Gen. Math. Notes, Vol. 16, No. 2, June, 2013, pp.32-47
ISSN 2219-7184; Copyright ©ICSRS Publication, 2013
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On the Invariant Regions and Global Existence of Solutions for Four Components Reaction-Diffusion Systems With a Full Matrix of Diffusion Cofficients and Nonhomogeneous Boundary Conditions

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(Received: 24-2-13 / Accepted: 31-3-13)

Abstract

In this work we consider a reaction-diffusion systems (four equations) with a full matrix of diffusion and with nonhomogeneous boundary conditions, by using the Invariant regions and Lyapunov functional method we established the global existence of solution.

Keywords: *Reaction-Diffusion systems, Invariant regions, Matrice of diffusion, Global Existence.*

1 Introduction

We consider the folowing reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v - c\Delta w - d\Delta z = f_1(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{1.1}$$
$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v - b\Delta z = f_2(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{1.2}$$
$$\frac{\partial w}{\partial t} - b\Delta u - a\Delta w - c\Delta z = f_3(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{1.3}$$
$$\frac{\partial z}{\partial t} - d\Delta u - c\Delta v - b\Delta w - a\Delta z = f_4(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{1.4}$$

with the bondary conditions

$$\begin{cases} \lambda u + (1-\lambda)\dfrac{\partial u}{\partial \eta} = \beta_1 & \text{in } R^+ \times \partial\Omega \\ \lambda v + (1-\lambda)\dfrac{\partial v}{\partial \eta} = \beta_2 & \text{in } R^+ \times \partial\Omega \\ \lambda w + (1-\lambda)\dfrac{\partial w}{\partial \eta} = \beta_3 & \text{in } R^+ \times \partial\Omega \\ \lambda z + (1-\lambda)\dfrac{\partial z}{\partial \eta} = \beta_4 & \text{in } R^+ \times \partial\Omega \end{cases} \tag{1.5}$$

and the intial data

$$u(0,x) = u_0(x); v(0,x) = v_0(x); w(0,x) = w_0(x) \; ; z(0,x) = z_0(x), \quad \text{in } \Omega \tag{1.6}$$

i. For nonhomogeneous Robin bondary conditions, we use

$$0 < \lambda < 1, \text{and } \beta_i \in R.i = 1,2,3,4.$$

ii. For nonhomogeneous Neumann bondary conditions, we use

$$\lambda = \beta_i = 0, \quad i = 1,2,3,4.$$

iii. For nonhomogeneous Dirichlet bondary conditions, we use

$$1 - \lambda = \beta_i = 0, \quad i = 1,2,3,4.$$

Where Ω is an open bounded domain of class C^1 in R^n , with bondary $\partial\Omega$ and $\dfrac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. a, b , c and d are positive constants satisfying the condition $\left(a + \frac{1}{2}(d-\mu)\right)(a-d)$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a & b & c & d \\ c & a & 0 & b \\ b & 0 & a & c \\ d & c & b & a \end{pmatrix} \tag{1.7}$$

is positive definite; that is the eingenvalues λ_1 , λ_2, $\lambda_3$and λ_4 ($\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$) of its transposed are positive.

The initial data $(u_0, v_0, w_0, z_0) \in R^4$ are assumed to be in the folowing region :

$$\sum = \left\{ \left\{ \begin{array}{l} u_0 \leq -\frac{2(b-c)}{d+\mu}(v_0 - w_0) + z_0 \\ u_0 \leq -\frac{2(b-c)}{d-\mu}(v_0 - w_0) + z_0 \\ u_0 + \frac{2(b+c)}{d-\mu}(v_0 + w_0) + z_0 \geq 0 \\ \text{and} \\ u_0 + \frac{2(b+c)}{d+\mu}(v_0 + w_0) + z_0 \geq 0 \end{array} \right. \text{ if } \left\{ \begin{array}{l} \beta_1 \leq -\frac{2(b-c)}{d+\mu}(\beta_2 - \beta_3) + \beta_4 \\ \beta_1 \leq -\frac{2(b-c)}{d-\mu}(\beta_2 - \beta_3) + \beta_4 \\ \beta_1 + \frac{2(b+c)}{d-\mu}(\beta_2 + \beta_3) + \beta_4 \geq 0 \\ \text{and} \\ \beta_1 + \frac{2(b+c)}{d+\mu}(\beta_2 + \beta_3) + \beta_4 \geq 0 \end{array} \right. \right\}$$

where

$$\mu = \sqrt{4b^2 + 8bc + 4c^2 + d^2}$$

We suppose that the reaction termes f_1, f_2 , f_3 and f_4 are continuously differentiable, polynomially bounded on Σ satisfying:

$$\left(-f_1 - \tfrac{2(b-c)}{d+\mu}(f_2 - f_3) + f_4\right)\left(-\tfrac{2(b-c)}{d+\mu}(v-w) + z, v, w, z\right) \geq 0 \quad (1.8)$$

$$\left(-f_1 - \tfrac{2(b-c)}{d-\mu}(f_2 - f_3) + f_4\right)\left(-\tfrac{2(b-c)}{d-\mu}(v-w) + z, v, w, z\right) \geq 0 \quad (1.9)$$

$$\left(f_1 + \tfrac{2(b+c)}{d-\mu}(f_2 + f_3) + f_4\right)\left(-\tfrac{2(b+c)}{d-\mu}(v+w) - z, v, w, z\right) \geq 0 \quad (1.10)$$

$$\left(f_1 + \tfrac{2(b+c)}{d+\mu}(f_2 + f_3) + f_4\right)\left(-\tfrac{2(b+c)}{d+\mu}(v+w) - z, v, w, z\right) \geq 0 \quad (1.11)$$

for all $v, w, z \geq 0$. And for positive constants $C_{11}, C_{12}, C_{13} \leq 1$.

$$(C_{11}f_1 + C_{12}f_2 + C_{13}f_3 + f_4)(u, v, w, z) \leq C_1(u + v + w + z + 1); \quad (1.12)$$

for all u, v, w, z in Σ where C_1 is a positive constant.

