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Certain Properties of Mixed Super Quasi Einstein Manifolds

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Abstract

In this paper we have defined mixed super quasi –Einstein manifold $MS(QE)_n$ which is more generalized form of Einstein manifold , quasi –Einstein manifold, generalized quasi –Einstein manifold and super quasi –Einstein manifold . In this paper it has been shown that $MS(QE)_n (n > 3)$ is a $MS(QC)_n$ if it is conformally flat. Moreover, it is shown that $MS(QC)_n (n > 3)$ is a conformally flat $MS(QE)_n$ and an example of mixed super quasi Einstein manifold is also given . Properties of the curvature tensor in a conformally flat, projectively flat and conharmonically flat manifold have been discussed.

It has also been shown that a totally umbilical hypersurface of a conharmonically flat $MS(QE)_n (n > 3)$ is a manifold of mixed super quasi-constant curvature.

Keywords: *Quasi- Einstein manifolds, Mixed super quasi-Einstein manifolds, Projectively flat, Conharmonically flat, Totally umbilical.*

1 Preliminaries

A non-flat Riemannian manifold $(M^n, g)(n \geq 3)$ is called quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies

(1.1) $$S(X,Y) = ag(X,Y) + bA(X)A(Y)$$

where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

(1.2) $$g(X,U) = A(X), \qquad \forall X \quad \text{tangents to } M^n$$

and U is a unit vector field. In such case a, b are called the associated 1-forms and U the generator of the manifold. Such an n-dimensional manifold is denoted by $(QE)_n$.

The notion of mixed generalized quasi Einstein manifold was introduced by A.Bhattacharya, T.De and D.Debnath in their paper [1].A non- flat Riemannian manifold $(M^n, g)(n \geq 3)$ is called mixed generalized quasi Einstein manifold if the Ricci tensor S of type (0,2) is not identically zero and satisfies

(1.3) $$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)]$$

where a, b, c, d are scalars , $b \neq 0, c \neq 0, d \neq 0$ and A, B are two non-zero 1-forms such that

(1.4) $$g(X,U) = A(X) \quad , \qquad g(X,V) = B(X), \qquad \forall X \text{ tangents to } M^n$$

and

$$g(U,V) = 0$$

where U, V are unit vector fields . In such a case $a.b, c, d$ are called associated scalars, A, B the associated 1-forms and U, V the generators of the manifold. Such n-dimensional manifold is denoted by $MG(QE)_n$.

As a further generalization of quasi-Einstein manifold we introduce the notion of mixed super quasi-Einstein manifold . A non flat Riemannian manifold $(M^n, g)(n \geq 3)$ will be called a mixed super quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

(1.5) $$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y)$$

where a, b, c, d, e are scalars and $b \neq 0, c \neq 0, d \neq 0, e \neq 0$. A, B are `two non zero 1-forms such that (1.4) is satisfied , U, V mutually orthogonal unit vector fields and D is a symmetric (0,2) type of tensor field with zero trace and satisfies

(1.6) $$D(X,U) = 0 \ , \qquad \forall X \text{ tangents to } M^n.$$

In such a case a, b, c, d, e are called associated scalars . A, B the associated 1-forms U, V the generators and D the associated tensor of the manifold . Such an n-dimensional manifold will be denoted by $MS(QE)_n$.

Chen and Yano [2] introduced the notion of a manifold of quasi constant curvature. According to them a non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be of quasi -constant curvature if its curvature tensor $'R$ of type (0,4) satisfies the condition

(1.7) $$'R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)$$

$$+g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)]$$

where a,b are scalars of which $b\neq 0$ and A is a non zero 1-form defined by (1.2) and U a unit vector field . In such a case a,b are called the associated scalars, A is called the associated 1-form and U the generator of the manifold . Such an n-dimensional manifold is denoted by the symbol $(QC)_n$.

The idea of mixed generalised quasi-constant curvature was introduced by Bhatt , De and Debnath in their paper [1].

