keywords:** *p-Kirchhoff, Critical exponent, Ricceri's variational principle.*

## 1 Introduction

In this article, we are concerned with the following nonlocal elliptic system of $(p, q)$-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u = \mu\left(\int_\Omega |u|^{p^\star} dx\right)^{p/p^\star-1} |u|^{p^\star-2}u + \lambda F_u(x, u, v) & \text{in} \quad \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q dx\right)\right]^{q-1} \Delta_q v = \mu\left(\int_\Omega |v|^{q^\star} dx\right)^{q/q^\star-1} |v|^{q^\star-2}v + \lambda F_v(x, u, v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $R^N$ with smooth boundary $\partial\Omega, 1 < p, q < N$, $\Delta_p u = div(|\nabla|^{p-2}\nabla u)$ the $p-$Laplace operator, $p^\star = \frac{Np}{N-p}$, $q^\star = \frac{Nq}{N-q}$ and $\lambda, \mu$ are two real parameters respectively positive and non-negative and $M_i$ : $R^+ \to R$ are continuous functions with the following condition.

(M) There exists $m_0 \geq 1$ such that for all $t \geq 0$ one has

$$M_i(t) \geq m_0, \text{ for } (i = 1, 2).$$

Furthermore, $F : \Omega \times R^2 \to R$ is function such that $F(x, s, t)$ is measurable in $x$ for all $(s, t) \in R^2$ and $F(x, s, t)$ is $C^1$ in $(s, t)$ for a.e. $x \in \Omega$, and $F_u$ denote the partial derivatives of $F$ with respect to $u$ such that $F_u(x, 0, 0) \neq 0$ or $F_v(x, 0, 0) \neq 0$ in $\Omega$. We assume that $F(x, s, t)$ satisfy the following conditions.

(F) There exist two positive constants $\gamma < p^\star$, $\beta < q^\star$ and a positive real function $\alpha \in L^\infty(\Omega)$ such that

$$|F(x, s, t)| \leq \alpha(x) \left(1 + |s|^\gamma + |t|^\beta\right), \text{ for a.e. } x \in \Omega \text{ and all } (s, t) \in R^2.$$

The problem (1) is related to the stationary problem of a model introduced by Kirchhoff [12]. More precisely, Kirchhoff introduced a model given by the following equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Latter (2) was developed to form

$$u_{tt} - M \left( \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u) \text{ in } \Omega. \tag{3}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-M \left( \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \tag{4}$$

Problems like (4) can be used for modeling several physical and biological systems where $u$ describes a process which depends on the average of it self, such as the population density , see [1]. Many interesting results for (4) were obtained see for example [6, 7, 13, 15]

The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian. In [9] Hamydy, Massar and Tsouli study the existence of solutions for the p-Kirchhoff type problem involving the critical Sobolev exponent

$$\begin{cases} -\left[ M \left( \int_\Omega |\nabla u|^p dx \right) \right]^{p-1} \Delta_p u &= |u|^{p^\star - 2} u + \lambda f(x, u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

the authors applied the concentration-compactness principle to deal the difficulty of the lack of compactness of the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$, and show the existence of nontrivial solution of the problem (5) under some suitable conditions on $f$ and $M$.

In [5] Corrêa and Nascimento considered the following nonlocal elliptic system of p-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u &= f(x,u) + h_1(x) \quad \text{in} \quad \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^p dx\right)\right]^{p-1} \Delta_p v &= g(x,v) + h_2(x) \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

Under suitable assumptions on $M_i, h_i$, $(i = 1, 2)$, $f$ and $g$, the authors proved the existence of a weak solution for (6). In [8] Cheng, Wu and Liu studied the following system

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u &= \lambda F_u(x,u,v) \quad \text{in} \quad \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q dx\right)\right]^{q-1} \Delta_q v &= \lambda F_v(x,u,v) \quad \text{in} \quad \Omega, \\ u &= v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where $p, q > N$. They established the existence of two solutions and three solutions via Bonanno's multiple critical points theorems without the Palais-Smale condition and Ricceri's three critical points theorem, respectively.

When $\mu \neq 0$, we look, in problem (1), the presence of the term with the critical exponents $p^\star$ and $q^\star$, and the classical Sobolev inequality, ensures that the embedding of the space $W_0^{1,r}(\Omega) \hookrightarrow L^{r^\star}(\Omega)$ is continuous but not compact. Due to this lack of compactness the classical methods cannot be used in order to prove the weak lower semicontinuity of the energy functional associated to (1). So we overcome this difficulty through a lower semicontinuity result obtained by Montefusco in [14].

In the present paper, we will prove the existence of well precise intervals of parameters $\lambda$ such that problem (1) admits at least one nontrivial weak solution. Our main tool is a general critical points theorem due to Bonanno and Molica Bisci [2] that is a generalization of a previous result of Ricceri [16].