2 Existence

2.1 Local Existence

The usual norms in spaces $\mathbb{L}^p(\Omega)$, $\mathbb{L}^\infty(\Omega)$ and $C(\Omega)$ are respectively denote by:

$$\|u\|_p^p = \frac{1}{\Omega}\int_\Omega |u(x)|^p \, dx$$

$$\|u\|_\infty = \max_{x\in\Omega} |u(x)| .$$

For any initial data in $C(\overline{\Omega})$ or $\mathbb{L}^p(\Omega)$, $p \in (1, +\infty)$; local existence and uniqueness of solutions to the initial value problem (1.1)-(1.5) folow from the basic existence theory for abstract semilinear differential equations (see A. Friedman[3], D.Henry[4] and A Pazzy[12]). The solutions are classical on $[0, T_{\max}[$; where $T_{\max}$ denotes the eventual blowing-up time in $\mathbb{L}^\infty(\Omega)$.

2.2 Invariant Regions

Proposition 1 *Soppose that the functions f_1, f_2, f_3 and f_4 points into the region $\sum$ on $\partial \sum$ then for any (u_0, v_0, w_0, z_0) in $\sum$, the solution $(u(t,.), v(t;.), w(t;.), z(t,.))$ of the problem (1.1)-(1.6) remains in $\sum$ for any time.*

Proof. The proof folows the same way to that in S.Kouachi[5]

• Multiplying successively (1.1) by -1, (1.2) by $-\frac{2(b-c)}{d+\mu}$, (1.3) by $\frac{2(b-c)}{d+\mu}$ and (1.4) by 1.

Ading the first result, the second, the third and the forth results we get (2.1).

• Multiplying successively (1.1) by -1, (1.2) by $-\frac{2(b-c)}{d-\mu}$, (1.3) by $\frac{2(b-c)}{d-\mu}$ and last one by 1.

Ading the first result, the second result, the third and the forth results we get (2.2).

• Multiplying successively (1.1) by 1, (1.2) and (1.3) by $\frac{2(b+c)}{d+\mu}$, and (1.4) by 1.

Ading the first result, the second result, the third and the forth results we get (2.3).

• Multiplying successively (1,1) by 1, the second and the third by $\frac{2(b+c)}{d-\mu}$ and last one by 1.

Ading the first result, the second result, the third and the forth results we get (2.4).

Finaly we obtain the system

$$\frac{\partial U}{\partial t} - \left(a - \frac{1}{2}(d+\mu)\right)\Delta U = F_1(U,V,W,Z) \tag{2.1}$$

$$\frac{\partial V}{\partial t} - \left(a - \frac{1}{2}(d-\mu)\right)\Delta V = F_2(U,V,W,Z) \tag{2.2}$$

$$\frac{\partial W}{\partial t} - \left(a + \frac{1}{2}(d-\mu)\right)\Delta W = F_3(U,V,W,Z) \tag{2.3}$$

$$\frac{\partial Z}{\partial t} - \left(a + \frac{1}{2}(d+\mu)\right)\Delta Z = F_4(U,V,W,Z) \tag{2.4}$$

with the boundary conditions

$$\begin{cases} \lambda U + (1-\lambda)\frac{\partial U}{\partial \eta} = \rho_1 & \text{in }]0,T^*[\times \partial\Omega \\ \lambda V + (1-\lambda)\frac{\partial V}{\partial \eta} = \rho_2 & \text{in }]0,T^*[\times \partial\Omega \\ \lambda W + (1-\lambda)\frac{\partial W}{\partial \eta} = \rho_3 & \text{in }]0,T^*[\times \partial\Omega \\ \lambda Z + (1-\lambda)\frac{\partial Z}{\partial \eta} = \rho_4 & \text{in }]0,T^*[\times \partial\Omega \end{cases} \tag{2.5}$$

and the initial data

$$U(0,x)=U_0(x); V(0,x)=V_0(x); W(0,x)=W_0(x); Z(0,x)=Z_0(x) \tag{2.6}$$

where

$$\begin{cases} U(t,x) = -u(t,x) + \dfrac{2(b-c)}{d+\mu}\left(w(t,x) - v(t,x)\right) + z(t,x) \\ V(t,x) = -u(t,x) + \dfrac{2(b-c)}{d-\mu}\left(w(t,x) - v(t,x)\right) + z(t,x) \\ W(t,x) = u(t,x) + \dfrac{2(b+c)}{d-\mu}\left(v(t,x) + w(t,x)\right) + z(t,x) \\ Z(t,x) = u(t,x) + \dfrac{2(b+c)}{d+\mu}\left(v(t,x) + w(t,x)\right) + z(t,x) \end{cases} \tag{2.7}$$