Let us call a non-flat Riemannian manifold $(M^n,g)(n\geq 3)$ a manifold of mixed super quasi-constant curvature if its curvature tensor $'R$ of type (0, 4) satisfies

(1.8)
$$\begin{aligned}'R(X,Y,Z,W) &= a[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\
&+b[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W)\\
&+g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)]\\
&+c[g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W)\\
&+g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)]\\
&+d[g(Y,Z)\{A(X)B(W)+A(W)B(X)\}\\
&-g(X,Z)\{A(Y)B(W)+A(W)B(Y)\}\\
&+g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\}\\
&-g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}]\\
&+e[g(Y,Z)D(X,W)-g(X,Z)D(Y,W)\\
&+g(X,W)D(Y,Z)-g(Y,W)D(X,Z)]\end{aligned}$$

where a,b,c,d,e are scalars such that $b\neq 0, c\neq 0, d\neq 0, e\neq 0$ and A,B are two non-zero1- forms defined by(1.4), U,V being two unit vector fields such that $g(U,V)=0$ and D is a symmetric tensor of type (0,2) defined as (1.6). Such an n-dimensional manifold shall be denoted by the symbol $MS(QC)_n$.

If in (1.8) $e=0$ then the manifold becomes a manifold of mixed generalized quasi-constant curvature $MS(QC)_n$.

2 Some Results on Mixed Super Quasi Einstein manifold $MS(QE)_n$

We consider an $MS(QE)_n(n>2)$ with associated scalars $a,b,c,d,e(b\neq 0, c\neq 0, d\neq o, e\neq 0)$ associated 1-forms A,B generators U,V and associated symmetric (0,2) tensor field D.

Then equations (1.4),(1.5) and (1.6) will hold good. Since U,V are mutually orthogonal unit vector fields, we have

(2.1) $g(U,U)=1,\quad g(V,V)=1,\quad g(U,V)=0$.

Further

(2.2) traceD=0 and $D(X,U)=0,\quad D(X,V)=0\quad \forall X$ tangents to M^n.

By virtue of the equation (1.4), we can write the equation (2.1) in the form

(2.3) $A(U)=1,\quad B(V)=1,\quad A(V)=0,\quad B(U)=0$

Now contracting (1.5) over X and Y we get

$r = na + b + c$, where r denotes the scalar curvature.

Again from (1.5) we get

$$S(U,U) = a + b, \qquad S(V,V) = a + c + eD(V,V)$$
$$S(U,V) = d$$

Let L and l be the symmetric endomorphisms of the tangent space at each point corresponding to the Ricci tensor S and the associated tensor D respectively. Then

$$g(LX,Y) = S(X,Y)$$

And $\forall X, Y$ tangents to M^n.

(2.4) $$g(lX,Y) = D(X,Y)$$

In an n-dimensional $(n > 2)$ Riemannian manifold the quasi-conformal curvature tensor is defined as [5]

(2.5) $$'C(X,Y,Z,W) = a_1\, 'R(X,Y,Z,W) + b_1[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] - \tfrac{r}{n}[\tfrac{a_1}{n-1} + 2b_1][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

where $$g(C(X,Y)Z,W) = \,'C(X,Y,Z,W)$$

If $a_1 = 1$, $b_1 = \frac{-1}{n-2}$ then (2.5) takes the form of conformal curvature tensor, where

(2.6) $$'C(X,Y,Z,W) = \,'R(X,Y,Z,W) + \tfrac{-1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] - \tfrac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

The conharmonic curvature tensor in an n-dimensional Riemannian manifold $(n > 2)$ is defined as

(2.7) $$'H(X,Y,Z,W) = \,'R(X,Y,Z,W) - \tfrac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)]$$

The projective curvature tensor in an n-dimensional Riemannian manifold $(n > 2)$ is defined as

(2.8) $$'P(X,Y,Z,W) = \,'R(X,Y,Z,W) - \tfrac{1}{(n-1)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]$$

3 Conformally Flat $MS(QE)_n (n > 3)$

In this section we consider a conformally flat $MS(QE)_n (n > 3)$ and it has been shown that such a $MS(QE)_n$ is a $MS(QC)_n$.

It is clear that [4] in a conformally flat Riemannian manifold $(M^n, g)(n>3)$ the curvature tensor $'R$ of type (0,4) has the following form :

(3.1) $$'R(X,Y,Z,W)=\tfrac{1}{(n-2)}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]-\tfrac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]$$

where $'R$ is defined earlier.
Now using (1.5) we can express (3.1) as follows

(3.2) $$\begin{aligned}'R(X,Y,Z,W)=\;& a_1[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &+b_1[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W)\\ &\quad+g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)]\\ &+c_1[g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W)\\ &\quad+g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)]\\ &+d_1[g(Y,Z)\{A(X)B(W)+A(W)B(X)\}\\ &\quad-g(X,Z)\{A(Y)B(W)+A(W)B(Y)\}\\ &\quad+g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\}\\ &\quad-g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}]\\ &+e_1[g(Y,Z)D(X,W)-g(X,Z)D(Y,W)\\ &\quad+g(X,W)D(Y,Z)-g(Y,W)D(X,Z)]\end{aligned}$$

where $a_1=\tfrac{2a(n-1)-r}{(n-1)(n-2)}$, $b_1=\tfrac{b}{(n-2)}$, $c_1=\tfrac{c}{(n-2)}$,

$d_1=\tfrac{d}{(n-2)}$, $e_1=\tfrac{e}{(n-2)}$.

