Gen. Math. Notes, Vol. 23, No. 1, July 2014, pp.15-21
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Module Amenability and Tensor Product of Banach Algebras

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(Received: 11-3-14 / Accepted: 27-4-14)

Abstract

Let Banach algebra $\mathcal{A}$ is a $\mathcal{U}-$module with compatible actions. In this paper we show that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is module amenable when $\mathcal{A}$ is module amenable. In particular, we investigate module amenability of unitization of $\mathcal{A}$.

Keywords: *module amenability, module derivation ,commutative action, Banach algebra.*

1 Introduction

A Banach algebra $\mathcal{A}$ is amenable if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module inner. This concept was introduced by Barry Johnson in [4]. He proved that if $\mathcal{A}$ and $\mathcal{B}$ are amenable Banach algebra, then so is $\mathcal{A}\hat{\otimes}\mathcal{B}$(see also [3]).

M. Amini in [1] introduced the concept of module amenability for a Banach algebra which is a Banach module on another Banach algebra with compatible actions. This could be considered as a generalization of the Johnson's amenability.

In this paper we prove that module amenability of $\mathcal{A}$ implies module amenability of $\mathcal{A}\hat{\otimes}\mathcal{A}$ as a $\mathcal{U}$-module, when $\mathcal{A}$ has a unite and it is a commutative $\mathcal{U}$-module. Also we show, when $\mathcal{A}$ is a $\mathcal{U}$-module with the compatible actions, also $\mathcal{A}^\sharp$ is a $\mathcal{U}$-module if and only if the actions are trivial, where $\mathcal{A}^\sharp$

is the unitization of $\mathcal{A}$.

2 Notation and Preliminaries

Throughout this paper, $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module with compatible action, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathcal{U}).$$

Moreover, if $\alpha \cdot a = a \cdot \alpha$, we say that $\mathcal{A}$ is commutative $\mathcal{U} - module$.
The Banach algebra $\mathcal{U}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathcal{U}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a \quad (a \cdot \alpha = f(\alpha)a)$, where f is a continuous character on $\mathcal{U}$.
Let X be a Banach $\mathcal{A}$-module and a Banach $\mathcal{U}$-module with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$
$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \ (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in X);$$

and the same for right or two-sided actions, then we say that X is a Banach $\mathcal{A}$-$\mathcal{U}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \qquad (\alpha \in \mathcal{U}, x \in X)$$

then X is called a commutative $\mathcal{A}$–$\mathcal{U}$-module. If X is a (commutative) Banach $\mathcal{A} - \mathcal{U}$-module then so is X^*, where the actions of $\mathcal{A}$ and $\mathcal{U}$ on X^* are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \quad (a \cdot f)(x) = f(x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in X, f \in X^*)$$

and the same for the right actions.
It is well known that the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach algebra with respect to the canonical multiplication defined by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd)$$

and extended by bi-linearity and continuity. Then $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{U}$-module by the following canonical actions:

$$\alpha \cdot (a \otimes b) = (\alpha \cdot a) \otimes b, \quad (a \otimes b) \cdot \alpha = (a \otimes b \cdot \alpha) \qquad (\alpha \in \mathcal{U}, a, b \in \mathcal{A})$$

similarity, for the right actions.
Let I be the closed ideal of the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathcal{U}, a, b \in \mathcal{A}$. Also we consider J, the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha \cdot a)b - a(b \cdot \alpha)$ for $\alpha \in \mathcal{U}$, $a, b \in \mathcal{A}$[5].

Let $\mathcal{A}$ and $\mathcal{U}$ be as in the above and X be a Banach $\mathcal{A}-\mathcal{U}$-module. $\mathcal{A}$ bounded map $D:\mathcal{A}\to X$ is called a module derivation if

$$\begin{aligned} D(a\pm b) &= D(a)\pm D(b) \;\;,\;\; D(ab)=Da\cdot b+a\cdot Db \qquad (a,b\in\mathcal{A}) \\ D(\alpha\cdot a) &= \alpha\cdot D(a) \;\;,\;\; D(a\cdot\alpha)=D(a)\cdot\alpha \qquad (a\in\mathcal{A},\alpha\in\mathcal{U}) \end{aligned}$$