for any (t,x) in $]0,T^*[\times\Omega$,and

$$\begin{cases} F_1(U,V,W,Z) = \left(-f_1 + \dfrac{2(b-c)}{d+\mu}\left(f_3 - f_2\right) + f_4\right)(u,v,w,z) \\ F_2(U,V,W,Z) = \left(-f_1 + \dfrac{2(b-c)}{d-\mu}\left(f_3 - f_2\right) + f_4\right)(u,v,w,z) \\ F_3(U,V,W,Z) = \left(f_1 + \dfrac{2(b+c)}{d-\mu}\left(f_2 + f_3\right) + f_4\right)(u,v,w,z) \\ F_4(U,V,W,Z) = \left(f_1 + \dfrac{2(b+c)}{d+\mu}\left(f_2 + f_3\right) + f_4\right)(u,v,w,z) \end{cases} \tag{2.8}$$

for all (u,v,w,z) in $\sum$,

$$\begin{cases} \lambda_1 = \left(a - \frac{1}{2}(d+\mu)\right) \\ \lambda_2 = \left(a - \frac{1}{2}(d-\mu)\right) \\ \lambda_3 = \left(a + \frac{1}{2}(d-\mu)\right) \\ \lambda_4 = \left(a + \frac{1}{2}(d+\mu)\right) \end{cases}$$

and

$$\begin{cases} \rho_1 = -\beta_1 + \dfrac{2(b-c)}{d+\mu}\left(\beta_3 - \beta_2\right) + \beta_4 \\ \rho_2 = -\beta_1 + \dfrac{2(b-c)}{d-\mu}\left(\beta_3 - \beta_2\right) + \beta_4 \\ \rho_3 = \beta_1 + \dfrac{2(b+c)}{d-\mu}\left(\beta_2 + \beta_3\right) + \beta_4 \\ \rho_4 = \beta_1 + \dfrac{2(b+c)}{d+\mu}\left(\beta_2 + \beta_3\right) + \beta_4 \end{cases}$$

First, let's notice that the condition of parabolicity of the system (1.1)-(1.4) implies the one of the (2.1)-(2.4) system; since $\left(a + \frac{1}{2}(d-\mu)\right)(a-d) \geqslant 0 \Longrightarrow \left(a + \frac{1}{2}(d-\mu)\right) > 0$; $\left(a - \frac{1}{2}(d+\mu)\right) > 0$ and $\left(a - \frac{1}{2}(d-\mu)\right) > 0$.

Now, it suffices to prove that the region

$$\Sigma = \{U \geq 0, V \geq 0, W \geq 0, Z \geq 0\} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,$$

is invariant region for the system (2.1)-(2.4).

Since , from $(1.8)-(1.9)-(1.10)-(1.11)$ we have $F_1(0,V,W,Z)\geq 0$ it suffices to be

$$-f_1(-\tfrac{2(b-c)}{d+\mu}(v-w)+z,v,w,z)-\tfrac{2(b-c)}{d+\mu}\Big((f_2-f_3)\Big((-\tfrac{2(b-c)}{d+\mu}(v-w)+z,v,w,z\Big)\Big)$$
$$+f_4(-\tfrac{2(b-c)}{d+\mu}(v-w)+z,v,w,z)\geq 0, \text{for all } V,W,Z\geq 0 \text{ and all } v,w,z\geq 0,$$

then

$$U(t,x)\geq 0 \text{ for all } (t,x) \text{ in }]0,T^*[\times\Omega,$$

thanks to the invariant region method (see Smoller[13]), and because $F_2(U,0,W,Z)\geq 0$ it suffices to be

$$-f_1(-\tfrac{2(b-c)}{d-\mu}(v-w)+z,v,w,z)-\tfrac{2(b-c)}{d-\mu}\Big((f_2-f_3)\Big((-\tfrac{2(b-c)}{d-\mu}(v-w)+z,v,w,z\Big)\Big)$$
$$+f_4(-\tfrac{2(b-c)}{d-\mu}(v-w)+z,v,w,z)\geq 0, \text{for all } U,W,Z\geq 0 \text{ and all } v,w,z\geq 0,$$

then

$$V(t,x)\geq 0 \text{ for all } (t,x)in]0,T^*[\times\Omega, \text{and } F_3(U,V,0,Z)\geq 0$$

it suffices to be

$$f_1(-\tfrac{2(b+c)}{d-\mu}(v+w)-z,v,w,z)+\tfrac{2(b+c)}{d-\mu}\Big((f_2+f_3)\Big((-\tfrac{2(b+c)}{d-\mu}(v+w)-z,v,w,z\Big)\Big)$$
$$+f_4(-\frac{2(b+c)}{d-\mu}(v+w)-z,v,w,z)\geq 0, \text{for all } U,V,Z\geq 0 \text{ and all } v,w,z\geq 0,$$

then

$$W(t,x)\geq 0 \text{ for all } (t,x)in]0,T^*[\times\Omega, \text{and } F_4(U,V,W,0)\geq 0$$

it suffices to be

$$f_1(-\tfrac{2(b+c)}{d+\mu}(v+w)-z,v,w,z)+\tfrac{2(b+c)}{d+\mu}\Big((f_2+f_3)\Big((-\tfrac{2(b+c)}{d+\mu}(v+w)-z,v,w,z\Big)\Big)$$
$$+f_4(-\tfrac{2(b+c)}{d+\mu}(v+w)-z,v,w,z)\geq 0, \text{for all } U,V,W\geq 0 \text{ and all } v,w,z\geq 0,$$

then

$$Z(t,x)\geq 0 \text{ for all } (t,x) \text{ in }]0,T^*[\times\Omega,$$

then $\sum$ is an invariant region for the system $(1.1)-(1.4)$.

Then the system $(1.1)-(1.4)$with boundary condition (1.5) and initial data in $\sum$ is equivalent to system $(2.1)-(2.4)$ with bondary conditions (2.5) and positive initiale data (2.6).