In view of (1.8) it follows from (3.2) that the manifold under consideration is a $MS(QC)_n(b_1\neq 0, c_1\neq 0, d_1\neq 0, e_1\neq 0)$ .Therefore we have the following theorem:

Theorem 1: *Every conformally flat* $MS(QE)_n(n>3)$ *is a* $MS(QC)_n$.
Also , we proved that every $MS(QC)_n(n\geq 3)$ *is a* $MS(QE)_n$. *Contracting (1.8) over* Y *and* Z *we get*

(3.3) $$S(X,W)=[a(n-1)+b+c]g(X,W)+b(n-2)A(X)A(W)+c(n-2)B(X)B(W)+d(n-2)[A(X)B(W)+A(W)B(X)]+e(n-2)D(X,W)$$

In virtue of (1.5), it follows that a $MS(QC)_n(n\geq 3)$ is a $MS(QE)_n$.(Since $b\neq 0, c\neq 0, d\neq 0, e\neq 0$) .

Again contracting (3.3) over X and W ,we have

(3.4) $$r=(n-1)(na+2b+2c)$$

In a Riemannian manifold $(M^n, g)(n>3)$ the conformal curvature tensor $'C$ of type (0,4) has the following form :

(3.5)
$$'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \tfrac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] + \tfrac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

Using (1.8), (3.3) and (3.4) it follows from (3.5) that

$$'C(X,Y,Z,W) = 0$$

i.e the manifold under consideration is conformally flat. Thus we can state the following theorem:

Theorem 2: *Every* $MS(QC)_n (n \geq 3)$ *is a* $MS(QE)_n$ *while every* $MS(QC)_n (n>3)$ *is a conformally flat* $MS(QE)_n$.

Example: A manifold of mixed super quasi constant curvature is a mixed super quasi Einstein manifold.

4 Projectively Flat $MS(QE)_n (n>2)$

Let R be the curvature tensor of type (1,3) of a projectively flat $MS(QE)_n (n>2)$. Then from (2.8) we have

(4.1)
$$'R(X,Y,Z,W) = \tfrac{1}{(n-1)}[g(X,W)S(Y,Z) - g(Y,W)S(X,Z)]$$

where $'R$ is defined earlier .
From (1.5) and (4.1) we have

(4.2)
$$\begin{aligned} 'R(X,Y,Z,W) = {} & a_2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + b_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ & + c_2[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ & + d_2[g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ & \quad - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\ & + e_2[g(X,W)D(Y,Z) - g(Y,W)D(X,Z)] \end{aligned}$$

where $a_2 = \tfrac{a}{(n-1)}$, $b_2 = \tfrac{b}{(n-1)}$, $c_2 = \tfrac{c}{(n-1)}$, $d_2 = \tfrac{d}{(n-1)}$, $e_2 = \tfrac{e}{(n-1)}$.

Let $U^{\perp}$ be the $(n-1)$ distribution orthogonal to U in a projectively flat $MS(QE)_n$. Then $g(X,U) = 0$ if $X \in U^{\perp}$.

Hence from (4.2) we have the following properties of R

(4.3) $$R(X,Y,Z) = a_2[g(Y,Z)X - g(X,Z)Y] \\ +c_2[B(Y)B(Z)X - B(X)B(Z)Y] \\ +e_2[D(Y,Z)X - D(X,Z)Y]$$

when $X,Y,Z \in U^{\perp}$ and $a_2 = \frac{a}{(n-1)}$, $c_2 = \frac{c}{(n-1)}$, $e_2 = \frac{e}{(n-1)}$. Also

(4.4) $$R(X,U,U) = a_2 X, \quad \text{when } X \in U^{\perp}.$$

Therefore we can state the following theorem:

