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β-γ-Irresolute and β-γ-Closed Graph

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Abstract

In this paper, we introduce the notion of β-γ-g.closed sets and some weak separation axioms. Also we show that some basic properties of β-γ-$T_{\frac{1}{2}}$, β-γ-T_i, β-γ-D_i for $i = 0, 1, 2$ spaces and we ofer a new class of functions called β-γ-irresolute, β-γ-continuous functions and a new notion of the graph of a function called a β-γ-closed graph and investigate some of their fundamental properties.

Keywords: *β-γ-open set, β-γ-g.closed set.*

1 Introduction

Ogata [3] introduced the notion of γ-open sets which are weaker than open sets. The concept of β-γ-open sets and β-γD-sets in topological spaces are introduced by Hariwan Z. Ibrahim [1].

In this paper, we introduce the notion of β-γ-g.closed sets and some weak separation axioms. Also we show that some basic properties of β-γ-$T_{\frac{1}{2}}$, β-γ-T_i, β-γ-D_i for $i = 0, 1, 2$ spaces and we ofer a new class of functions called β-γ-irresolute, β-γ-continuous functions and a new notion of the graph of a function called a β-γ-closed graph and investigate some of their fundamental properties.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. An operation γ [3] on a topology τ is a mapping from τ in to power set $P(X)$ of X such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [3] if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Then, τ_γ denotes the set of all γ-open set in X. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ-open sets are called γ-closed. The τ_γ-interior [2] of A is denoted by τ_γ-$Int(A)$ and defined to be the union of all γ-open sets of X contained in A. A subset A of a space X is said to be β-γ-open [1] if $A \subseteq Cl(\tau_\gamma\text{-}Int(Cl(A)))$.

3 β-γ-g Closed Sets, β-γ-$T_{\frac{1}{2}}$ Spaces and β-γ-Irresolute

Definition 3.1 *A subset A of X is called β-γ-closed if and only if its complement is β-γ-open.*

Moreover, β-$\gamma O(X)$ denotes the collection of all β-γ-open sets of (X, τ) and β-$\gamma C(X)$ denotes the collection of all β-γ-closed sets of (X, τ).

Definition 3.2 *Let A be a subset of a topological space (X, τ). The intersection of all β-γ-closed sets containing A is called the β-γ-closure of A and is denoted by β-$\gamma Cl(A)$.*

Definition 3.3 *Let (X, τ) be a topological space. A subset U of X is called a β-γ-neighbourhood of a point $x \in X$ if there exists a β-γ-open set V such that $x \in V \subseteq U$.*

Theorem 3.4 *For the β-γ-closure of subsets A, B in a topological space (X, τ), the following properties hold:*

1. *A is β-γ-closed in (X, τ) if and only if $A = \beta$-$\gamma Cl(A)$.*

2. *If $A \subseteq B$ then β-$\gamma Cl(A) \subseteq \beta$-$\gamma Cl(B)$.*

3. *β-$\gamma Cl(A)$ is β-γ-closed, that is β-$\gamma Cl(A) = \beta$-$\gamma Cl(\beta$-$\gamma Cl(A))$.*

4. *$x \in \beta$-$\gamma Cl(A)$ if and only if $A \cap V \neq \phi$ for every β-γ-open set V of X containing x.*

Proof. It is obvious.

Definition 3.5 *A subset A of the space (X,τ) is said to be β-γ-g.closed if β-$\gamma Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a β-γ-open set in (X,τ).*

It is clear that every β-γ-closed subset of X is also a β-γ-g.closed set. The following example shows that a β-γ-g.closed set need not be β-γ-closed.

