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Fuzzy rw-Connectedness and Fuzzy rw-Disconnectedness in Fuzzy Topological Spaces

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Abstract

In this paper, using the concept of fuzzy rw-open set we shall define various notions of fuzzy rw-connectedness and fuzzy rw-disconnectedness in fuzzy topological spaces and some of their properties and characterizations of such spaces are also investigated.

Keywords: *Fuzzy rw-connected, fuzzy super rw-connected, fuzzy strongly rw-connected, extremally fuzzy rw-disconnectedness, totally fuzzy rw-disconnectedness.*

1 Introduction and Preliminaries

Fuzzy relations were first introduced by Lofti A. Zadeh in 1971 [22]. Fuzzy sets have application in applied fields such as information [17], control [18, 19] and pattern recognition [13, 16]. This concept has been applied by many authors to several branches of mathematics particuarly in the fields such as fuzzy numbers [5], fuzzy groups [20], fuzzy topological groups [7], L-fuzzy sets [9], fuzzy linear spaces [10], fuzzy algebra [12] fuzzy vector spaces [14] and fuzzy proximity space [15]. One of the applications was the study of fuzzy topological spaces [4] introduced and studied by chang in 1968. Ever since the introducting of fuzzy topological spaces notions like fuzzy regular open [1] and fuzzy regular semiopen [21] were extended from general topological structures.

The concept of rw-open sets and fuzzy rw-open sets was introduced and studied by Wali [21] motivated us to study on fuzzy rw-connectedness in fuzzy topological space. In this paper, using the concept of fuzzy rw-open sets. We shall defined varies notions of fuzzy rw-connectedness and rw-disconnectedness in fuzzy topological spaces and some of their properties and characterzation of such spaces and also investigated.

Throughtout the paper I will denote the unit interval $[0, 1]$ of the real line R. X, Y, Z will be nonempty sets. The symbols λ, μ, γ, η ... are used to denote fuzzy sets and all other symbols have their usual meaning unless explicity stated.

Definition 1.1. *A fuzzy set λ in a fts (X, τ) is said to be fuzzy regular open set [1] if $int(cl\lambda) = \lambda$ and a fuzzy regular closed set if $cl(int(\lambda)) = \lambda$,*
(iii) fuzzy regular semi open [23] if there exists fuzzy regular open set σ in X such that $\sigma \leq \alpha \leq cl(\sigma)$.

Definition 1.2. *A fts (X, T) is said to be fuzzy connected [11] if it has no proper fuzzy clopen set. Otherwise it is called fuzzy disconnected.*

Definition 1.3. *A fuzzy set λ in fts (X, T) is proper if $\lambda \neq 0$ and $\lambda \neq 1$.*

Definition 1.4. *A fts (X, T) is called fuzzy super connected [6] if it has no proper fuzzy regular open set.*

Definition 1.5. *A fts (X, T) is called fuzzy strongly connected [6] if it has no non-zero fuzzy closed sets λ and μ such that $\lambda + \mu \leq 1$.*

Definition 1.6. *A fts (X, T) is said to be fuzzy extremally disconnected [2] if $\lambda \in T$ implies $Cl(\lambda) \in T$.*

Definition 1.7. *A fts (X, T) is said to be fuzzy totally disconnected [2] if and only if for every pair of fuzzy points p, q with $p \neq q$ in (X, T) there exists non-zero fuzzy open sets λ, μ such that $\lambda + \mu = 1$, λ contains p and μ contains q. Suppose $A \subset X$. A is said to be a fuzzy totally disconnected subset of X if A as a fuzzy subspace of (X, T) is fuzzy totally disconnected.*

Definition 1.8. *Let (X, T) be fts and Y be an ordinary subset of X. Then $T/Y = \{\lambda/Y : \lambda \in T\}$ is a fuzzy topology on Y and is called the induced or relative fuzzy topology. The pair $(Y, T/Y)$ is called a fuzzy subspace [8] of (X, T). $(Y, T/Y)$ is called fuzzy open (fuzzy closed) subspace if the characteristic function of Y viz., χ_Y is fuzzy open (fuzzy closed).*

Definition 1.9. *A fuzzy set λ in a fts $(X, \ \tau)$ is said to be a fuzzy regular weakly closed set (briefly, frw-closed) [21] if $cl(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is fuzzy regular semi-open in X.*

From this definition it is clear that λ is fuzzy rw-open $\Leftrightarrow$ $1 - \lambda$ is fuzzy rw-closed. Also we define $\lambda_0 = \vee\{\delta |\ \delta \leq \lambda, \delta$ is fuzzy rw -open$\}$, $\underline{\lambda} = \wedge\{\delta |\ \delta \geq \lambda, \delta$ is fuzzy rw -closed$\}$. λ_0 and $\underline{\lambda}$ are called fuzzy rw-interior and fuzzy rw-closure of λ respectively. A fuzzy set λ which is both fuzzy rw-open and fuzzy rw-closed is called fuzzy rw-clopen.

It is easy to verify the following relations between fuzzy rw-interior and fuzzy rw-closure: (a) $1 - \lambda_0 = (\underline{1 - \lambda})$, (b) $1 - \underline{\lambda} = (1 - \lambda)_0$ where λ is any fuzzy set in (X, T).

We shall denote the class of all fuzzy rw-open sets of the fuzzy topological space (X, T) by $FRWO(X, T)$.