Theorem 3: *A projectively flat $MS(QE)_n(n>2)$ is a manifold of mixed super quasi constant cuvature and the curvature tensor R of type (1,3) satisfies the properties given by (4.3) and (4.4).*

5 Conharmonically Flat $MS(QE)_n(n>3)$

Let R be the curvature tensor of type (1,3) of a Conharmonically flat $MS(QE)_n(n>3)$. Then from (2.7) we have

(5.1) $$'R(X,Y,Z,W) = \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ +g(Y,Z)S(X,W) - g(X,Z)S(Y,W)]$$

where $'R$ is defined earlier .
From (1.5) and (5.1) we have

(5.2) $$'R(X,Y,Z,W) = a_3[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ +b_3[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ +g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ +c_3[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ +g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ +d_3[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ -g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ +g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ -g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\ +e_3[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ +g(X,W)D(Y,Z) - g(Y,W)D(X,Z)]$$

where $a_3 = \frac{2a}{(n-2)}$, $b_3 = \frac{b}{(n-2)}$, $c_3 = \frac{c}{(n-2)}$, $d_3 = \frac{d}{(n-2)}$, $e_3 = \frac{e}{(n-2)}$.

Let $U^{\perp}$ be the $(n-1)$ distribution orthogonal to U in a conharmonically flat $MS(QE)_n(n>3)$. Then $g(X,U) = 0$ if $X \in U^{\perp}$.

Hence from (5.2) we get the following properties of R

(5.3)
$$\begin{aligned} R(X,Y,Z) &= a_3[Xg(Y,Z)-Yg(X,Z)] \\ &+c_3[XB(Y)B(Z)-YB(X)B(Z) \\ &\quad +g(Y,Z)B(X)V-g(X,Z)B(Y)V] \\ &+d_3[g(Y,Z)UB(X)-g(X,Z)UB(Y)] \\ &+e_3[XD(Y,Z)-YD(X,Z) \\ &\quad +g(Y,Z)lX-g(X,Z)lY] \end{aligned}$$

when $X,Y,Z \in U^{\perp}$ and $a_3 = \frac{2a}{(n-2)}$, $c_3 = \frac{c}{(n-2)}$, $d_3 = \frac{d}{(n-2)}$, $e_3 = \frac{e}{(n-2)}$. Also

(5.4) $$R(X,U,U) = a_3X + c_3B(X)V + d_3B(X)U + e_3lX,$$ where $X \in U^{\perp}$.

Therefore we can state the following theorem:

Theorem 4: *A conharmonically flat $MS(QE)_n (n>3)$ is a manifold of mixed super quasi constant curvature and the curvature tensor R of type (1,3) satisfies the property given by (5.3) and (5.4).*

6 Totally Umbilical Hypersurfaces of a Conharmonically Flat $MS(QE)_n (n>3)$

In this section we consider a hypersurface $(\bar{M}^{n-1}, \bar{g})$ of a conharmonically flat $MS(QE)_n (n>3)$ and denote its curvature tensor of the hypersurface by $\bar{R}$. Then we have the following theorem of Gauss [4].

(6.1)
$$\begin{aligned} g(R(X,Y,Z),W) &= \bar{g}(\bar{R}(X,Y,Z),W) - \bar{g}[h(X,W),h(Y,Z)] \\ &+\bar{g}[h(Y,W),h(X,Z)] \end{aligned}$$

where R is the curvature tensor of $MS(QE)_n$, $\bar{g}$ is the metric tensor of the hypersurface and h is the second fundamental form of the hypersurface and X,Y,Z,W are vector fields tangent to the hypersurface.

If $$h(X,Y) = \bar{g}(X,Y)\mu$$

where μ is mean curvature of $\bar{M}$, then hypersurface is said to be totally umbilical [7].

Let us suppose that hypersurface $\bar{M}$ under consideration is totally umbilical then using (5.2) we can express (6.1) as follows