Example 3.6 *let $X = \{a,b,c\}$, $\tau = \{\phi,\ X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$, define an operation $\gamma : \tau \to P(X)$ such that $\gamma(A) = X$. Then $\{b\}$ is β-γ-g.closed but it is not β-γ-closed.*

Proposition 3.7 *A subset A of (X,τ) is β-γ-g.closed if and only if β-$\gamma Cl(\{x\}) \cap A \neq \phi$ holds for every $x \in \beta$-$\gamma Cl(A)$.*

Proof. Let U be a β-γ-open set such that $A \subseteq U$. Let $x \in \beta$-$\gamma Cl(A)$. By assumption there exists a $z \in \beta$-$\gamma Cl(\{x\})$ and $z \in A \subseteq U$. It follows from Theorem 3.4 that $U \cap \{x\} \neq \phi$. Hence $x \in U$. This implies β-$\gamma Cl(A) \subseteq U$. Therefore A is β-γ-g.closed set in (X,τ).
Conversely, let A be a β-γ-g.closed subset of X and $x \in \beta$-$\gamma Cl(A)$ such that β-$\gamma Cl(\{x\}) \cap A = \phi$. Since, β-$\gamma Cl(\{x\})$ is β-γ-closed set in (X,τ). Therefore by Definition 3.1, $X-(\beta$-$\gamma Cl(\{x\}))$ is a β-γ-open set. Since $A \subseteq X-(\beta$-$\gamma Cl(\{x\}))$ and A is β-γ-g.closed implies that β-$\gamma Cl(A) \subseteq X-(\beta$-$\gamma Cl(\{x\}))$ holds, and hence $x \notin \beta$-$\gamma Cl(A)$. This is a contradiction. Hence β-$\gamma Cl(\{x\}) \cap A \neq \phi$.

Theorem 3.8 *If β-$\gamma Cl(\{x\}) \cap A \neq \phi$ holds for every $x \in \beta$-$\gamma Cl(A)$, then β-$\gamma Cl(A) - A$ does not contain a non empty β-γ-closed set.*

Proof. Suppose there exists a non empty β-γ-closed set F such that $F \subseteq \beta$-$\gamma Cl(A) - A$. Let $x \in F$, $x \in \beta$-$\gamma Cl(A)$ holds. It follows that $F \cap A = \beta$-$\gamma Cl(F) \cap A \supseteq \beta$-$\gamma Cl(\{x\}) \cap A \neq \phi$. Hence $F \cap A \neq \phi$. This is a contradiction.

Corollary 3.9 *A is β-γ-g.closed if and only if $A = F - N$, where F is β-γ-closed and N contains no non-empty β-γ-closed subsets.*

Proof. Necessity follows from Proposition 3.7 and Theorem 3.8 with $F = \beta$-$\gamma Cl(A)$ and $N = \beta$-$\gamma Cl(A) - A$.
Conversely, if $A = F - N$ and $A \subseteq O$ with O is β-γ-open, then $F \cap (X - O)$ is a β-γ-closed subset of N and thus is empty. Hence β-$\gamma Cl(A) \subseteq F \subseteq O$.

Theorem 3.10 *If a subset A of X is β-γ-g.closed and $A \subseteq B \subseteq \beta$-$\gamma Cl(A)$, then B is a β-γ-g.closed set in X.*

Proof. Let A be a β-γ-g.closed set such that $A \subseteq B \subseteq \beta$-$\gamma Cl(A)$. Let U be a β-γ-open set of X such that $B \subseteq U$. Since A is β-γ-g.closed, we have β-$\gamma Cl(A) \subseteq U$. Now β-$\gamma Cl(A) \subseteq \beta$-$\gamma Cl(B) \subseteq \beta$-$\gamma Cl[\beta$-$\gamma Cl(A)] = \beta$-$\gamma Cl(A) \subseteq U$. That is β-$\gamma Cl(B) \subseteq U$, U is β-γ-open. Therefore B is a β-γ-g.closed set in X.

Theorem 3.11 *Let $\gamma : \tau \to P(X)$ be an operation. Then for each $x \in X$, either $\{x\}$ is β-γ-closed or $X - \{x\}$ is β-γ-g.closed set in (X, τ).*

Proof. Suppose that $\{x\}$ is not β-γ-closed, then by Definition 3.1, $X - \{x\}$ is not β-γ-open. Let U be any β-γ-open set such that $X - \{x\} \subseteq U$, so $U = X$. Hence β-$\gamma Cl(X - \{x\}) \subseteq U$. Therefore $X - \{x\}$ is β-γ-g.closed.