2 Fuzzy rw-Connectedness

Definition 2.1. *A fuzzy topological space (X, T) is said to be fuzzy rw-connected if (X, T) has no proper fuzzy set λ which is both fuzzy rw-open and fuzzy rw-closed.*

Proposition 2.1. *A fuzzy topological space (X, T) is fuzzy rw-connected $\Leftrightarrow$ it has no non-zero fuzzy rw-open sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$.*

Proof. If such λ_1 and λ_2 exist, then λ_1 is a proper fuzzy set which is both fuzzy rw-open and fuzzy rw-closed. To prove the converse suppose that (X, T) is not fuzzy rw-connected. Then it has a proper fuzzy set λ_1 (say) which is both fuzzy rw-open and fuzzy rw-closed. Now put $\lambda_2 = 1 - \lambda_1$. Then λ_2 is a fuzzy rw-open set such that $\lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 = 1$.

Corollary 2.1. *A fuzzy topological space (X, T) is fuzzy rw-connected $\Leftrightarrow$ it has no non-zero fuzzy sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = \underline{\lambda_1} + \lambda_2 = \lambda_1 + \underline{\lambda_2} = 1$.*

Definition 2.2. *If $A \subset X$, then A is said to be fuzzy rw-connected subset of (X, T) when $(A, T/A)$ is fuzzy rw-connected as a fuzzy subspace of (X, T).*

Proposition 2.2. *Let A be a fuzzy rw-connected subset of X and λ_1 and λ_2 be non-zero fuzzy rw-open sets in (X, T) such that $\lambda_1 + \lambda_2 = 1$. Then either $\lambda_1/A = 1$ or $\lambda_2/A = 1$.*

Proof. Follows from proposition 2.1.

Proposition 2.3. *Let $(A, T/A)$ be a fuzzy subspace of the fuzzy topological space (X, T) and let λ be a fuzzy set in A. Further let δ be the fuzzy set in X defined as*

$$\delta(x) = \begin{cases} \lambda(x) & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A \end{cases}$$

Then $(\underline{\lambda})_{T/A} = (\underline{\delta})_{T/A}$ where $(\underline{\lambda})_{T/A}$ is the fuzzy rw- closure of λ with respect to T/A and $(\underline{\delta})_T$ is the fuzzy rw-closure of δ with respect to T.

Definition 2.3. *Fuzzy sets λ_1 and λ_2 in a fuzzy topological space (X, T) are said to be fuzzy rw-separated if $\underline{\lambda}_1 + \lambda_2 \leqslant 1$ and $\lambda_1 + \underline{\lambda}_2 \leqslant 1$.*

Proposition 2.4. *Let $\{ A_k\}_{k \in \Gamma}$ be a family of fuzzy rw-connected subsets of (X, T) such that for each $k, \ell \in \Gamma$ and $k \neq \ell, \chi_{A_k}$ and χ_{A_ℓ} are not fuzzy rw-separated from each other. Then $\bigcup\limits_{k \in \Gamma} A_k$ is a fuzzy rw-connected subset of (X, T).*

Proof. Suppose $Y = \bigcup\limits_{k \in \Gamma} A_k$ is not fuzzy rw-connected subset of (X, T). Then there exist non zero fuzzy rw-open sets δ and σ in Y such that $\delta + \sigma = 1$. Fix $k_0 \in \Gamma$. Then A_{k_0} is a fuzzy rw-connected subset of Y as it is so in (X, T). Then by Proposition 2.2., either $^{\delta}/_{A_{k_0}} = 1$ or $^{\sigma}/_{A_{k_0}} = 1$. Without loss of generality we can assume that

$$\delta/A_{k_0} = 1 = \chi_{A_{k_0}}/A_{k_0} \qquad (i)$$

Define two fuzzy sets λ_1 and λ_2 in X as follows:

$$\lambda_1(x) = \begin{cases} \delta(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in X \setminus Y; \end{cases}$$
$$\lambda_2(x) = \begin{cases} \sigma(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in X \setminus Y. \end{cases}$$

Then by Proposition 2.3.,

$$(\underline{\delta})_{T/Y} = \left(\underline{\lambda_1}\right)_T/Y \text{ and } (\underline{\sigma})_{T/Y} = \left(\underline{\lambda_2}\right)_T/Y \qquad (ii)$$

Now (i) implies that

$$\chi_{A_{k_0}} \leqslant \lambda_1 \text{ and so } \underline{\chi_{A_{k_0}}} \leqslant \underline{\lambda_1} \qquad (iii)$$

Let $k \in \Gamma - \{k_0\}$. Since A_k is a fuzzy rw- connected subset of Y either $\delta/A_k = 1$ or $\sigma/A_k = 1$. We shall show that $^{\chi_{A_k}}/_{A_k} \neq {}^{\sigma}/_{A_k}$. Suppose that $^{\chi_{A_k}}/_{A_k} = {}^{\sigma}/_{A_k}$.

Then

$$\chi_{A_k} \leq \lambda_2 \text{ and hence } \underline{\chi_{A_k}} \leq \underline{\lambda_2} \qquad (iv)$$

Since $\delta + \sigma = \underline{\delta} + \sigma = \delta + \underline{\sigma} = 1, \lambda_1 + \underline{\lambda_2} \leq 1$ and $\underline{\lambda_1} + \lambda_2 \leq 1$ (by (ii) and definitions of λ_1 and λ_2). Now (iii) and (iv) imply that

$$\underline{\chi_{A_{k_0}}} + \chi_{A_k} \leq \underline{\lambda_1} + \lambda_2 \leq 1 \text{ and } \chi_{A_{k_0}} + \underline{\chi_{A_k}} \leq \lambda_1 + \underline{\lambda_2} \leq 1.$$

This gives a contradiction as $\chi_{A_{k_0}}$ and χ_{A_k} are not rw- separated from each other. This contradiction shows that ${\chi_{A_k}}/{A_k} \neq {\sigma}/{A_k}$ and hence ${\chi_{A_k}}/{A_k} = {\delta}/{A_k}$ for $k \in \Gamma$ implies $\delta = {\chi_{A_k}}/{Y}$. But $\delta + \sigma = 1$. So $\sigma(x) = 0$ for all $x \in Y$. That is $\sigma = 0$, which is a contradiction since $\sigma \neq 0$. So our assumption is wrong. Hence the proposition is proved.