Once invariant regions are constructed, one can apply Lyapunov technique and estabish global existence of unique solutions for (1.1)-(1.6) ■

2.3 Global Existence

As the determinant of the linear algebraic system (2.7) with regard to variables u,v,w and z, is different from zero, then to prove global existence of solutions of problem $(1.1)-(1.6)$, one needs to prove it for problem $(2.1)-(2.6)$. To this subject ; it is well-known that (see Henry[4]) it sufficies to derive an uniform estimate of

$\|F_1(U,V,W,Z)\|_p$, $\|F_2(U,V,W,Z)\|_p$,$\|F_3(U,V,W,Z)\|_p$ and $\|F_4(U,V,W,Z)\|_p$ on $[0,T^*[$ for some $p>\dfrac{N}{2}$ $(N=\dim\Omega)$.

Let's put $A_{12}=\dfrac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}$, $A_{13}=\dfrac{\lambda_1+\lambda_3}{2\sqrt{\lambda_1\lambda_3}}$, $A_{14}=\dfrac{\lambda_1+\lambda_4}{2\sqrt{\lambda_1\lambda_4}}$, $A_{23}=\dfrac{\lambda_2+\lambda_3}{2\sqrt{\lambda_2\lambda_3}}$, $A_{24}=\dfrac{\lambda_2+\lambda_4}{2\sqrt{\lambda_2\lambda_4}}$; θ_1,θ_2 and θ_3 are three positive constants sufficiently large such that

$$\theta_1>A_{12}, \tag{2.9}$$

$$\left(\theta_1^2-A_{12}^2\right)\left(\theta_2^2-A_{23}^2\right)>\left(A_{13}-A_{12}A_{23}\right)^2 \tag{2.10}$$

$$\left(\theta_1^2-A_{12}^2\right)\left(\theta_2^2-A_{23}^2\right)\left(\theta_3^2-A_{34}^2\right)>\left(A_{13}-A_{12}A_{23}\right)^2+\left(A_{14}-A_{13}A_{24}\right)^2 \tag{2.11}$$

The main result of the paper is as follows:

Theorem 1. Let $(U(t,.),\ V(t,.),\ W(t,.),\ Z(t,.))$ be any positive solution of $(2.1)-(2.6)$ and let the functional

$$L(t)=\int_\Omega H_n(U(t,x),V(t,x),W(t,x),Z(t,.))dx, \tag{2.12}$$

where

$$H_n(U,V,W,Z)=\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j}C_n^k\ C_k^j\ C_j^i\ \theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{k^2}\ U^i\ V^{j-i}\ W^{k-j}\ Z^{n-k}.$$

Thene, the functional L is uniformly bounded on the interval $[0,T^*]$, $T^*<T_{\max}$.

Corollary 1. Suppose that the functions f_1, f_2, f_3 and f_4 are countinuously differentiable on $\sum$, point into $\partial\sum$ and satisfy condition (1.12). Then all solutions of $(1.1)-(1.6)$ with the initial data in $\sum$ and uniformly bounded on Ω are in $L^\infty(0,T^*;L^p(\Omega))$ for all $p\geq 1$

Proposition 2. Under the hypothesis of corolly 1, if the reactions f_1, f_2, f_3 and f_4 are polynomially bounded, then all solutions of $(1.1)-(1.6)$ with the initial data in $\sum$ and uniformly bounded on Ω are global time.

2.4 Proofs

For the proof of theorem 1, we need some preparatory lemmas.

Lemma 1. Let H_n be the homogeneous polynomial defined by (2.11). Then

$$\frac{\partial H_n}{\partial U}=n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^kC_k^j\ C_j^i\ \theta_1^{(i+1)^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+1)^2}U^i\ V^{j-i}\ W^{k-j}\ Z^{(n-1)-k},$$

$$\frac{\partial H_n}{\partial V}=n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^kC_k^jC_j^i\theta_1^{i^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+1)^2}\ U^i\ V^{j-i}\ W^{k-j}\ Z^{(n-1)-k},$$

$$\frac{\partial H_n}{\partial W}=n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^kC_k^jC_j^i\theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{(k+1)^2}U^i\ V^{j-i}\ W^{k-j}\ Z^{(n-1)-k},$$

$$\frac{\partial H_n}{\partial Z}=n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^kC_k^jC_j^i\theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{k^2}\ U^i\ V^{j-i}\ W^{k-j}\ Z^{(n-1)-k},$$

Proof of lemma 1. see Said Kouachi[6].

Lemma 2. The second partial derivatives of H_n are given by

$$\begin{aligned}
\frac{\partial^2 H_n}{\partial U^2} &= n(n\text{-}1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}C_{j}^{i}\theta_1^{(i+2)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial UV} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}C_{j}^{i}\theta_1^{(i+1)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2}U^iV^{j\text{-}i}W^{k\text{-}j}Z^{(n\text{-}2)\text{-}k},\\
\frac{\partial^2 H_n}{\partial UW} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{(i+1)^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+2)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial UZ} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{(i+1)^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+1)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k}\\
\frac{\partial^2 H_n}{\partial V^2} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{(j+2)^2}\ \theta_3^{(k+2)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial VW} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+2)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial VZ} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{(j+1)^2}\ \theta_3^{(k+1)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial W^2} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{(k+2)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial WZ} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{(k+1)^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},\\
\frac{\partial^2 H_n}{\partial Z^2} &= n(n-1)\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ \theta_1^{i^2}\ \theta_2^{j^2}\ \theta_3^{k^2}U^iV^{j-i}W^{k-j}Z^{(n-2)-k},
\end{aligned}$$

Proof of lemma 2. see Said Kouachi[6].