although D is not necessary linear, but still boundedness implies its norm continuity (since it preserves subtraction). When X is commutative, each $x\in X$ defines a module derivation

$$\delta_x(a)=a\cdot x-x\cdot a \qquad (a\in\mathcal{A})$$

which is called inner module derivations.
The Banach algebra $\mathcal{A}$ is called module amenable (as an $\mathcal{U}$-module) if for any commutative Banach $\mathcal{A}-\mathcal{U}$-module X, each module derivation $D:\mathcal{A}\to X^*$ is inner.
When the Banach algebra $\mathcal{U}$ acts trivially on $\mathcal{A}$ from left(right) and $\mathcal{A}$ is module amenable we say that $\mathcal{A}$ is module amenable with the trivial left(right) action.
Bodaghi in [2] proved that if $\mathcal{A}$ is a Banach $\mathcal{U}$-module with trivial left action and $\mathcal{A}\hat{\otimes}\mathcal{A}$ is module amenable, then $\mathcal{A}/J\hat{\otimes}\mathcal{A}/J$ is amenable.

3 Module Amenability of Tensor Product of Banach Algebras

Suppose that $\mathcal{A}$ has a unit, we consider the projective tensor products $\mathcal{A}\hat{\otimes}e$ and $e\hat{\otimes}\mathcal{A}$ which are Banach $\mathcal{U}$-modules by the following usual actions:

$$\alpha\cdot(a\otimes e)=\alpha\cdot a\otimes e \;\;,\;\; (a\otimes e)\circ\alpha=a\cdot\alpha\otimes e \qquad (\alpha\in\mathcal{U},a\in\mathcal{A}),$$

$$\alpha\bullet(e\otimes a)=e\otimes\alpha\cdot a \;\;,\;\; (e\otimes a)\cdot\alpha=e\otimes a\cdot\alpha \qquad (\alpha\in\mathcal{U},a\in\mathcal{A}).$$

Note that although $\mathcal{A}\hat{\otimes}e$ and $e\hat{\otimes}\mathcal{A}$ are subalgebra of $\mathcal{A}\hat{\otimes}A$ but by the module actions defined on $\mathcal{A}\hat{\otimes}A$ they are not $\mathcal{U}$-modules.

Lemma 3.1. *If $\mathcal{A}$ is module amenable, then $\mathcal{A}\hat{\otimes}e$ and $e\hat{\otimes}\mathcal{A}$ are module amenable.*

Proof. Consider a module homomorphism $\varphi:\mathcal{A}\to\mathcal{A}\hat{\otimes}e$ defined by $\varphi(a)=a\hat{\otimes}e$. By Proposition 2.5 of [1], the module amenability of $\mathcal{A}$ implies the module amenability of $\mathcal{A}\hat{\otimes}e$. Similarly $e\hat{\otimes}\mathcal{A}$ is module amenable. □

Theorem 3.2. *Let $\mathcal{A}$ be a commutative $\mathcal{U}$-module with a unit. If $\mathcal{A}$ is module amenable, then so is $\mathcal{A}\hat{\otimes}\mathcal{A}$.*

Proof. Let X be a commutative $\mathcal{A}\hat{\otimes}\mathcal{A}$-$\mathcal{U}$-module and $D : \mathcal{A}\hat{\otimes}\mathcal{A} \to X^*$ be a module derivation. Since X is a commutative $\mathcal{U}$-module, we have

$$\begin{aligned}(a \otimes e) \cdot (\alpha \cdot x) &= (a \otimes e) \cdot (x \cdot \alpha)\\ &= ((a \otimes e) \cdot x) \cdot \alpha\\ &= \alpha \cdot ((a \otimes e) \cdot x)\\ &= (\alpha \cdot (a \otimes e)) \cdot x\\ &= (\alpha \cdot a \otimes e) \cdot x\\ &= (a \cdot \alpha \otimes e) \cdot x\\ &= ((a \otimes e) \circ \alpha) \cdot x.\end{aligned}$$