Corollary 2.2. *Let $\{A_k\}_{k\in\Gamma}$ be a family of fuzzy rw-connected subsets of a fuzzy topological space (X, T) and $\bigcap_{k\in\Gamma} A_k \neq \phi$. Then $\bigcup_{k\in\Gamma} A_k$ is a fuzzy rw-connected subset of (X, T).*

Corollary 2.3. *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of fuzzy rw-connected subsets of a fuzzy topological space (X, T) such that χ_{A_n} and $\chi_{A_{n+1}}$ are not fuzzy rw-separated from each other for $n = 1, 2, 3, \cdots$, then $\bigcup_{n=1}^{\infty} A_n$ is a fuzzy rw-connected subset of (X, T).*

The following proposition is easy to establish.

Proposition 2.5. *Let A and B be subsets of a fuzzy topological space (X, T) such that $\chi_A \leq \chi_B \leq \underline{\chi_A}$. If A is a fuzzy rw-connected subset of (X, T), then B is also a fuzzy rw-connected subset of (X, T).*

3 Fuzzy Super rw-Connectedness

Definition 3.1. *A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy rw-regular open set if $\lambda = (\underline{\lambda})_0$.*

Definition 3.2. *A fuzzy topological space (X, T) is called fuzzy super rw-connected if there is no proper fuzzy rw-regular open set.*

In the following proposition we give several characterizations of fuzzy super rw-connected spaces.

Proposition 3.1. *The following are equivalent for a fuzzy topological space (X, T).*

1. (X, T) *is fuzzy super* rw*-connected.*

2. $\underline{\lambda} = 1$ *whenever* λ *is any non-zero fuzzy* rw*- open set in* (X, T).

3. $\lambda_0 = 0$ *whenever* λ *is a fuzzy* rw*-closed set in* (X, T) *such that* $\lambda \neq 1$.

4. (X, T) *does not have non-zero fuzzy* rw*-open sets* λ_1 *and* λ_2 *such that* $\lambda_1 + \lambda_2 \leq 1$.

5. (X, T) *does not have non-zero fuzzy sets* λ_1 *and* λ_2 *such that* $\underline{\lambda_1} + \lambda_2 = \lambda_1 + \underline{\lambda_2} = 1$.

6. (X, T) *does not have fuzzy closed sets* δ_1 *and* δ_2 *such that* $\delta_1 + \delta_2 \geq 1$.

Definition 3.3. *A subset* A *of a fuzzy topological space* (X, T) *is called fuzzy super* rw*- connected subset of* (X, T) *if it is fuzzy super* rw*- connected topological space as a fuzzy subspace of* (X, T).

Definition 3.4. *Let* (X, T) *be a fuzzy topological space. If* $A \subset Y \subset X$, *then* A *is a fuzzy super* rw*-connected subset of* $(X, T) \Leftrightarrow$ *it is a fuzzy super* rw*-connected subset of the fuzzy subspace* $(Y, T/Y)$ *of* (X, T).

Proposition 3.2. *Let* A *be a fuzzy super* rw*-connected subset of a fuzzy topological space* (X, T). *If there exists fuzzy* rw*-closed sets* δ_1 *and* δ_2 *in* X *such that* $(\delta_1)_0 + \delta_2 = \delta_1 + (\delta_2)_0 = 1$, *then* $\delta_1/A = 1$ *or* $\delta_2/A = 1$.

Proof. Suppose that A is not a fuzzy super rw-connected space. Then there exists fuzzy rw-closed sets δ_1, δ_2 in X such that
$\delta_1/A \neq 0$, $\delta_2/A \neq 0$,
and $\delta_1/A + \delta_2/A \leq 1$,

Then X is not a fuzzy super rw-connected space, a contradication.

Proposition 3.3. *Let* (X, T) *be a fuzzy topological space and* $A \subset X$ *be a fuzzy super* rw*- connected subset of* X *such that* χ_A *is fuzzy* rw*-open set in* X. *If* λ *is a fuzzy regular* rw*-open set in* (X, T), *then either* $\chi_A \leq \lambda$ *or* $\chi_A \leq 1 - \lambda$.

Proof. Follows from proposition 3.2.

Proposition 3.4. *Let* $\{A_k\}_{k \in \Gamma}$ *be a family of subsets of a fuzzy topological space* (X, T) *such that each* χ_{A_k} *is fuzzy* rw*-open . If* $\bigcap_{k \in \Gamma} A_k \neq \phi$ *and each* A_k *is a fuzzy super* rw*-connected subset of* X, *then* $\bigcup_{k \in \Gamma} A_k$ *is also a fuzzy super* rw*- connected subset of* (X, T).