Proof of Theorem 1. Differentiating L with respect to t yields

$$\begin{aligned}
L^{'}(t) &= \int_{\Omega}\left(\partial_U H_n\frac{\partial U}{\partial t}+\partial_V H_n\frac{\partial V}{\partial t}+\partial_W H_n\frac{\partial W}{\partial t}+\partial_Z H_n\frac{\partial Z}{\partial t}\right)dx\\
&= \int_{\Omega}\left(\lambda_1\partial_U H_n\Delta U+\lambda_2\partial_V H_n\Delta V+\lambda_3\partial_W H_n\Delta W+\lambda_4\partial_Z H_n\Delta Z\right)dx\\
&\quad +\int_{\Omega}\left(F_1\partial_U H_n+F_2\partial_V H_n+F_3\partial_W H_n+F_4\partial_Z H_n\right)dx\\
&= I+J.
\end{aligned}$$

Using Green's formula and applying lemma 1 we get $I = I_1 + I_2$, where

$$I_1=\int_{\partial\Omega}\left(\lambda_1\partial_U H_n\frac{\partial U}{\partial\eta}+\lambda_2\partial_V H_n\frac{\partial V}{\partial\eta}+\lambda_3\partial_W H_n\frac{\partial W}{\partial\eta}+\lambda_4\partial_Z H_n\frac{\partial Z}{\partial\eta}\right)dx \qquad (2.13)$$

$$I_2=-n(n-1)\int_{\Omega}\sum_{k=0}^{n-2}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{n-2}^{k}C_{k}^{j}\ C_{j}^{i}\ (TB_{ijk})T^t dx, \qquad ((2.14))$$

where

$$B_{ijk} = \begin{bmatrix} \lambda_1\theta_1^{(i+2)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \frac{\lambda_1+\lambda_2}{2}a_{12} & \frac{\lambda_1+\lambda_3}{2}a_{13} & \frac{\lambda_1+\lambda_4}{2}a_{14} \\ \frac{\lambda_1+\lambda_2}{2}a_{12} & \lambda_2\theta_1^{i^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \frac{\lambda_2+\lambda_3}{2}a_{23} & \frac{\lambda_2+\lambda_4}{2}a_{24} \\ \frac{\lambda_1+\lambda_3}{2}a_{13} & \frac{\lambda_2+\lambda_3}{2}a_{23} & \lambda_3\theta_1^{i^2}\theta_2^{j^2}\theta_3^{(k+2)^2} & \frac{\lambda_3+\lambda_4}{2}a_{34} \\ \frac{\lambda_1+\lambda_4}{2}a_{14} & \frac{\lambda_2+\lambda_4}{2}a_{24} & \frac{\lambda_3+\lambda_4}{2}a_{34} & \lambda_4\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2} \end{bmatrix}$$

where

$a_{12} = \theta_1^{(i+1)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2}; a_{13} = \theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+2)^2}; a_{14} = \theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2};$
$a_{23} = \theta_1^{i^2}\theta_2^{(j+1)^2}\theta_3^{(k+2)^2}; a_{24} = \theta_1^{i^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}; a_{34} = \theta_1^{i^2}\theta_2^{j^2}\theta_3^{(k+1)^2},$
for $i = 0, .., j; j = 0, .., k; k = 0, ..., n-2;$ and $T = (\nabla U, \nabla V, \nabla W, \nabla Z)$

We prove that there exists a positive constant C_2 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \leq C_2 \text{ for all } t \in [0, T_{\max}[\tag{2.15}$$

and that

$$I_2 \leq 0 \tag{2.16}$$

for several boundary conditions.

i) If $0 < \lambda < 1$, using the boundary conditions (2.5) we get

$$I_1 = \int_{\partial\Omega} (\lambda_1\partial_U H_n(\gamma_1 - \alpha U) + \lambda_2\partial_V H_n(\gamma_2 - \alpha V) + \lambda_3\partial_W H_n(\gamma_3 - \alpha W) + \lambda_4\partial_Z H_n(\gamma_4 - \alpha Z))\, dx,$$

where $\alpha = \dfrac{\lambda}{1-\lambda}$ and $\gamma_i = \dfrac{\rho_i}{1-\lambda}$, $i = 1, 2, 3, 4.$

Since

$$\begin{aligned} K(U, V, W, Z) &= \lambda_1\partial_U H_n(\gamma_1 - \alpha U) + \lambda_2\partial_V H_n(\gamma_2 - \alpha V) + \lambda_3\partial_W H_n(\gamma_3 - \alpha W) \\ &+\lambda_4\partial_Z H_n(\gamma_4 - \alpha Z) = P_{n-1}(U, V, W, Z) - Q_n(U, V, W, Z) \end{aligned}$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $(n-1)$ and n and since the solution is positive, then,

$$\limsup_{(|U|+|V|+|W|)\longrightarrow\infty} K(U, V, W, Z) = -\infty \tag{2.17}$$

which proves that K is uniformly bounded on $\mathbb{R}_+^4$ and consequently (2.15).

(ii) If $\lambda = 0$, then $I_1 = 0$ on $[0, T_{\max}[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{\max}[\times\Omega$ implies $\partial_\eta U \leq 0$, $\partial_\eta V \leq 0$, $\partial_\eta W \leq 0$,and $\partial_\eta Z \leq 0$ on $[0, T_{\max}[\times\partial\Omega$. Consequently one gets again (2.15) with $C_2 = 0$.