(6.2)
$$\begin{aligned} \bar{g}(\bar{R}(X,Y,Z),W) &= g(R(X,Y,Z),W) + \bar{g}[h(X,W),h(Y,Z)] \\ &-\bar{g}[h(Y,W),h(X,Z)] \\ &= (a_3 + |\mu|^2)[\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)] \\ &+b_3[\bar{g}(X,W)A(Y)A(Z) - \bar{g}(Y,W)A(X)A(Z) \end{aligned}$$

$$
\begin{aligned}
&+\bar{g}(Y,Z)A(X)A(W)-\bar{g}(X,Z)A(Y)A(W)] \\
&+c_3[\bar{g}(X,W)B(Y)B(Z)-\bar{g}(Y,W)B(X)B(Z) \\
&\quad+\bar{g}(Y,Z)B(X)B(W)-\bar{g}(X,Z)B(Y)B(W)] \\
&+d_3[\bar{g}(X,W)\{A(Y)B(Z)+A(Z)B(Y)\} \\
&\quad-\bar{g}(Y,W)\{A(X)B(Z)+A(Z)B(X)\} \\
&\quad+\bar{g}(Y,Z)\{A(X)B(W)+A(W)B(X)\} \\
&\quad-\bar{g}(X,Z)\{A(Y)B(W)+A(W)B(Y)\}] \\
&+e_3[\bar{g}(X,W)D(Y,Z)-\bar{g}(Y,W)D(X,Z) \\
&\quad+\bar{g}(Y,Z)D(X,W)-\bar{g}(X,Z)D(Y,W)]
\end{aligned}
$$

where $a_3=\frac{2a}{n-2}$, $b_3=\frac{b}{n-2}$, $c_3=\frac{c}{n-2}$, $d_3=\frac{d}{n-2}$, $e_3=\frac{e}{n-2}$.

In virtue of (1.8) it follows from (6.2) that the hypersurface under consideration is mixed super quasi constant curvature. Hence we have the following theorem:

Theorem 5: *A totally umbilical hypersurface of a conharmonically flat $MS(QE)_n (n>3)$ is a manifold of mixed super quasi constant curvature.*

7 Necessary Condition for the Validity of the Relation R(X,Y).S=0

In a $MS(QE)_n$ we can write

(7.1) $$[R(X,Y).S](Z,W)=-S(R(X,Y)Z,W)-S(Z,R(X,Y)W)$$

Making use of the equation (1.5), we can write (1.7) as

$$
\begin{aligned}
[R(X,Y).S](Z,W)=&-[ag(R(X,Y)Z,W)+bA(R(X,Y)Z)A(W) \\
&+cB(R(X,Y)Z)B(W)+dA(R(X,Y)Z)B(W) \\
&+dA(W)B(R(X,Y)Z)+eD(R(X,Y)Z,W)] \\
&-[ag(Z,R(X,Y)W)+bA(Z)A(R(X,Y)W) \\
&+cB(Z)B(R(X,Y)W)+dA(Z)B(R(X,Y)W) \\
&+dA(R(X,Y)W)B(Z)+eD(Z,R(X,Y)W)] \\
=&-[bA(R(X,Y)Z)A(W)+cB(R(X,Y)Z)B(W) \\
&+dA(R(X,Y)Z)B(W)+dA(W)B(R(X,Y)Z) \\
&+eD(R(X,Y)Z,W)] \\
&-[bA(Z)A(R(X,Y)W)+cB(Z)B(R(X,Y)W) \\
&+dA(Z)B(R(X,Y)W)+dA(R(X,Y)W)B(Z) \\
&+eD(Z,R(X,Y)W)]
\end{aligned}
$$

In general R(X,Y).S=0 does not hold good. The necessary condition for the validity of the relation R(X,Y).S=0 for any vector fields X and Y is

(7.2)
$$\begin{aligned} &bA(R(X,Y)Z)A(W)+cB(R(X,Y)Z)B(W)\\ &+dA(R(X,Y)Z)B(W)+dA(W)B(R(X,Y)Z)\\ &+eD(R(X,Y)Z,W)]+bA(Z)A(R(X,Y)W)\\ &+cB(Z)B(R(X,Y)W)+dA(Z)B(R(X,Y)W)\\ &+dA(R(X,Y)W)B(Z)+eD(Z,R(X,Y)W)=0 \end{aligned}$$

Putting $W=U$ in (7.2) we get

(7.3)
$$bA(R(X,Y)Z)+dB(R(X,Y)Z)-eA(R(X,Y)lZ)=0$$

Therefore we can state the following theorem:

Theorem 6: *In a* $MS(QE)_n$ *the necessary condition for the validity of the relation* $R(X,Y).S=0$ *is given by equation (7.3).*

8 Necessary Condition for the Validity of the Relation $R(X,Y).D=0$

(8.1)
$$\begin{aligned}[R(X,Y).D](Z,W)&=D(R(X,Y)Z,W)+D(R(X,Y)W,Z)\\ &=g(lR(X,Y)Z,W)+g(lR(X,Y)W,Z)\end{aligned}$$ (using (2.4))