Proposition 3.5. *If A and B are subsets of a fuzzy topological space (X,T) and $\chi_A \leq \chi_B \leq \underline{\chi_A}$ and if A is a fuzzy super rw- connected subset of (X,T), then so is B.*

Proposition 3.6. *Let (X,T) and (Y,S) be any two fuzzy super rw- connected spaces. Assume (X,T) and (Y,S) are product related. Then $(X \times Y, T \times S)$ is a fuzzy super rw-connected space.*

Proof. Let (X,T) and (Y,S) be fuzzy super rw-connected topological spaces. Assume (X,T) and (Y,S) are product related. Suppose now that $(X \times Y, T \times S)$ is not fuzzy super rw- connected. Then there exists $\lambda_1,\ \lambda_2 \in FRWO(X,T)$ and $\eta_1, \eta_2 \in FRWO(Y,S)$ such that

$$\lambda_1 \times \eta_1 \neq 0,\ \lambda_2 \times \eta_2 \neq 0$$

and

$$(\lambda_1 \times \eta_1)(x,y) + (\lambda_2 \times \eta_2)(x,y) \leq 1 \text{ for every } (x,y) \in X \times Y \qquad (i)$$

As (X,T) and (Y,S) are product related $\lambda_1 \times \eta_1$ and $\lambda_2 \times \eta_2$ are fuzzy rw-open sets in $(X \times Y, T \times S)$.

Now $\lambda_1 \times \eta_1 = P_X^{-1}(\lambda_1) \wedge P_Y^{-1}(\eta_1)$; P_X is the projection map of $X \times Y$ onto X etc., So $\min\{(\lambda_1(x), \eta_1(y)\} + \min\{(\lambda_2(x), \eta_2(y)\} \leq 1$ for every $(x,y) \in X \times Y$. Now from (i) we have that for any $(x,y) \in X \times Y$ either (i) $\lambda_1(x) + \lambda_2(x) \leq 1$ or (ii) $\lambda_1(x) + \eta_1(y) \leq 1$ or (iii) $\eta_1(y) + \lambda_2(x) \leq 1$ or (iv) $\eta_1(y) + \eta_2(y) \leq 1$. Now $\lambda_1 \wedge \lambda_2 \in FRWO(X,T)$ and $\eta_1 \wedge \eta_2 \in FRWO(Y,S)$. As (X,T) and (Y,S) are fuzzy super rw- connected topological spaces, if $\lambda_1 \wedge \lambda_2 \neq 0, \eta_1 \wedge \eta_2 \neq 0$, then there exists $x_1 \in X$ and $y_1 \in Y$ such that $(\lambda_1 \wedge \lambda_2)(x_1) > 1/2$ and $(\eta_1 \wedge \eta_2)(y_1) > 1/2$. So $\lambda_1(x_1) > 1/2, \lambda_2(x_1) > 1/2, \eta_1(y_1) > 1/2, \eta_2(y_1) > 1/2$. Therefore, if $x = x_1, y = y_1$ then none of the above four possibilities will be true. If $\lambda_1 \wedge \lambda_2 = 0$, then for each $x \in X$ either $\lambda_1(x) = 0$ or $\lambda_2(x) = 0$. So for every $x \in X, \lambda_1(x) + \lambda_2(x) \leq 1$. Note that $\lambda_1 \neq 0, \lambda_2 \neq 0$ as both $\lambda_1 \times \eta_1 \neq 0$ and $\lambda_2 \times \eta_2 \neq 0$, which implies that (X,T) is not fuzzy super rw-connected. Similarly, $\eta_1 \wedge \eta_2 = 0$ will imply that (Y,S) is not fuzzy super rw- connected. This gives a contradiction as both (X,T) and (Y,S) are fuzzy super rw- connected spaces. So our assumption that $(X \times Y, T \times S)$ is not fuzzy super rw-connected is wrong.

4 Fuzzy Strongly rw-Connectedness

Definition 4.1. *A fuzzy topological space (X,T) is said to be fuzzy strongly rw- connected if it has no non-zero fuzzy rw- closed sets λ and μ such that $\lambda + \mu \leq 1$.*

If (X, T) is not fuzzy strongly rw- connected, then it is called fuzzy weakly rw-connected.

Proposition 4.1. *(X, T) is fuzzy strongly rw-connected $\Leftrightarrow$ it has no non-zero fuzzy rw-open sets λ_1 and λ_2 such that $\lambda_1 \neq 1, \lambda_2 \neq 1$ and $\lambda_1 + \lambda_2 \geq 1$.*

Proposition 4.2. *Let $(A, T/A)$ be a fuzzy subspace of a fuzzy strongly rw-connected space (X, T). Then A is fuzzy strongly rw-connected $\Leftrightarrow$ for any fuzzy rw-open sets λ_1 and λ_2 in $(X, T), \chi_A \leq \lambda_1 + \lambda_2$ implies either $\chi_A \leq \lambda_1$ or $\chi_A \leq \lambda_2$.*

Proof. If A is not a fuzzy strongly rw-connected subset of X, then there exist fuzzy rw-closed sets f and k in X such that (i) $f/A \neq 0$ (ii) $k/A \neq 0$ and (iii) $f/A + k/A \leq 1$. If we put $\lambda_1 = 1 - f$ and $\lambda_2 = 1 - k$ then $\lambda_1/A = 1 - f/A$, $\lambda_2/A = 1 - k/A$. So (i), (ii) and (iii) imply that $\chi_A \leq \lambda_1 + \lambda_2$ but $\chi_A \nleq \lambda_1$ and $\chi_A \nleq \lambda_2$. Conversely if there exist fuzzy rw-open sets λ_1 and λ_2 such that $\chi_A \leq \lambda_1 + \lambda_2$ but $\chi_A \nleq \lambda_1$ and $\chi_A \nleq \lambda_2$ then $\lambda_1/A \neq 1, \lambda_2/A \neq 1$ and $\lambda_1/A + \lambda_2/A \geq 1$. So A is not fuzzy strongly rw-connected.