Now we prove (2.16). The quadratic forms (with respect to ∇U, ∇V, ∇W and ∇Z) associated with the matrixes B_{ijk} , $i=0,...,j$; $j=0,...,k$ and $k=0,...,n-2$ are positive since their main determinants Δ_1 , Δ_2 ,Δ_3 and Δ_4 are positive , too . To see this, we have :

1. $\Delta_1 = \lambda_1\theta_1^{(i+2)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} > 0$ for $i=0,...,j$; $j=0,...,k$; and $k=0,...,n-2$

2.

$$\Delta_2 = \begin{vmatrix} \lambda_1\theta_1^{(i+2)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \frac{\lambda_1+\lambda_2}{2}\theta_1^{(i+1)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} \\ \frac{\lambda_1+\lambda_2}{2}\theta_1^{(i+1)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \lambda_2\theta_1^{i^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} \end{vmatrix}$$
$$= \lambda_1\lambda_2\theta_1^{2(i+1)^2}\theta_2^{2(j+2)^2}\theta_3^{2(k+2)^2}(\theta_1^2 - A_{12}^2),$$

for $i=0,...,j$; $j=0,...,k$ and $k=0,...,n-2$. Using (2.9), we get $\Delta_2 > 0$.

3.

$$\Delta_3 = \begin{vmatrix} \lambda_1\theta_1^{(i+2)^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \frac{\lambda_1+\lambda_2}{2}a_{12} & \frac{\lambda_1+\lambda_3}{2}a_{13} \\ \frac{\lambda_1+\lambda_2}{2}a_{12} & \lambda_2\theta_1^{i^2}\theta_2^{(j+2)^2}\theta_3^{(k+2)^2} & \frac{\lambda_2+\lambda_3}{2}a_{23} \\ \frac{\lambda_1+\lambda_3}{2}a_{13} & \frac{\lambda_2+\lambda_3}{2}a_{23} & \lambda_3\theta_1^{i^2}\theta_2^{j^2}\theta_3^{(k+2)^2} \end{vmatrix}$$
$$= \lambda_1\lambda_2\lambda_3\theta_1^{2(i+2)^2}\theta_1^{2(i+1)^2}\theta_2^{2(j+2)^2}\theta_2^{2(j+1)^2}\theta_3^{2(k+1)^2}\theta_3^{k^2}$$
$$\times\left[\left(\theta_1^2 - A_{12}^2\right)\left(\theta_2^2 - A_{23}^2\right) - \left(A_{13} - A_{12}A_{23}\right)^2\right]$$

for $i=0,...,j$; $j=0,...,k$ and $k=0,...,n-2$.Using (2.10), we get $\Delta_3 > 0$.

4.

$$\Delta_4 = |B_{ijk}| = \left(\lambda_1\lambda_2\lambda_3\lambda_{43}\ \theta_1^{2(i+2)^2}\ \theta_1^{2(i+1)^2}\ \theta_2^{2(j+1)^2}\theta_2^{j^2}\ \theta_3^{2(k+1)^2}\ \theta_3^{k^2}\right)$$
$$\times\left[\left(\theta_1^2 - A_{12}^2\right)\left(\theta_2^2 - A_{23}^2\right)\left(\theta_3^2 - A_{34}^2\right) - \left(A_{13} - A_{12}A_{23}\right)^2 - \left(A_{14} - A_{13}A_{24}\right)^2\right]$$

for $i=0,...,j$; $j=0,...,k$ and $k=0,...,n-2$.Using (2.11) , we get $\Delta_4 > 0$.

Substituting the expressions of the partial derivatives given by lemma 1 in the second integral, yields

$$J=\int_{\Omega}n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^{k}C_{k}^{j}C_{j}^{i}\left(\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}F_1+\theta_1^{i^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}F_2+\theta_1^{i^2}\theta_2^{j^2}\theta_3^{(k+1)^2}F_3\right.$$
$$\left.+\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}F_4\right)U^iV^{j-i}W^{k-j}Z^{(n-1)-k}dx=\int\limits_{\Omega}\left(n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^{k}C_{k}^{j}C_{j}^{i}U^iV^{j-i}W^{k-j}Z^{(n-1)-k}\right)$$
$$\times\left(\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}F_1+\frac{\theta_1^{i^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}F_2+\frac{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}F_3+F_4\right)\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}dx.$$
$$=\int_{\Omega}\left(n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^{k}C_{k}^{j}C_{j}^{i}U^iV^{j-i}W^{k-j}Z^{(n-1)-k}\right)\left(\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}F_1+\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}F_2\right.$$
$$\left.+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}F_3+F_4\right)\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}dx.$$

Using the expressions (2.8), we obtain

$$\begin{aligned}J=\ &\int_{\Omega}\left(n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^{k}C_{k}^{j}C_{j}^{i}U^iV^{j-i}W^{k-j}Z^{(n-1)-k}\right)\\
&\times\left[\left(-\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}-\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1\right)f_1\right.\\
&+\left(-\frac{2(b-c)}{d+\mu}\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}-\frac{2(b-c)}{d-\mu}\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b+c)}{d-\mu}\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+\frac{2(b+c)}{d+\mu}\right)f_2\\
&+\left(\frac{2(b-c)}{d+\mu}\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b-c)}{d-\mu}\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b+c)}{d-\mu}\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+\frac{2(b+c)}{d+\mu}\right)f_3\\
&\left.+\left(\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1\right)f_4\right]\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}dx\end{aligned}$$

