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Algebras Generated by Invertible Elements

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Abstract

It is well known that the linear span of invertible elements of a unital Banach algebra A equals to A. In this note we characterize this fact and prove the same result for some classes of topological algebras.

Keywords: *Fundamental algebra, FLM algebra, Strongly bounded algebra, invertible element, spectrum.*

1 Introduction

The fact that a Banach algebra with unit element can be generated by its invertible elements is proved in [5]. It is also easy to see that if A has not unit, then it can be generated by its quasi invertible elements. In this note we study and characterize this facts and recall the definitions of some complete metrizable non- normable topological algebras and prove the same results for these algebras. We also give an example of a non-complete normed algebra for which some elements have non-empty compact spectrum.

As usual by $InvA$ and $q-InvA$ we mean the invertible and quasi invertible elements of algebra A, respectively by $A =< S >$, we mean that A equals to the linear span of elements of S.

2 Definitions and Related Theorems

Definition 2.1 *A topological algebra A is said to be fundamental one if there exists $b > 1$ such that for every sequence (x_n) of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that (x_n) is cauchy.*

Definition 2.2 *A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. We call such an algebra, an FLM algebra.*

Definition 2.3 *Let A be a topological algebra. An element a of A is said to be bounded if for some non-zero complex number λ, the sequence $(\lambda^{-n}a^n)$ converges to zero. A topological algebra A is called strongly bounded if all elements of A are bounded.*

Theorem 2.4 *Let A be a complete metrizable fundamental topological algebra, and $x \in A$. If for some $b > 1, b^n\ x^n \to 0$ in A, then:*
$i)$ x is quasi-invertible and $x^0 = -\sum_{n=1}^{\infty} x^n$,
$ii)$ If A possesses a unit element, then $1 - x$ is invertible and

$$(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n.$$

Proof. See [3].

Theorem 2.5 *$i)$ The element a of algebra A has q-inverse b, if and only if $(0,1) - (a,0)$ has inverse $(0,1) - (b,0)$ in $A + F$.*
$ii)$ If A has a unit, the element $a \in A$, has q-inverse b, if and only if $1 - a$ has the inverse $1 - b$

Proof. See [5].

3 New Results

Proposition 3.1 *Let A be an algebra. If for all $a \in A$, $Sp(a) \neq C$, then:*
$i)$ If A has a unit element, then $A =< InvA >$ also $A =< \{1\} \cup q - InvA >$.
$ii)$ If A doesn't have unit element, then $A =< q - InvA >$.

Proof. *i*. Suppose $\lambda \notin Sp(a)$. If $\lambda \neq 0$, then $\lambda 1 - a \in InvA$. So $1 - \lambda^{-1}a \in InvA$. By Theorem 2.5, $\lambda^{-1}a \in q - InvA$. So $a \in < q - InvA >$.
If $\lambda = 0$, $a \in InvA$. By Theorem 2.5, $1 - a \in q - InvA$, and then $a \in < \{1\} \cup q - InvA >$. Similarly we can see $A =< InvA >$.

ii. In this case, $Sp(a)$ has a non-zero element λ and so $\lambda^{-1}a \in q - InvA$. Therefor, $A =< q - InvA >$.

Theorem 3.2 *Let A be a topological algebra such that for all $a \in A$, $Sp(a)$ is bounded or compact, then*
i. If A has a unit element, then $A =< InvA >$ and $A =< \{1\} \cup q - InvA >$.
ii. If A doesn't have unit element, then $A =< q - InvA >$.

The well known fact that Banach algebras as well as locally bounded topological algebras can be generated by the invertible or q-invertible elements follows from theorem 3.2.

For FLM algebras which are introduced in [3], it is proved that every complete metrizable FLM algebra with unit element is locally bounded algebra and there are also FLM algebras without unit. Since every element of a complete locally bounded topological algebra has a compact spectrum so by theorem 3.2, every complete metrizable FLM algebra with unit element can be generated by its invertible elements.
For FLM algebras without unit we have the following theorem.

Theorem 3.3 *Every complete metrizable FLM algebra without unit can be generated by its q-invertible elements.*

Proof. For $a \in A$ with $b^n a^n \to 0$, where b is determined in definition of fundamental algebra, by theorem 2.4, $a \in< q - InvA >$. If $b^n a^n \not\to 0$, by the definition of FLM algebra, there is a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. Since U_0 is absorbing there is a non-zero λ such that $ba \in \lambda U_0$, or $b\lambda^{-1}a \in U_0$. So for sufficiently large powers of $b\lambda^{-1}a$ lies in V. Then $b^n\lambda^{-n}a^n \to 0$. So by theorem 2.4, $\lambda^{-1}a \in q - InvA$ and $a \in< q - InvA >$.