Proposition 4.3. *Let (X, T) be a fuzzy strongly rw- connected space. Let A be a subset of (X, T) such that χ_A is fuzzy rw- closed in (X, T). Then A is fuzzy strongly rw- connected sub set of (X, T).*

Proof. Suppose that A is not so. Then there exist fuzzy rw-closed sets f and k in X such that (i) $f/A \neq 0$ (ii) $k/A \neq 0$ and (iii) $f/A + k/A \leq 1$. (iii) implies that $(f \wedge \chi_A) + (K \wedge \chi_A) \leq 1$ where by (i) and (ii) $f \wedge \chi_A \neq 0$, $k \wedge \chi_A \neq 0$. So X is not fuzzy strongly rw-connected, which is a contradiction.

Regarding the product of fuzzy strongly rw-connected spaces we prove the following.

Proposition 4.4. *Let (X, T) and (Y, S) be fuzzy strongly rw-connected spaces. Assume (X, T) and (Y, S) are product related. Then $(X \times Y, T \times S)$ is a fuzzy strongly rw-connected space.*

Proof. Let (X, T) and (Y, S) be fuzzy strongly rw-connected spaces such that (X, T) and (Y, S) are product related. We claim that $(X \times Y, T \times S)$ is fuzzy strongly rw- connected. Suppose not. Since members of $Frwo(X \times Y, T \times S)$ are precisely of the type (by Theorem 1.5 of [3]) "$\lambda \times \delta$" where $\lambda \in Frwo(X, T), \delta \in Frwo(Y, S)$, there exist non-zero fuzzy sets $\lambda_1, \lambda_3 \in Frwo(X, T)$ and $\lambda_2, \lambda_4 \in Frwo(Y, S)$ such that $\lambda_1 \times \lambda_2 \neq 1, \lambda_3 \times \lambda_4 \neq 1$ and for every $x \in X, y \in Y$,

$$\min\{\lambda_1(x), \lambda_2(y)\} + \min\{\lambda_3(x), \lambda_4(y)\} \geq 1. \tag{1}$$

Clearly $\lambda_1 \vee \lambda_3 \in Frwo(X,T)$ and $\lambda_2 \vee \lambda_4 \in Frwo(Y,S)$. Given that (X,T) and (Y,S) are fuzzy strongly rw-connected, so if $\lambda_1 \vee \lambda_3 \neq 1$, and $\lambda_2 \vee \lambda_4 \neq 1$, then there is $x_1 \in X$ and $y_1 \in Y$ such that $(\lambda_1 \vee \lambda_3)(x_1) < \frac{1}{2}$ and $(\lambda_2 \vee \lambda_4)(y_1) < \frac{1}{2}$ which implies $\lambda_1(x_1) < \frac{1}{2}, \lambda_3(x_1) < \frac{1}{2}, \lambda_2(y_1) < \frac{1}{2}$ and $\lambda_4(y_1) < \frac{1}{2}$. So for $x = x_1$ and $y = y_1$, (1) does not hold. If $\lambda_1 \vee \lambda_3 = 1$, then for each $x \in X$,

$$\lambda_1(x) = 1 \text{ or } \lambda_3(x) = 1. \tag{2}$$

Now we show that $\lambda_1 \neq 1$. Suppose $\lambda_1 = 1$. Then $\lambda_1 \times \lambda_2 \neq 1$ and (Y,S) is fuzzy strongly rw -connected implies that there exists $y_0 \in Y$ such that $\lambda_2(y_0) < \frac{1}{2}$. Now $\lambda_3 \times \lambda_4 \neq 1$. So either $\lambda_3 \neq 1$ or $\lambda_4 \neq 1$.

Case 1: If $\lambda_3 \neq 1$, then as (X,T) is fuzzy strongly rw -connected there is $x_0 \in X$ such that $\lambda_3(x_0) < \frac{1}{2}$. So for $x = x_0, y = y_0$, (1) is not true.

Case 2: If $\lambda_4 \neq 1$, then since $\lambda_2 \neq 1$ and (Y,S) is fuzzy strongly rw - connected there is $y_1 \in Y$ such that $\lambda_2(y_1) + \lambda_4(y_1) < 1$. So for any $x \in X$ and $y = y_1$,

$$\min\{\lambda_1(x), \lambda_2(y)\} + \min\{\lambda_3(x), \lambda_4(y)\} \leq \lambda_2(y_1) + \lambda_4(y_1) < 1.$$

This is a contradiction because of (1). Thus $\lambda_1 = 1$ is not possible. Similarly, we can prove that $\lambda_3 \neq 1$. By (2) $\lambda_1 + \lambda_3 \geq 1$. So (X,T) is not fuzzy strongly rw-connected, which is a contradiction. Therefore $\lambda_1 \vee \lambda_3 = 1$ is not possible. Similarly, we can show that $\lambda_2 \vee \lambda_4 = 1$ is not possible. This proves that our assumption is wrong. Hence the proposition is proved.