## 2 Preliminaries

Denote $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, which is a reflexive Banach space endowed with the norm

$$||(u,v)|| = ||u||_p + ||v||_q,$$

where

$$||u||_p = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p} \text{ and } \quad ||v||_q = \left(\int_\Omega |\nabla v|^q dx\right)^{1/q}.$$

Recall that $(u, v) \in X$ is called a weak solution of system (1) if

$$\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \left[M_2\left(\int_\Omega |\nabla v|^q dx\right)\right]^{q-1} \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi dx$$
$$-\frac{\mu}{p^\star}\left(\int_\Omega |u|^{p^\star} dx\right)^{p/p^\star-1} \int_\Omega |u|^{p^\star-2}u\varphi dx - \frac{\mu}{q^\star}\left(\int_\Omega |v|^{q^\star} dx\right)^{q/q^\star-1} \int_\Omega |v|^{p^\star-2}v\psi dx$$
$$-\lambda\int_\Omega F_u(x, u, v)\varphi dx - \lambda\int_\Omega F_v(x, u, v)\psi dx = 0,$$

for all $(\varphi, \psi) \in X$.

We see that weak solutions of system (1) are critical points of the functional $I : X \to R$ given by

$$I(u, v) = \Phi_\mu(u, v) - \lambda\Psi(u, v),$$

where

$$\Phi_\mu(u,v) = \frac{1}{p}\widehat{M_1}\left(\int_\Omega |\nabla u|^p dx\right) + \frac{1}{q}\widehat{M_2}\left(\int_\Omega |\nabla v|^q dx\right) - \frac{\mu}{p}\left(\int_\Omega |u|^{p^\star} dx\right)^{p/p^\star} - \frac{\mu}{q}\left(\int_\Omega |v|^{q^\star} dx\right)^{q/q^\star}$$

$$\Psi(u, v) = \int_\Omega F(x, u, v)dx,$$

and

$$\widehat{M_1}(t) = \int_0^t [M_1(s)]^{p-1}ds, \quad \widehat{M_2}(t) = \int_0^t [M_2(s)]^{q-1}ds. \tag{8}$$

Following we consider the well-know inequalities

$$\left(\int_\Omega |u|^{p^\star} dx\right)^{1/p^\star} \le \frac{1}{S_p^{1/p}}\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \quad \forall u \in W_0^{1,p}(\Omega)$$
$$\left(\int_\Omega |v|^{q^\star} dx\right)^{1/q^\star} \le \frac{1}{S_q^{1/q}}\left(\int_\Omega |\nabla v|^q dx\right)^{1/q}, \quad \forall v \in W_0^{1,q}(\Omega), \tag{9}$$

where, $S_p$ and $S_q$ are the best constants in the Sobolev inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^{q^\star}(\Omega)$ respectively.

Fixing $\mu \in [0, \min(S_p, S_q)[$ , from the definition of $\Phi_\mu$ and (9), we have

$$\Phi_\mu(u, v) \ge m(\mu, p, q)\left(\frac{||u||_p^p}{p} + \frac{||v||_q^q}{q}\right), \tag{10}$$

for every $(u, v) \in X$, where

$$m(\mu, p, q) := \min\left(1 - \frac{\mu}{S_p}, 1 - \frac{\mu}{S_q}\right).$$

In [14], Montefusco showed the results in following theorem, which are essential for our work.

**Theorem 2.1** *Assum that $\mu \in [0, S_p[$, then the functional*

$$\mathcal{S}_\mu(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p}\left(\int_\Omega |u|^{p^\star} dx\right)^{p/p^\star}$$

*is sequentially weakly lower semicontinuous in $W_0^{1,p}(\Omega)$.*

The fact that $\widehat{M}_i$ is continuous and monotone for $(i = 1, 2)$, by $(M)$ and theorem 2.1 we get that $\Phi_\mu$ is sequentially weakly lower semicontinuous for $\mu \in [0, \min(S_p, S_q)[$ and it is also a coercive functional, as well as the map $\Psi$ is continuously Gâteaux differentiable and sequentially weakly upper semicontinuous. Moreover by the assumptions $(M)$ and $(F)$, it is standard to see that $I \in C^1(X, R)$ and a critical point of $I$ corresponds to a weak solution of problem (1).

We denote by $C_\gamma$ and $C_\beta$ the embedding constants of Sobolev $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^\beta(\Omega)$ respectively.

Let us here recall for the reader's convenience a smooth version of a previous result of Ricceri [16].

**Theorem 2.2** *Let $X$ be a reflexive real Banach space, let $\Phi, \Psi : X \to R$ be two Gâteaux differentiable functionals such that $\Phi$ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive and $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}.$$

*Then, for every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $I_\lambda$ in $X$.*

# 3 Main Results

Our main result is the following theorem.