Definition 3.12 *A space X is said to be β-γ-$T_{\frac{1}{2}}$ space if every β-γ-g.closed set in (X, τ) is β-γ-closed.*

Theorem 3.13 *A space X is a β-γ-$T_{\frac{1}{2}}$ space if and only if $\{x\}$ is β-γ-closed or β-γ-open in (X, τ).*

Proof. Suppose $\{x\}$ is not β-γ-closed. Then it follows from assumption and Theorem 3.11 that $\{x\}$ is β-γ-open.
Conversely, Let F be β-γ-g.closed set in (X, τ). Let x be any point in β-$\gamma Cl(F)$, then $\{x\}$ is β-γ-open or β-γ-closed.

1. Suppose $\{x\}$ is β-γ-open. Then by Theorem 3.4, we have $\{x\} \cap F \neq \phi$, hence $x \in F$. This implies β-$\gamma Cl(F) \subseteq F$, therefore F is β-γ-closed.

2. Suppose $\{x\}$ is β-γ-closed. Assume $x \notin F$, then $x \in \beta$-$\gamma Cl(F) - F$. This is not possible by Theorem 3.8. Thus we have $x \in F$. Therefore β-$\gamma Cl(F) = F$ and hence F is β-γ-closed.

Definition 3.14 *[1] A topological space (X, τ) with an operation γ on τ is said to be*

1. *β-γ-T_0 if for each pair of distinct points x, y in X, there exists a β-γ-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*

2. *β-γ-T_1 if for each pair of distinct points x, y in X, there exist two β-γ-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

3. *β-γ-T_2 if for each distinct points x, y in X, there exist two disjoint β-γ-open sets U and V containing x and y respectively.*

Definition 3.15 *[1] A subset A of a topological space X is called a β-γD-set if there are two β-γ-open sets U and V such that $U \neq X$ and $A = U - V$.*

Definition 3.16 *[1] A topological space (X, τ) with an operation γ on τ is said to be*

1. *β-γ-D_0 if for any pair of distinct points x and y of X there exists a β-γD-set of X containing x but not y or a β-γD-set of X containing y but not x.*

2. *β-γ-D_1 if for any pair of distinct points x and y of X there exist two β-γD-sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

3. *β-γ-D_2 if for any pair of distinct points x and y of X there exist disjoint β-γD-sets G and E of X containing x and y, respectively.*

Definition 3.17 *A topological space (X, τ) with an operation γ on τ, is said to be β-γ-symmetric if for x and y in X, $x \in \beta$-$\gamma Cl(\{y\})$ implies $y \in \beta$-$\gamma Cl(\{x\})$.*

Proposition 3.18 *If (X, τ) is a topological space with an operation γ on τ, then the following are equivalent:*

1. *(X, τ) is a β-γ-symmetric space.*

2. *$\{x\}$ is β-γ-g.closed, for each $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\{x\} \subseteq U \in \beta$-$\gamma O(X)$, but β-$\gamma Cl(\{x\}) \nsubseteq U$. Then β-$\gamma Cl(\{x\}) \cap X - U \neq \phi$. Now, we take $y \in \beta$-$\gamma Cl(\{x\}) \cap X - U$, then by hypothesis $x \in \beta$-$\gamma Cl(\{y\}) \subseteq X - U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is β-γ-g.closed, for each $x \in X$.
(2) $\Rightarrow$ (1). Assume that $x \in \beta$-$\gamma Cl(\{y\})$, but $y \notin \beta$-$\gamma Cl(\{x\})$. Then $\{y\} \subseteq X - \beta$-$\gamma Cl(\{x\})$ and hence β-$\gamma Cl(\{y\}) \subseteq X - \beta$-$\gamma Cl(\{x\})$. Therefore $x \in X - \beta$-$\gamma Cl(\{x\})$, which is a contradiction and hence $y \in \beta$-$\gamma Cl(\{x\})$.