Example 4.1 The following example [6] shows that an infinite product of fuzzy strongly rw-connected spaces need not be fuzzy strongly rw -connected. Let $X_n = [0,1], n = 1,2,\cdots$ and $T_n = \left\{0, 1, \frac{n}{2(n+1)}\right\}$, $n = 1,2,\cdots$. Clearly (X_n, T_n) is fuzzy strongly rw -connected for all $n = 1,2,\cdots$. Let T be the product fuzzy topology on $X = \prod_{n=1}^{\infty} X_n$. Then (X,T) is not strongly fuzzy rw-connected, since T contains a member $\bigvee_{n=1}^{\infty} P_n^{-1}(\lambda_n) \neq 1$ such that $\bigvee_{n=1}^{\infty} P_n^{-1}(\lambda_n)(x) = \frac{1}{2}$ for $x \in X$, where $\lambda_n = \frac{n}{2(n+1)}$ and $P_n : X \to X_n$ is the projection mapping.

Definition 4.2. *Let (X,T) and (Y,S) be any two fuzzy topological spaces and let $F : (X,T) \to (Y,S)$ be a mapping. For each $\alpha \in I^Y$, the fuzzy dual F^* of F is defined as $F^*(\alpha) = sup\{\beta \in I^X, \alpha \nleq F(\beta)\}$.*

Definition 4.3. *A mapping F from a fuzzy topological space (X,τ) to another fuzzy topological space (Y,S) is called fuzzy upper* (lower*) rw-continuous iff for every fuzzy open (closed) set λ in Y, the set $\mu \in I^X$ such that $F(\mu) \leq \lambda$ is fuzzy rw-open (closed) in X.*

Proposition 4.5. *Let $F : (X, T) \to (Y, S)$ be any mapping from a fuzzy topological space (X, T) to another fuzzy topological space (Y, S). Then the following are equivalent.*

1. *For $\lambda \in I^Y$, the subsets $\{A_\beta\}_{\beta \in \Gamma}$ with $\chi_{A_\beta} = F^*(\beta)$ such that $\beta \leq \lambda$ are fuzzy rw-connected.*

2. *For every pair of fuzzy sets (λ_1, λ_2) in $(X \times X,\ T(X \times X))$, there exists a fuzzy rw-connected subset C with $\chi_C \leq \lambda_1$ or $\chi_C \leq \lambda_2$ such that $F(\mu) \leq F(\lambda_1) \leq F(\lambda_1) \vee F(\lambda_2)$ for all $\mu \geq \chi_C$.*

3. *For every pair of fuzzy sets (λ_1, λ_2) in $(X \times X,\ T(X \times X))$, the subset A such that $F(\chi_A) \leq F(\lambda_1) \vee F(\lambda_2)$ is fuzzy rw-connected.*

Proof. (a) $\Rightarrow$ (c): Let $\lambda = F(\lambda_1) \vee F(\lambda_2)$ and let $\{A_\beta\}_{\beta \in \Gamma}$ be subsets with $\chi_{A_\beta} = F^*(\beta)$ for all $\beta \leq \lambda$.

Let $\mu = \bigwedge \chi_{A_\beta}$
$\Leftrightarrow \mu \leq F^*(\beta)$ for all $\beta \leq \lambda$
$\Leftrightarrow$ for those μ, $\beta \not\leq F(\gamma)$, for all $\beta \leq \lambda$, $\gamma \leq F^*(\beta)$
$\Leftrightarrow$ for those μ, $F(\gamma) \leq F(\lambda_1) \vee F(\lambda_2)$
$\Leftrightarrow$ for those μ, $F(\mu) \leq F(\lambda_1) \vee F(\lambda_2)$
$\Leftrightarrow \mu = \chi_A$ such that $F(\chi_A) \leq F(\lambda_1) \vee F(\lambda_2)$

Thus, $\bigwedge \chi_{A_\beta} = \chi_A$ with $F(\chi_A) \leq F(\lambda_1) \vee F(\lambda_2)$
$\Rightarrow A = \bigcap A_\beta$ with $F(\chi_A) \leq F(\lambda_1) \vee F(\lambda_2)$.

Hence (c) follows.

(c) $\Rightarrow$ (b): Let C be a subset such that if $F(\eta) \leq F(\lambda_1) \vee F(\lambda_2)$ then $\eta \geq \chi_C$. Clearly $\chi_C \leq \lambda_1$ or $\chi_C \leq \lambda_2$ and C is fuzzy rw-connected.

(b) $\Rightarrow$ (a): For any $\lambda \in I^Y$, let D be a subset with $\chi_D = \bigwedge \chi_{A_\beta} = \bigwedge F^*(\beta)$ for all $\beta \leq \lambda$ and $\chi_D \leq \gamma_1$ or $\chi_D \leq \gamma_2$. Then $\alpha \not\leq F(\gamma_1)$ for all $\alpha \leq \lambda$ and $\alpha \not\leq F(\gamma_2)$ for all $\alpha \leq \lambda$.

So $\alpha \not\leq F(\gamma_1) \vee F(\gamma_2)$ for all $\alpha \leq \lambda$. By hypotheis there exists a fuzzy rw-connected subset C such that $\chi_C \leq \gamma_1$ or $\chi_C \leq \gamma_2$ with $F(\eta) \leq F(\gamma_1) \vee F(\gamma_2)$ for all $\chi_C \leq \eta$. Therefore, for any $\alpha \leq \lambda$, $\alpha \not\leq F(\eta)$ we have $\eta \leq F^*(\alpha)$ for all $\alpha \leq \lambda$. Thus $\eta \leq \bigwedge F^*(\alpha) = \chi_D$. Hence $\chi_C \leq \chi_D$. Therefore, D is fuzzy rw-connected.