And so we get the following inequality:

$$\begin{aligned}J\le\ &\int_{\Omega}\left(n\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{j}C_{n-1}^{k}C_{k}^{j}C_{j}^{i}U^iV^{j-i}W^{k-j}Z^{(n-1)-k}\right)\\
&\times\left[\left(\frac{-\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}-\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1}{\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1}\right)f_1\right.\\
&+\left(\frac{-\frac{2(b-c)}{d+\mu}\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}-\frac{2(b-c)}{d-\mu}\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b+c)}{d-\mu}\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+\frac{2(b+c)}{d+\mu}}{\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1}\right)f_2\\
&\left.+\left(\frac{\frac{2(b-c)}{d+\mu}\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b-c)}{d-\mu}\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{2(b+c)}{d-\mu}\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+\frac{2(b+c)}{d+\mu}}{\frac{\theta_1^{(i+1)^2}\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_2^{(j+1)^2}\theta_3^{(k+1)^2}}{\theta_2^{j^2}\theta_3^{k^2}}+\frac{\theta_3^{(k+1)^2}}{\theta_3^{k^2}}+1}\right)f_3+f_4\right]\\
&\times\theta_1^{i^2}\theta_2^{j^2}\theta_3^{k^2}dx,\end{aligned}$$

using condition (1.10) , and relation (2.6) successively, we get

$$J \leq C_4 \int_\Omega \int_\Omega n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q U^q V^{p-q} W^{(n-1)-p} (U+V+W+1)\, dx.$$

Following the same reasoning as in S. Kouachi[6], a straight forward calculation shows that

$$L^{'}(t) \leq C_5 L(t) + C_6 L^{\frac{n-1}{n}}(t) \text{ on } [0,T^*],$$

which for $Z = L^{\frac{1}{n}}$ can be written as

$$nZ^{'} \leq C_5 Z + C_6.$$

A simple integration gives the uniform bound of the functional L on the interval $[0,T^*[$; this ends the proof of the theorem.

Proof of corollary 1. The proof of this corolarly is an immediate consequence of the theorem and the inequality

$$\int_\Omega (U+V+W+Z)^P dx \leq C_8 L(t) \quad \text{on } [0,T^*[,$$

for some $P \geq 1$, taking into consideration expressions (2.7).

Proof of proposition 2. As it has been mentioned above; it suffices to derive a uniforme estimate of $\|F_1(U,V,W,Z)\|_p$, $\|F_2(U,V,W,Z)\|_p$, $\|F_3(U,V,W,Z)\|_p$and $\|F_4(U,V,W,Z)\|_p$ on $[0,T^*[$ for some $p > \dfrac{N}{2}$. Since the functions f_1, f_2, f_3 and f_4 are polynomially bounded on $\sum$, then using relations (2.5), (2.7) and (2.8), we get that F_1, F_2, F_3 and F_4 are polynomially bounded, too and the proof becomes an immediate consequence of corollary 1.

3 Final Remarks

1. In the case when the system (1.1) – (1.4) rewriten as follows:

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v - c\Delta w = f_1(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{3.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v - d\Delta w - b\Delta z = f_2(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{3.2}$$

$$\frac{\partial w}{\partial t} - b\Delta u - d\Delta v - a\Delta w - c\Delta z = f_3(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{3.3}$$

$$\frac{\partial z}{\partial t} c\Delta v - b\Delta w - a\Delta z = f_4(u,v,w,z) \quad \text{in } R^+ \times \Omega \tag{3.4}$$

with the same boundary conditions (1.5) and initial data (1.6), and the diffusion matrix

$$\begin{pmatrix} a & b & c & d \\ c & a & b & 0 \\ 0 & b & a & c \\ d & c & b & a \end{pmatrix},$$

All the previous results remain valid in the region

$$\sum = \Big\{ (u_0, v_0, w_0, z_0) \in \mathbb{R}^4 \text{ such that:} u_0 \leq -\frac{2(b-c)}{d-\mu}(v_0 - w_0) + z_0 \\ , u_0 \leq -\frac{2(b-c)}{d+\mu}(v_0 - w_0) + z_0, , u_0 + \frac{2(b+c)}{d-\mu}(v_0 + w_0) + z_0 \geq 0 \\ \text{and } u_0 + \frac{2(b+c)}{d+\mu}(v_0 + w_0) + z_0 \geq 0, \Big\} \tag{3.5}$$

and system $(2.1) - (2.4)$ becomes

$$\begin{cases} \frac{\partial U}{\partial t} - \left(a - \frac{1}{2}(d-\mu)\right)\Delta U = F_{11}(U,V,W,Z) \\ \frac{\partial V}{\partial t} - \left(a - \frac{1}{2}(d+\mu)\right)\Delta V = F_{21}(U,V,W,Z) \\ \frac{\partial W}{\partial t} - \left(a + \frac{1}{2}(d-\mu)\right)\Delta W = F_{31}(U,V,W,Z) \\ \frac{\partial Z}{\partial t} - \left(a + \frac{1}{2}(d+\mu)\right)\Delta Z = F_{41}(U,V,W,Z) \end{cases} \tag{3.6}$$

where

$$\begin{cases} U(t,x) = -u(t,x) + \frac{2(b-c)}{d+\mu}(w(t,x) - v(t,x)) + z(t,x) \\ V(t,x) = -u(t,x) + \frac{2(b-c)}{d-\mu}(w(t,x) - v(t,x)) + z(t,x) \\ W(t,x) = u(t,x) + \frac{2(b+c)}{d-\mu}(v(t,x) + w(t,x)) + z(t,x) \\ Z(t,x) = u(t,x) + \frac{2(b+c)}{d+\mu}(v(t,x) + w(t,x)) + z(t,x) \end{cases}$$