Proposition 3.19 *A topological space (X, τ) is β-γ-T_1 if and only if the singletons are β-γ-closed sets.*

Proof. Let (X, τ) be β-γ-T_1 and x any point of X. Suppose $y \in X - \{x\}$, then $x \neq y$ and so there exists a β-γ-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq X - \{x\}$, that is $X - \{x\} = \cup\{U : y \in X - \{x\}\}$ which is β-γ-open.
Conversely, suppose $\{p\}$ is β-γ-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X - \{x\}$. Hence $X - \{x\}$ is a β-γ-open set contains y but not x. Similarly $X - \{y\}$ is a β-γ-open set contains x but not y. Accordingly X is a β-γ-T_1 space.

Corollary 3.20 *If a topological space (X, τ) with an operation γ on τ is a β-γ-T_1 space, then it is β-γ-symmetric.*

Proof. In a β-γ-T_1 space, every singleton is β-γ-closed (Proposition 3.19) and therefore is β-γ-g.closed. Then by Proposition 3.18, (X, τ) is β-γ-symmetric.

Corollary 3.21 *For a topological space (X, τ) with an operation γ on τ, the following statements are equivalent:*

1. *(X, τ) is β-γ-symmetric and β-γ-T_0.*
2. *(X, τ) is β-γ-T_1.*

Proof. By Corollary 3.20 and Remark 3.8 [1], it suffices to prove only (1) $\Rightarrow$ (2).
Let $x \neq y$ and as (X, τ) is β-γ-T_0, we may assume that $x \in U \subseteq X - \{y\}$ for some $U \in \beta$-$\gamma O(X)$. Then $x \notin \beta$-$\gamma Cl(\{y\})$ and hence $y \notin \beta$-$\gamma Cl(\{x\})$. There exists a β-γ-open set V such that $y \in V \subseteq X - \{x\}$ and thus (X, τ) is a β-γ-T_1 space.

Remark 3.22 *Let (X, τ) be a topological space and γ be an operation on τ, then the following statements are hold:*

1. *Every β-γ-T_1 space is β-γ-$T_{\frac{1}{2}}$.*
2. *Every β-γ-$T_{\frac{1}{2}}$ space is β-γ-T_0.*

Proposition 3.23 *If (X, τ) is a β-γ-symmetric space with an operation γ on τ, then the following statements are equivalent:*

1. *(X, τ) is a β-γ-T_0 space.*
2. *(X, τ) is a β-γ-$T_{\frac{1}{2}}$ space.*
3. *(X, τ) is a β-γ-T_1 space.*

Proof. (1) $\Leftrightarrow$ (3). Obvious from Corollary 3.21.
(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). Directly from Remark 3.22.

Corollary 3.24 *For a β-γ-symmetric space (X, τ), the following are equivalent:*

1. *(X, τ) is β-γ-T_0.*
2. *(X, τ) is β-γ-D_1.*
3. *(X, τ) is β-γ-T_1.*

Proof. (1) $\Rightarrow$ (3). Follows from Corollary 3.21.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1). Follows from Remark 3.8 [1]and Corollary 3.11 [1].

Definition 3.25 *Let A be a subset of a topological space (X, τ) and γ be an operation on τ. The β-γ-kernel of A, denoted by β-$\gamma ker(A)$ is defined to be the set*

$$\beta\text{-}\gamma ker(A) = \cap\{U \in \beta\text{-}\gamma O(X) \colon A \subseteq U\}.$$

Proposition 3.26 *Let (X, τ) be a topological space with an operation γ on τ and $x \in X$. Then $y \in \beta$-$\gamma ker(\{x\})$ if and only if $x \in \beta$-$\gamma Cl(\{y\})$.*

Proof. Suppose that $y \notin \beta$-$\gamma ker(\{x\})$. Then there exists a β-γ-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \beta$-$\gamma Cl(\{y\})$. The proof of the converse case can be done similarly.