Proposition 4.6. *Let F be a mapping from a fuzzy topological space (X, T) to another fuzzy topological space (Y, S). Then*

*(a) F is fuzzy upper*rw continuous iff $\bigwedge\{F^*(\alpha) : \alpha \leq \lambda\}$ is fuzzy rw-open in X for fuzzy closed set λ in Y.*
*(b) F is fuzzy lower*rw continuous iff $\bigwedge\{F^*(\alpha) : \alpha \leq \mu\}$ is fuzzy rw-closed in X for fuzzy open set μ in Y.*

Proof. Consider λ is fuzzy closed in Y. $1-\lambda$ is fuzzy open in Y. F is fuzzy upper*rw-continuous $\Leftrightarrow \mu \in I^X$ such that $F(\mu) \leq 1-\lambda$ is fuzzy rw-open in X.
$\Leftrightarrow \{\mu \in I^X$ such that $\gamma \not\leq F(\mu)\}$ is fuzzy rw-open for any $\gamma \leq \lambda$.
$\Leftrightarrow \bigwedge\{\mu \in I^X$ such that $\gamma \not\leq F(\mu), \gamma \leq \lambda\}$ is fuzzy rw-open.
$\Leftrightarrow \bigwedge\{F^*(\gamma), \gamma \leq \lambda\}$ is fuzzy rw-open for any fuzzy closed set λ in Y.
(b) The proof is similar to (a).

5 Extremally Fuzzy rw-Disconnectedness

Definition 5.1. *Let (X,T) be a fuzzy topological space. (X,T) is said to be extremally fuzzy rw-disconnected $\Leftrightarrow$ $\underline{\lambda}$ is fuzzy rw -open for every $\lambda \in Frwo(X,T)$.*

Using the techniques adopted in [2] we present the characterizations and properties of these spaces as follows:-

Proposition 5.1. *For any fuzzy topological space (X,T) the following are equivalent.*

(a) (X,T) is extremally fuzzy rw-disconnected.

(b) For each fuzzy rw -closed set λ, λ_0 is fuzzy rw -closed.

(c) For each fuzzy rw-open set λ, we have $\underline{\lambda} + \underline{(1-\underline{\lambda})} = 1$.

(d) For every pair of fuzzy rw- open sets λ, δ in (X,T) with $\underline{\lambda} + \delta = 1$, we have $\underline{\lambda} + \underline{\delta} = 1$.

Proof. (a) $\Rightarrow$ (b). Let λ be any fuzzy rw-closed set. We claim λ_0 is fuzzy rw-closed. Now $1-\lambda_0 = \underline{1-\lambda}$. Since λ is fuzzy rw-closed ,$1-\lambda$ is fuzzy rw-open and $1-\lambda_0 = \underline{(1-\lambda)}$ and by (a) we get $\underline{(1-\lambda)}$ is fuzzy rw- open. That is λ_0 is fuzzy rw-closed.
(b) $\Rightarrow$ (c). Suppose that λ is any fuzzy rw-open set. Now $1-\underline{\lambda} = (1-\lambda)_0$. Therefore

$$\begin{aligned}\underline{\lambda} + \underline{(1-\underline{\lambda})} &= \underline{\lambda} + \underline{(1-\lambda)}_{0} \\ &= \underline{\lambda} + (1-\lambda)_0 \qquad [by(b)] \\ &= \underline{\lambda} + (1-\underline{\lambda}) = 1.\end{aligned}$$

(c) $\Rightarrow$ (d). Suppose λ and δ be any two fuzzy rw-open sets in (X,T) such that

$$\underline{\lambda} + \delta = 1. \quad (1)$$

Then by (c),

$$\underline{\lambda} + \overline{\underline{(1-\lambda)}} = 1 = \underline{\lambda} + \delta \Rightarrow \delta = \overline{\underline{(1-\lambda)}}.$$

But from (1) $\delta = 1 - \underline{\lambda}$ and $1 - \underline{\lambda} = \overline{\underline{(1-\lambda)}}$. That is $1 - \underline{\lambda}$ is fuzzy rw-closed and so $\underline{\delta} = 1 - \underline{\lambda}$. That is $\underline{\delta} + \underline{\lambda} = 1$.

(d) $\Rightarrow$ (a). Let λ be any fuzzy rw-open set in (X,T) and put $\delta = 1 - \underline{\lambda}$. From the construction of δ it follows that $\underline{\lambda} + \delta = 1$. Therefore by (d) we have $\underline{\lambda} + \underline{\delta} = 1$ and hence $\underline{\lambda}$ is fuzzy rw-open in (X,T). That is (X,T) is extremally fuzzy rw-disconnected.

6 Totally Fuzzy rw-Disconnectedness

Definition 6.1. *A fuzzy topological space (X,T) is said to be totally fuzzy rw-disconnected $\Leftrightarrow$ for every pair of fuzzy points p, q in X, there exist non-zero fuzzy rw-open sets λ, δ such that $\lambda + \delta = 1$, λ contains p and δ contains q. Suppose $A \subset X$. A is said to be totally fuzzy rw-disconnected subset of X if $(A, T/A)$ as a fuzzy subspace of (X,T) is totally fuzzy rw-disconnected.*

Proposition 6.1. *The maximal fuzzy rw-connected subsets of a totally fuzzy rw-disconnected space (X,T) are singleton sets.*

Proof. Let $(Y, T/Y)$ be a subspace of (X,T). It suffices to show that $(Y, T/Y)$ is totally fuzzy rw-disconnected whenever it contains more than one point. Let x_1 and x_2 be any two distinct points in Y. Define

$$p : Y \to I \text{ as } p(x) = \begin{cases} \frac{1}{3} & x = x_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q : Y \to I \text{ as } q(x) = \begin{cases} \frac{2}{3} & x = x_2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly p and q are distinct fuzzy points in Y.