Clearly $\alpha \cdot ((a \otimes e) \cdot x) = (\alpha \cdot (a \otimes e)) \cdot x, (\alpha \cdot x) \cdot (a \otimes e) = \alpha \cdot (x \cdot (a \otimes e))$. And the same for right or two-sided actions. Thus X is a commutative $\mathcal{A}\hat{\otimes}e$-$\mathcal{U}$-module with the compatible actions. Consider $\tilde{D} : \mathcal{A}\hat{\otimes}e \to X^*$ defined by $\tilde{D}(a \otimes e) = D(a \otimes e)$ for all $a \in \mathcal{A}$. By previous lemma, there exist $x^* \in X^*$ such that $\tilde{D} = \delta_{x^*}$. Hence $D|_{\mathcal{A}\hat{\otimes}e} = \delta_{x^*}$. Now consider $\tilde{\tilde{D}} := D - \delta_{x^*}$. Thus $\tilde{\tilde{D}}|_{\mathcal{A}\hat{\otimes}e} = 0$. Let F be the closed linear span of elements of the form

$$\{(a \otimes e) \cdot x - x \cdot (a \otimes e) : a \in \mathcal{A}, x \in X\}.$$

We prove in three step that F is a $e\hat{\otimes}\mathcal{A} - \mathcal{U}$-module with module actions given by

$$(e \otimes b) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e)) := (a \otimes e) \cdot y - y \cdot (a \otimes e)$$

such that $y = (e \otimes b) \cdot x$ and

$$\alpha \diamond ((a \otimes e) \cdot x - x \cdot (a \otimes e)) = (\alpha \cdot a \otimes e) \cdot x - x \cdot (\alpha \cdot a \otimes e),$$

and the same for right or two-sided actions.

Step 1. we show that

$$\alpha \diamond ((e \otimes b) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = (\alpha \bullet (e \otimes b)) \star (a \otimes e) \cdot x - x \cdot (a \otimes e)).$$

We have

$$
\begin{aligned}
\alpha \diamond ((e \otimes b) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e))) &= \alpha \diamond ((a \otimes e) \cdot ((e \otimes b) \cdot x) \\
&- ((e \otimes b) \cdot x) \cdot (a \otimes e)) \\
&= (\alpha \cdot a \otimes e) \cdot ((e \otimes b) \cdot x) \\
&- ((e \otimes b) \cdot x) \cdot (\alpha \cdot a \otimes e) \\
&= (\alpha \cdot (a \otimes e)) \cdot ((e \otimes b) \cdot x) \\
&- ((e \otimes b) \cdot x) \cdot (\alpha \cdot (a \otimes e)) \\
&= \alpha \cdot ((a \otimes e) \cdot ((e \otimes b) \cdot x)) \\
&- ((e \otimes b) \cdot (x \cdot \alpha)) \cdot (a \otimes e) \\
&= ((a \otimes e) \cdot ((e \otimes b) \cdot x)) \cdot \alpha \\
&- ((e \otimes b) \cdot (\alpha \cdot x)) \cdot (a \otimes e) \\
&= (a \otimes e) \cdot ((e \otimes b) \cdot (x \cdot \alpha)) \\
&- ((e \otimes b \cdot \alpha) \cdot x) \cdot (a \otimes e) \\
&= (a \otimes e) \cdot ((e \otimes b) \cdot (\alpha \cdot x)) \\
&- ((e \otimes \alpha \cdot b) \cdot x) \cdot (a \otimes e) \\
&= (a \otimes e) \cdot ((\alpha \bullet (e \otimes b)) \cdot x) \\
&- ((\alpha \bullet (e \otimes b)) \cdot x) \cdot (a \otimes e)
\end{aligned}
$$

and proof of step 1 is complete.