Proposition 3.27 *Let (X, τ) be a topological space with an operation γ on τ and A be a subset of X. Then, β-$\gamma ker(A) = \{x \in X$: β-$\gamma Cl(\{x\}) \cap A \neq \phi\}$.*

Proof. Let $x \in \beta$-$\gamma ker(A)$ and suppose β-$\gamma Cl(\{x\}) \cap A = \phi$. Hence $x \notin X - \beta$-$\gamma Cl(\{x\})$ which is a β-γ-open set containing A. This is impossible, since $x \in \beta$-$\gamma ker(A)$. Consequently, β-$\gamma Cl(\{x\}) \cap A \neq \phi$. Next, let $x \in X$ such that β-$\gamma Cl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \beta$-$\gamma ker(A)$. Then, there exists a β-γ-open set V containing A and $x \notin V$. Let $y \in \beta$-$\gamma Cl(\{x\}) \cap A$. Hence, V is a β-γ-neighbourhood of y which does not contain x. By this contradiction $x \in \beta$-$\gamma ker(A)$ and the claim.

Proposition 3.28 *If a singleton $\{x\}$ is a β-γD-set of (X, τ), then β-$\gamma ker(\{x\}) \neq X$.*

Proof. Since $\{x\}$ is a β-γD-set of (X, τ), then there exist two subsets $U_1, U_2 \in \beta$-$\gamma O(X, \tau)$ such that $\{x\} = U_1 - U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that β-$\gamma ker(\{x\}) \subseteq U_1 \neq X$ and so β-$\gamma ker(\{x\}) \neq X$.

Proposition 3.29 *For a β-γ-$T_{\frac{1}{2}}$ topological space (X, τ) with at least two points, (X, τ) is a β-γ-D_1 space if and only if β-$\gamma ker(\{x\}) \neq X$ holds for every point $x \in X$.*

Proof. Necessity. Let $x \in X$. For a point $y \neq x$, there exists a β-γD-set U such that $x \in U$ and $y \notin U$. Say $U = U_1 - U_2$, where $U_i \in \beta$-$\gamma O(X, \tau)$ for each $i \in \{1, 2\}$ and $U_1 \neq X$. Thus, for the point x, we have a β-γ-open set U_1 such that $\{x\} \subseteq U_1$ and $U_1 \neq X$. Hence, β-$\gamma ker(\{x\}) \neq X$.
Sufficiency. Let x and y be a pair of distinct points of X. We prove that there exist β-γD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Theorem 3.13, we can take the subsets A and B for the following four cases for two points x and y.
Case1. $\{x\}$ is β-γ-open and $\{y\}$ is β-γ-closed in (X, τ). Since β-$\gamma ker(\{y\}) \neq X$, then there exists a β-γ-open set V such that $y \in V$ and $V \neq X$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = V - (X - \{y\})$, then V is a β-γ-open set with $V \neq X$ and $X - \{y\}$ is β-γ-open, and B is a required β-γD-set containing y such that $x \notin B$. Obviously, A is a required β-γD-set containing x such that $y \notin A$.

Case 2. $\{x\}$ is β-γ-closed and $\{y\}$ is β-γ-open in (X,τ). The proof is similar to Case 1.
Case 3. $\{x\}$ and $\{y\}$ are β-γ-open in (X,τ). Put $A=\{x\}$ and $B=\{y\}$.
Case 4. $\{x\}$ and $\{y\}$ are β-γ-closed in (X,τ). Put $A=X-\{y\}$ and $B=X-\{x\}$.
For each case of the above, the subsets A and B are the required β-γD-sets. Therefore, (X,τ) is a β-γ-D_1 space.

Definition 3.30 *Let (X,τ) and (Y,σ) be two topological spaces and γ, β operations on τ, σ, respectively. A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be β-γ-irresolute if for each $x\in X$ and each β-β-open set V containing $f(x)$, there is a β-γ-open set U in X containing x such that $f(U)\subseteq V$.*

Theorem 3.31 *Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a mapping, then the following statements are equivalent:*

1. *f is β-γ-irresolute.*
2. *$f(\beta\text{-}\gamma Cl(A))\subseteq\beta\text{-}\beta Cl(f(A))$ holds for every subset A of (X,τ).*
3. *$f^{-1}(B)$ is β-γ-closed in (X,τ), for every β-β-closed set B of (Y,σ).*