**Theorem 3.1** *Assume that $(M)$, $(F)$ hold and*

*(i) $F(x, s, t) \geq 0$ for every $(x, s, t) \in \Omega \times [0, +\infty)^2$.*

*Then, for every $\mu \in [0, \min(S_p, S_q)[$ there exists a positive number $\delta^\star(p, q)$ given by*

$$\delta^\star(p,q) := \sup_{\xi>0}\left(\frac{\xi^{pq}}{||\alpha||_\infty\left(|\Omega| + C_\gamma^\gamma\left(\frac{p}{m(\mu,p,q)}\right)^{\gamma/p}\xi^{\gamma q} + C_\beta^\beta\left(\frac{q}{m(\mu,p,q)}\right)^{\beta/q}\xi^{\beta p}\right)}\right)$$

*such that, for every*

$$\lambda \in \Lambda := ]0, \delta^\star(p,q)[\,,$$

*problem (1) admits at least one nontrivial weak solution.*

**Proof.** The functionals introduced before $\Phi_\mu$ and $\Psi$ satisfy the required conditions and clearly $\inf_{(u,v)\in X} \Phi_\mu(u,v) = 0$. In order to apply Theorem 2.2, fix $\mu \in ]0, \min(S_p, S_q)[$ and $\lambda \in ]0, \delta^\star(p,q)[$ , then there exists $\overline{\xi} > 0$ such that

$$\lambda < \frac{\overline{\xi}^{pq}}{||\alpha||_\infty \left(|\Omega| + C_\gamma^\gamma \left(\frac{p}{m(\mu,p,q)}\right)^{\gamma/p} \overline{\xi}^{\gamma q} + C_\beta^\beta \left(\frac{q}{m(\mu,p,q)}\right)^{\beta/q} \overline{\xi}^{\beta p}\right)}. \tag{11}$$

Now, we set

$$\varphi(r) := \inf_{(u,v)\in \Phi_\mu^{-1}(]-\infty,r[)} \frac{\left(\sup_{(w,z)\in\Phi_\mu^{-1}(]-\infty,r[)} \Psi(w,z)\right) - \Psi(u,v)}{r - \Phi_\mu(u,v)}$$

Note that $\Phi_\mu(0,0) = 0$, and by $(i)$, $\Psi(0,0) \geq 0$. Therefore, for every $r > 0$,

$$\begin{aligned} \varphi(r) &= \inf_{(u,v)\in \Phi_\mu^{-1}(]-\infty,r[)} \frac{\left(\sup_{(w,z)\in\Phi_\mu^{-1}(]-\infty,r[)} \Psi(w,z)\right) - \Psi(u,v)}{r - \Phi_\mu(u,v)} \\ &\leq \frac{\sup_{\Phi_\mu^{-1}(]-\infty,r[)} \Psi}{r} \\ &= \frac{\sup_{\Phi_\mu(u,v)<r} \int_\Omega F(x,u,v)dx}{r}. \end{aligned} \tag{12}$$

Moreover, from Assumption $(F)$, we have

$$\Psi(u,v) = \int_\Omega F(x,u,v)dx \leq ||\alpha||_\infty \left(|\Omega| + ||u||^\gamma_{L^\gamma(\Omega)} + ||v||^\beta_{L^\beta(\Omega)}\right). \tag{13}$$

Then, by (10) we get

$$||u||_p \leq \left(\frac{pr}{m(\mu,p,q)}\right)^{1/p} \text{ and } ||u||_q \leq \left(\frac{qr}{m(\mu,p,q)}\right)^{1/q},$$

for every $(u,v) \in \Phi^{-1}(]-\infty, r[)$.
Hence, from the embedding of Sobolev, we obtain

$$\begin{aligned} \Psi(u,v) &\leq ||\alpha||_\infty \left(|\Omega| + C_\gamma^\gamma ||u||_p^\gamma + C_\beta^\beta ||v||_q^\beta\right) \\ &\leq ||\alpha||_\infty \left(|\Omega| + C_\gamma^\gamma \left(\frac{pr}{m(\mu,p,q)}\right)^{\gamma/p} + C_\beta^\beta \left(\frac{qr}{m(\mu,p,q)}\right)^{\beta/q}\right). \end{aligned} \tag{14}$$

It follows from (12) and (14) that

$$\varphi(r) \leq ||\alpha||_\infty \left( \frac{|\Omega|}{r} + C_\gamma^\gamma \left( \frac{p}{m(\mu,p,q)} \right)^{\gamma/p} r^{\gamma/p-1} + C_\beta^\beta \left( \frac{q}{m(\mu,p,q)} \right)^{\beta/q} r^{\beta/q-1} \right) \tag{15}$$

for every $r > 0$. In particular, we infer

$$\varphi(\overline{\xi}^{pq}) \leq \frac{||\alpha||_\infty}{\overline{\xi}^{pq}} \left( |\Omega| + C_\gamma^\gamma \left( \frac{p}{m(\mu,p,q)} \right)^{\gamma/p} \overline{\xi}^{\gamma q} + C_\beta^\beta \left( \frac{q}{m(\mu,p,q)} \right)^{\beta/q} \overline{\xi}^{\beta p} \right). \tag{16}$$

Thus, by (11) and the above inequality we get

$$\lambda \in \Lambda \subseteq \left] 0, \frac{1}{\varphi(\overline{\xi}^{pq})} \right[. \tag{17}$$

So, Theorem 2.2 assures the existence of a global minimum of the restriction of $I$ to $\Phi_\mu^{-1}(]-\infty, \overline{\xi}^{pq}[)$ which is a critical point of $I$ in $X$. Further, by the assumption on the function $F$, the solution cannot be trivial and the proof is complete.

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