for all (t,x) in $]0, T^*[\times \Omega$

$$\begin{cases} F_{11}(U,V,W,Z) = \left(-f_1 + \frac{2(b-c)}{d-\mu}(f_3 - f_2) + f_4\right)(u,v,w,z) \\ F_{12}(U,V,W,Z) = \left(-f_1 + \frac{2(b-c)}{d+\mu}(f_3 - f_2) + f_4\right)(u,v,w,z) \\ F_{13}(U,V,W,Z) = \left(f_1 + \frac{2(b+c)}{d-\mu}(f_2 + f_3) + f_4\right)(u,v,w,z) \\ F_{14}(U,V,W,Z) = \left(f_1 + \frac{2(b+c)}{d+\mu}(f_2 + f_3) + f_4\right)(u,v,w,z) \end{cases} \tag{3.7}$$

for all (u,v,w,z) in $\sum$
With the boundary conditions

$$\begin{cases} \lambda U + (1-\lambda)\frac{\partial U}{\partial \eta} = \rho_1 & \text{in }]0, T^*[\times \partial\Omega \\ \lambda V + (1-\lambda)\frac{\partial V}{\partial \eta} = \rho_2 & \text{in }]0, T^*[\times \partial\Omega \\ \lambda W + (1-\lambda)\frac{\partial W}{\partial \eta} = \rho_3 & \text{in }]0, T^*[\times \partial\Omega \\ \lambda Z + (1-\lambda)\frac{\partial Z}{\partial \eta} = \rho_4 & \text{in }]0, T^*[\times \partial\Omega \end{cases} \tag{3.8}$$

where

$$\begin{cases} \rho_1 = -\beta_1 + \dfrac{2(b-c)}{d-\mu}(\beta_3 - \beta_2) + \beta_4 \\ \rho_2 = -\beta_1 + \dfrac{2(b-c)}{d+\mu}(\beta_3 - \beta_2) + \beta_4 \\ \rho_3 = \beta_1 + \dfrac{2(b+c)}{d-\mu}(\beta_2 + \beta_3) + \beta_4 \\ \rho_4 = \beta_1 + \dfrac{2(b+c)}{d+\mu}(\beta_2 + \beta_3) + \beta_4 \end{cases} \tag{3.9}$$

and initial data (1.5).

The conditions (1.8) – (1.12) become respectively:

$$\begin{array}{c} \left(f_1 + \frac{2(b+c)}{d-\mu}(f_2 + f_3) + f_4\right)(-\frac{2(b+c)}{d-\mu}(v+w) - z, v, w, z) \geq 0 \\ \left(-f_1 - \frac{2(b-c)}{d+\mu}(f_2 - f_3) + f_4\right)(-\frac{2(b-c)}{d+\mu}(v-w) + z, v, w, z) \geq 0 \\ \left(f_1 + \frac{2(b+c)}{d-\mu}(f_2 + f_3) + f_4\right)(-\frac{2(b+c)}{d-\mu}(v+w) - z, v, w, z) \geq 0 \\ \left(f_1 + \frac{2(b+c)}{d+\mu}(f_2 + f_3) + f_4\right)(-\frac{2(b+c)}{d+\mu}(v+w) - z, v, w, z) \geq 0 \end{array}$$

2. When the system (1.1) – (1.4) rewritten as follws:

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v - c\Delta w - d\Delta z = f_1(u,v,w,z) \\ \frac{\partial v}{\partial t} - c\Delta u - a\Delta v - b\Delta w = f_2(u,v,w,z) \\ \frac{\partial w}{\partial t} - b\Delta v - a\Delta w - c\Delta z = f_3(u,v,w,z) \\ \frac{\partial z}{\partial t} - d\Delta u - c\Delta v - b\Delta w - a\Delta z = f_4(u,v,w,z) \end{cases}$$

or

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v - c\Delta w - d\Delta z = f_1(u,v,w,z) \\ \frac{\partial v}{\partial t} - a\Delta v - c\Delta w - b\Delta z = f_2(u,v,w,z) \\ \frac{\partial w}{\partial t} - b\Delta u - c\Delta v - a\Delta w = f_3(u,v,w,z) \\ \frac{\partial z}{\partial t} - d\Delta u - c\Delta v - b\Delta w - a\Delta z = f_4(u,v,w,z) \end{cases},$$

all the previous results remain valid in the region

$$\sum = \Big\{(u_0, v_0, w_0, z_0) \in \mathbb{R}^4 \text{ such that:} u_0 \leq -\frac{2(b-c)}{d-\mu}(v_0 - w_0) + z_0,$$
$$u_0 \leq -\frac{2(b-c)}{d+\mu}(v_0 - w_0) + z_0, u_0 + \frac{2(b+c)}{d-\mu}(v_0 + w_0) + z_0 \geq 0$$
$$\text{and } u_0 + \frac{2(b+c)}{d+\mu}(v_0 + w_0) + z_0 \geq 0\Big\},$$

for the first system.

$$\sum = \Big\{(u_0, v_0, w_0, z_0) \in \mathbb{R}^4 \text{ such that:} u_0 \leq -\frac{2(b-c)}{d-\mu}(v_0 - w_0) + z_0,$$
$$u_0 \leq -\frac{2(b-c)}{d+\mu}(v_0 - w_0) + z_0, u_0 + \frac{2(b+c)}{d-\mu}(v_0 + w_0) + z_0 \geq 0$$
$$\text{and } u_0 + \frac{2(b+c)}{d+\mu}(v_0 + w_0) + z_0 \geq 0\Big\},$$

for the second system

Acknowledgements
The authors are grateful to the referees for their valuable suggestions to improve the quality of paper.

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