Also define

$$p^{\#} : X \to I \text{ as } p^{\#}(x) = \begin{cases} \frac{1}{3} & x = x_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q^{\#} : X \to I \text{ as } q^{\#}(x) = \begin{cases} \frac{2}{3} & x = x_2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $p^{\#}$ and $q^{\#}$ are two distinct fuzzy points in X such that $p^{\#}/Y = p$ and $q^{\#}/Y = q$. Since (X, T) is totally fuzzy rw-disconnected space there exists non-zero fuzzy rw -open sets λ, δ in (X, T) such that λ contains $p^{\#}, \delta$ contains $q^{\#}$ and $\lambda + \delta = 1$. Then $\lambda_1 = \lambda/Y, \delta_1 = \delta/Y$ are non-zero fuzzy rw-open sets in $(Y, T/Y)$ such that $\lambda_1 + \delta_1 = 1$, $\lambda_1$contains p and δ_1 contains q. This shows that $(Y, T/Y)$ is totally fuzzy rw-disconnected.

Proposition 6.2. *Every fuzzy subspace of a totally fuzzy rw-disconnected space is totally fuzzy rw-disconnected.*

Proposition 6.3. *Let (X, T) be a fuzzy rw-open $\mathbf{T}_1$- space. If (X, T) has a base whose members are fuzzy rw-clopen, then (X, T) is totally fuzzy rw-disconnected.*

Proposition 6.4. *Let (X, T) be fuzzy rw-open compact and fuzzy rw-open $\mathbf{T}_1$-space. Then (X, T) is totally fuzzy rw-disconnected $\Leftrightarrow$ (X, T) has a base whose members are fuzzy rw-clopen.*

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References

[1] K.K. Azad, On fuzzy semi continuity, fuzzy almost continuiy and fuzzy weekly continuity, *J. Math. Anal. Appl.*, 82(1981), 14.

[2] G. Balasubramanian, Fuzzy disconnectedness and its stronger forms, *Indian J. Pure Appl. Math.*, 24(1) (1993), 27-30.

[3] A.S.B. Shahna, On fuzzy strong semi-continuity and fuzzy pre-continuity, *Fuzzy Sets and Systems*, 44(1991), 303-308.

[4] C.L. Chang, Fuzzy topological spaces, *J. Math Anal. Appl.*, 24(1968), 182-190.

[5] P.D. Filev and R.R. Yager, Operations on fuzzy numbers via fuzzy reasoning, *Fuzzy Sets and Systems*, 91(1997), 137-142.

[6] U.V. Fatteh and D.S. Bassan, Fuzzy connectedness and its stronger forms, *J. Math. Anal. Appl.*, 111(1985), 449-464.

[7] D.N. Foster, Fuzzy topological groups, *J. Math. Anal. Appl.*, 67(1979), 549-569.

[8] M.H. Ghanim, E.E. Kerre and A.S. Mashour, Separation axioms, subspaces and sums in fuzzy topology, *J. Math. Anal.Appl.*, 102(1984), 189-202.

[9] J.A. Goguen, *L*-fuzzy sets, *J. Math. Anal. Appl.*, 18(1) (1967), 145-174.

[10] W.X. Gu and T. Lu, Fuzzy sets, fuzzy linear spaces, *Fuzzy Sets and Systems*, 49(3) (1992), 377-380.

[11] B. Hutton, Normality in fuzzy topological spaces, *J. Math. Anal. Appl.*, 50(1975), 74-79.

[12] A.K. Kandel and W.J. Byatt, Fuzzy sets, fuzzy algebra and fuzzy statistics, *Proc. IEEE*, 66(12) (1978), 1619-1639.

[13] A. Kandel, *A Fuzzy Techniques in Pattern Recognition*, John Wiley, New York, (1982).

[14] A.K. Katsaras and D.B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, *J. Math. Anal. Appl.*, 58(1977), 135-146.

[15] A.K. Katsars, Fuzzy proximity spaces, *J. Math. Anal. Appl.*, 68(1979), 100-110.

[16] H.K. Kwan and Y. Cai, A fuzzy neural network and its applications to pattern recognition, *IEEE Trans. on Fuzzy Systems*, 2(3) (1994), 185-193.

[17] P. Smets, The degree of belief in fuzzy event, *Inform. Sci.*, 25(1981), 1-19.

[18] M. Sugeno, An introductory survey of fuzzy control, *Inform. Sci.*, 36(1985), 59-83.

[19] R.R. Yager and P.D. Filev, Generalizing the modeling of fuzzy logic controllers by parametrized aggregation operators, *Fuzzy Sets and Systems*, 70(1995), 303-313.

[20] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, 35(3) (1971), 512-517.

[21] R.S. Wali and S.S. Benchalli, Some topics in general and fuzzy topological spaces, *Ph. D Thesis*, Karnatak University Dharwd, (2006).

[22] L.A. Zadeh, Fuzzy sets, *Inform and Control*, 8(1965), 338-353.

[23] A.N. Zahren, Fuzzy reguar semiopen sets ans S-closed spaces, *J. Fuzzy Math.*, 2(1994), 579-586.