Proof. (1)$\Rightarrow$(2). Let $y\in f(\beta\text{-}\gamma Cl(A))$ and V be any β-β-open set containing y. Then there exists a point $x\in X$ and a β-γ-open set U such that $f(\text{x})=y$ and $x\in U$ and $f(\text{U})\subseteq V$. Since $x\in\beta\text{-}\gamma Cl(A)$, we have $U\cap A\neq\phi$ and hence $\phi\neq f(U\cap A)\subseteq f(U)\cap f(A)\subseteq V\cap f(A)$. This implies $y\in\beta\text{-}\beta Cl(f(A))$. Therefore we have $f(\beta\text{-}\gamma Cl(A))\subseteq\beta\text{-}\beta Cl(f(A))$.
(2)$\Rightarrow$(3). Let B be a β-β-closed set in (Y,σ). Therefore $\beta\text{-}\beta Cl(B)=B$. By using (2) we have $f(\beta\text{-}\gamma Cl(f^{-1}(B))\subseteq\beta\text{-}\beta Cl(B)=B$. Therefore we have $\beta\text{-}\gamma Cl(f^{-1}(B))\subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is β-γ-closed.
(3)$\Rightarrow$(1). Obvious.

Definition 3.32 *A mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be β-γ-closed if for any β-γ-closed set A of (X,τ), $f(A)$ is a β-β-closed in (Y,σ).*

Theorem 3.33 *Suppose that f is β-γ-irresolute mapping and f is β-γ-closed. Then:*

1. *For every β-γ-g.closed set A of (X,τ) the image $f(A)$ is β-β-g.closed.*
2. *For every β-β-g.closed set B of (Y,σ) the inverse set $f^{-1}(B)$ is β-γ-g.closed.*

Proof.

1. Let V be any β-β-open set in (Y, σ) such that $f(\mathrm{A}) \subseteq V$. By using Theorem 3.31 $f^{-1}(V)$ is β-γ-open. Since A is β-γ-g.closed and $A \subseteq f^{-1}(V)$, we have β-$\gamma Cl(A) \subseteq f^{-1}(V)$, and hence $f(\beta\text{-}\gamma Cl(A)) \subseteq V$. By assumption $f(\beta\text{-}\gamma Cl(A))$ is a β-β-closed set. Therefore $\beta\text{-}\beta Cl(f(A)) \subseteq \beta\text{-}\beta Cl(f(\beta\text{-}\gamma Cl(A))) = f(\beta\text{-}\gamma Cl(A)) \subseteq V$. This implies $f(\mathrm{A})$ is β-β-g.closed.

2. Let U be β-γ-open set of (X, τ) such that $f^{-1}(B) \subseteq U$. Let $F = \beta\text{-}\gamma Cl(f^{-1}(B)) \cap (X - U)$, then F is β-γ-closed set in (X, τ). Since f is β-γ-closed this implies $f(\mathrm{F})$ is β-β-closed in (Y, σ). Since $f(F) \subseteq f(\beta\text{-}\gamma Cl(f^{-1}(B))) \cap f(X - U) \subseteq \beta\text{-}\beta Cl(f(f^{-1}(B))) \cap f(X - U) \subseteq \beta\text{-}\beta Cl(B) \cap (Y - B)$. This implies $f(F) = \phi$, and hence $F = \phi$. Therefore $\beta\text{-}\gamma Cl(f^{-1}(B)) \subseteq U$. Hence $f^{-1}(B)$ is β-γ-g.closed in (X, τ).

Theorem 3.34 *Let $f : (X, \tau) \to (Y, \sigma)$ is β-γ-irresolute and β-γ-closed. Then:*

1. *If f is injective and (Y, σ) is β-β-$T_{\frac{1}{2}}$, then (X, τ) is β-γ-$T_{\frac{1}{2}}$.*

2. *If f is surjective and (X, τ) is β-γ-$T_{\frac{1}{2}}$, then (Y, σ) is β-β-$T_{\frac{1}{2}}$.*

Proof.