Allan proved in [1] that every element of a strongly bounded complete metrizable locally convex topological algebra has a non-empty and compact spectrum. So by theorem 3.2, every such algebra can be generated by its invertible or q-invertible elements.
In [4] the strongly bounded complete metrizable fundamental topological algebras are introduced and an example of such algebras is given which is not locally convex and not locally bounded. It is proved in [4] that every element of such algebra is non-empty and compact and so by theorem 3.2, it can be generated by its invertible or q-invertible elements.

Here we give an independent proof for complete metrizable strongly bounded fundamental topological algebras which can be generated by their invertible or q-invertible elements.

Theorem 3.4 *Let A be a complete metrizable fundamental strongly bounded algebra, then $A =< q - InvA >$.*

Proof. Let A doesn't have a unit element. For $a \in A$, by definition of strongly bounded algebra there is a non-zero $\lambda \in C$ which $\lambda^{-n}a^n \to 0$. Suppose $\gamma = b\lambda$, where b is determined in definition of fundamental algebra. So $b^n\gamma^{-n}a^n \to 0$. By theorem 2.4, $\gamma^{-1}a \in q - InvA$ and $a \in < q - InvA >$.
Now let A has a unit element. When $b^n\gamma^{-n}a^n \to 0$, we have $1 - \gamma^{-1}a$ is invertible. By 2.5, $\gamma^{-1}a$ is q-invertible and $a \in < q - InvA >$.

In the well known result about this matter that a topological algebra can be generated by its invertible elements or quasi invertible elements, we use the completeness as a sufficient condition. Here we give an example to show that this condition is not necessary. This example also shows that the more famous result about non-empty and compactness of the spectrum of elements of a Banach algebra [5; Theorem 8.5.1], may hold in the absence of completeness.

Example 3.5 *Let A be an infinite dimensional normed space which is not complete and φ be a continuous linear functional on A such that $\|\varphi\| = 1$. Then A can be made to a normed algebra by the multiplication $a.b = \varphi(a)b$. This algebra doesn't have any unit element e.(otherwise, the algebra can be generated by the single element e and it is a contradiction.) Now we show that this algebra can be generated by its quasi-invertible elements. Let $a \in A$ such that $\varphi(a) \neq 1$, one may verify that the quasi-product of a and $(1/(\varphi(a) - 1))a$ is equal to zero i.e. a is quasi-invertible. If $\varphi(a) = 1$, then there is a multiple of a such as λa such that $\varphi(\lambda a) \neq 1$ and so λa is quasi-invertible, hence a belongs to $< q - InvA >$. To see the other fact, let $a \in A$ and $\lambda \notin \{0, \varphi(a)\}$. Then $\varphi(\lambda^{-1}a) \neq 1$ and $\lambda^{-1}a \in q - InvA$, and so $Sp(a) \subseteq \{0, \varphi(a)\}$.*

Remark 3.6 *There are topological algebras which can not be generated by its invertible elements, for example the algebra of all complex polynomials on the closed interval $[a, b]$, with the supremum norm and pointwise multiplication as its product can not be generated by its invertible elements, because In this algebra just the constant polynomials are invertible. But this topological algebra is not complete. So immediately there is a question. Is there any complete topological algebra for which its invertible (or quasi invertible)elements do not generate the algebra? It is clear that we must investigate the topological algebras for which there exists at least one element with non-compact spectrum. Here We give an example of a Frechet algebra with some non-compact spectrum element. But we don't know whether this algebra can be generated by its invertible elements till now.*

Example 3.7 *Let H be the set of all analytic functions on the open unit disc S_1 in the complex plane C. With respect to the pointwise operations and the topology generated by the separating family of multiplicative semi-norms $p_n(x) = \sup |x(C_n)|$, where $C_n = \{\mu \in C : |\mu| \leq 1 - \frac{1}{n}\}$ becomes a Frechet* *algebra* [6]. *In* [6] *it is also proved that for* $x \in H$, $Sp(x) = x(S_1)$. *So it is clear that the spectrum of some elements in this complete* LMC *algebra need not to be compact.*

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