It is clear from equation (8.1) that in the $MS(QE)_n$ the relation $R(X,Y).D=0$ does not hold good.
Therefore necessary condition for the validity of the relation R(X,Y).D=0 for any vector fields X and Y is

(8.2)
$$g(lR(X,Y)Z,W)+g(lR(X,Y)W,Z)=0$$

Putting $W=U$ in (8.2) we have

$$g(lR(X,Y)Z,U)+g(lR(X,Y)U,Z)=0$$

or, $$g(R(X,Y)U,lZ)=0$$ (using(1.6))

or, $$-g(R(X,Y)lZ,U)=0$$

From above equation it follows that

(8.3)
$$A(R(X,Y)lZ)=0$$ (using(1.2))

Therefore we can state the following theorem:
Theorem7: In a $MS(QE)_n$ the necessary condition for the validity of the relation $R(X,Y).D=0$ is given by equation (8.3).

9 Physical Interpretation

P.Chakraborti, M.Bandyopadhyay and M.Barua in [9] and S.Guha in [10] found that a perfect fluid space- time satisfying Einstein's equation without cosmological constant is a 4-dimensional semi-Riemannian quasi-Einstein manifold and a non-viscous fluid space time admitting heat flux satisfying Einstein's equation without cosmological constant is a 4-dimensional semi-Riemannian generalized quasi-Einstein manifold .

The importance of $MS(QE)_n$ is that such a four dimensional semi-Riemannian manifold is relevant to the study of general relativistic viscous fluid space- time admitting heat flux , where U is taken as the velocity vector field of the fluid , V is taken as the heat flux vector field and D as the anisotropic pressure tensor of the fluid.

The energy momentum tensor of type (0,2) in [8] representing the matter distribution of a viscous fluid space-time admitting heat flux is of the form

(9.1) $$T(X,Y)=(\sigma+\rho)\{A(X)A(Y)+B(X)B(Y)\} \\ +\rho g(X,Y)+[A(X)B(Y)+A(Y)B(X)]+D(X,Y)$$

where σ,ρ denote the density and isotropic pressure and D denotes the anisotropic pressure tensor of the fluid , U is the unit time like velocity vector field of the fluid such that $g(X,U)=A(X)$ and V is the heat flux vector field such that $g(X,V)=B(X)$,U and V being mutually orthogonal.Then

(9.2) $g(U,U)=-1$, $g(V,V)=1$, g(U,V)=0

(9.3) $D(X,Y)=D(Y,X)$ trace $D=O$ and $D(X,U)=0$ $\forall X$.

Now, Einstein's equation without cosmological constant is of the form [11]

(9.4) $$S(X,Y)-\tfrac{1}{2}rg(X,Y)=kT(X,Y)$$

where k is the gravitational constant.

Using equations (9.1) and (9.4) we get

$$S(X,Y)-\tfrac{1}{2}rg(X,Y)=k(\sigma+\rho)\{A(X)A(Y)+B(X)B(Y)\} \\ +k\rho g(X,Y)+k[A(X)B(Y)+A(Y)B(X)]+kD(X,Y)$$

Therefore

(9.5) $$S(X,Y)=(k\rho+\tfrac{1}{2}r)g(X,Y)+k(\sigma+\rho)\{A(X)A(Y)+B(X)B(Y)\} \\ +k[A(X)B(Y)+A(Y)B(X)]+kD(X,Y)$$

Using (9.2), (9.3) and (9.5) we get

(9.6) $$S(U,U)=\tfrac{k}{2}(\sigma+3\rho)$$

(9.7) $$S(V,V)=\tfrac{k}{2}(\sigma+\rho)+kD(V,V)$$

(9.8) $$S(U,V)=-k$$

Solving equations (9.6), (9.7) and (9.8) we get

(9.9) $$\sigma = \frac{S(U,U)}{S(U,V)} - 3\frac{S(V,V)}{S(U,V)} - 3D(V,V)$$

And

(9.10) $$\rho = -\frac{S(U,U)}{S(U,V)} + \frac{S(V,V)}{S(U,V)} + D(V,V)$$

Therefore we can state the following theorem:

Theorem 8: *In a viscous fluid space-time admitting heat flux with an anisotropic tensor field D and satisfying Einstein's equation without cosmological constant the energy density and the isotropic pressure are given by (9.9) and (9.10) respectively.*

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