Step 2. Now we prove that

$$(e \otimes b) \star (\alpha \diamond ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = ((e \otimes b) \cdot \alpha) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e)).$$

We have

$$\begin{aligned}
(e\otimes b)\star(\alpha\diamond((a\otimes e)\cdot x-x\cdot(a\otimes e))) &= (e\otimes b)\star((\alpha\cdot a\otimes e)\cdot x \\
&- x\cdot(\alpha\cdot a\otimes e)) \\
&= (\alpha\cdot a\otimes e)\cdot((e\otimes b)\cdot x) \\
&- ((e\otimes b)\cdot x)\cdot(\alpha\cdot a\otimes e) \\
&= (\alpha\cdot(a\otimes e))\cdot((e\otimes b)\cdot x) \\
&- ((e\otimes b)\cdot x)\cdot(\alpha\cdot(a\otimes e)) \\
&= \alpha\cdot((a\otimes e)\cdot((e\otimes b)\cdot x)) \\
&- (((e\otimes b)\cdot x)\cdot\alpha)\cdot(a\otimes e) \\
&= ((a\otimes e)\cdot((e\otimes b)\cdot x))\cdot\alpha \\
&- ((e\otimes b)\cdot(x\cdot\alpha))\cdot(a\otimes e) \\
&= (a\otimes e)\cdot((e\otimes b)\cdot(x\cdot\alpha)) \\
&- ((e\otimes b)\cdot(\alpha\cdot x))\cdot(a\otimes e) \\
&= (a\otimes e)\cdot((e\otimes b)\cdot(\alpha\cdot x)) \\
&- (((e\otimes b)\cdot\alpha)\cdot x)\cdot(a\otimes e) \\
&= (a\otimes e)\cdot(((e\otimes b)\cdot\alpha)\cdot x) \\
&- (((e\otimes b)\cdot\alpha)\cdot x)\cdot(a\otimes e)
\end{aligned}$$

Step 3. Finally, we show that

$(\alpha\diamond((a\otimes e)\cdot x-x\cdot(a\otimes e)))\star(e\otimes b)=\alpha\diamond(((a\otimes e)\cdot x-x\cdot(a\otimes e))\star(e\otimes b))$.

$$\begin{aligned}
(\alpha\diamond((a\otimes e)\cdot x-x\cdot(a\otimes e)))\star(e\otimes b) &= ((\alpha\cdot a\otimes e)\cdot x \\
&- x\cdot(\alpha\cdot a\otimes e))\star(e\otimes b) \\
&= (\alpha\cdot a\otimes e)\cdot(x\cdot(e\otimes b)) \\
&- (x\cdot(e\otimes b))\cdot(\alpha\cdot a\otimes e).
\end{aligned}$$

Similarly for the right and two sided actions, these relations are correct.
For each $a,b\in\mathcal{A}$ and $x\in X$ we have

$\tilde{\tilde{D}}(e\otimes b)((a\otimes e)\cdot x-x\cdot(a\otimes e))=(\tilde{\tilde{D}}(e\otimes b)\cdot(a\otimes e)-(a\otimes e)\cdot\tilde{\tilde{D}}(e\otimes b))(x)$.

Since $\tilde{\tilde{D}}|_{\mathcal{A}\hat{\otimes}e}=0$,

$$\tilde{\tilde{D}}(a\otimes b)=(a\otimes e)\cdot\tilde{\tilde{D}}(e\otimes b)=\tilde{\tilde{D}}(e\otimes b)\cdot(a\otimes e).$$

Therefore $\tilde{\tilde{D}}(e\hat{\otimes}\mathcal{A})\subseteq F^{\perp}=(X/F)^{*}$ and $\tilde{\tilde{D}}|e\hat{\otimes}\mathcal{A}:e\hat{\otimes}\mathcal{A}\to(X/F)^{*}$ is a module derivation. By the previous lemma, there exists $f^{*}\in F^{\perp}$ Such that $\tilde{\tilde{D}}|_{e\hat{\otimes}\mathcal{A}}=\delta_{f^{*}}$. Then we have $D-\delta_{x^{*}}|_{e\hat{\otimes}\mathcal{A}}=\delta_{f^{*}}$. Since $\delta_{f^{*}}|_{\mathcal{A}\hat{\otimes}e}=0$, $D-\delta_{x^{*}}=\delta_{f^{*}}$ and proof is complete. □

Corollary 3.3. *Let $\mathcal{A}$ be a $\mathcal{U}$-module and $\mathcal{A}/J$ has a unit. Then $\mathcal{A}/J\hat{\otimes}\mathcal{A}/J$ is module amenable.*

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