1. Let A be a β-γ-g.closed set of (X, τ). By Theorem 3.33, $f(A)$ is β-β-g.closed. Since (Y, σ) is β-β-$T_{\frac{1}{2}}$, this implies that $f(A)$ is β-β-closed. Since f is β-γ-irresolute, then by Theorem 3.31, we have $A = f^{-1}(f(A))$ is β-γ-closed. Hence (X, τ) is β-γ-$T_{\frac{1}{2}}$.

2. Let B be a β-β-g.closed set of (Y, σ). By Theorem 3.33, $f^{-1}(B)$ is β-γ-g.closed in X. Since (X, τ) is β-γ-$T_{\frac{1}{2}}$, so $f^{-1}(B)$ is β-γ-closed. Since f is surjective and β-γ-closed, so $f(f^{-1}(B)) = B$ is β-β-closed.

Theorem 3.35 *If $f : (X, \tau) \to (Y, \sigma)$ is a β-γ-irresolute surjective function and E is a β-βD-set in Y, then the inverse image of E is a β-γD-set in X.*

Proof. Let E be a β-βD-set in Y. Then there are β-β-open sets U_1 and U_2 in Y such that $E = U_1 - U_2$ and $U_1 \neq Y$. By the β-γ-irresolute of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are β-γ-open in X. Since $U_1 \neq Y$ and f is surjective, we have $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a β-γD-set.

Theorem 3.36 *If (Y, σ) is β-β-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is β-γ-irresolute bijective, then (X, τ) is β-γ-D_1.*

Proof. Suppose that Y is a β-β-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is β-β-D_1, there exist β-βD-set G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(x) \notin G_y$ and $f(y) \notin G_x$. By Theorem 3.35, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are β-γD-set in X containing x and y, respectively, such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. This implies that X is a β-γ-D_1 space.

Theorem 3.37 *A topological space (X, τ) is β-γ-D_1 if for each pair of distinct points $x, y \in X$, there exists a β-γ-irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a β-β-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists a β-γ-irresolute, surjective function f of a space X onto a β-β-D_1 space Y such that $f(x) \neq f(y)$. Then, there exist disjoint β-βD-set G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is β-γ-irresolute and surjective, by Theorem 3.35, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint β-γD-sets in X containing x and y, respectively. Hence, X is β-γ-D_1 space.

4 β-γ-Continuous and β-γ-Closed Graphs

Definition 4.1 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be β-γ-continuous if for every open set V of Y, $f^{-1}(V)$ is β-γ-open in X.*

Theorem 4.2 *The following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

1. *f is β-γ-continuous.*
2. *The inverse image of every closed set in Y is β-γ-closed in X.*
3. *For each subset A of X, $f(\beta\text{-}\gamma Cl(A)) \subseteq Cl(f(A))$.*
4. *For each subset B of Y, $\beta\text{-}\gamma Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.*

Proof. (1) $\Leftrightarrow$ (2). Obvious.
(3) $\Leftrightarrow$ (4). Let B be any subset of Y. Then by (3), we have $f(\beta\text{-}\gamma Cl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) \subseteq Cl(B)$. This implies $\beta\text{-}\gamma Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (4), we have, $\beta\text{-}\gamma Cl(f^{-1}(f(A))) \subseteq f^{-1}(Cl(f(A)))$. Thus, $f(\beta\text{-}\gamma Cl(A)) \subseteq Cl(f(A))$.
(2) $\Rightarrow$ (4). Let $B \subseteq Y$. Since $f^{-1}(Cl(B))$ is β-γ-closed and $f^{-1}(B) \subseteq f^{-1}(Cl(B))$, then $\beta\text{-}\gamma Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.
(4) $\Rightarrow$ (2). Let $K \subseteq Y$ be a closed set. By (4), $\beta\text{-}\gamma Cl(f^{-1}(K)) \subseteq f^{-1}(Cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is β-γ-closed.

Theorem 4.3 *If $f : X \to Y$ is a β-γ-continuous injective function and Y is T_2, then X is β-γ-T_2.*

Proof. Let x and y in X be any pair of distinct points, then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since f is β-γ-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are β-γ-open in X containing x and y respectively, we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Thus, X is β-γ-T_2.

Definition 4.4 *For a function $f : (X, \tau) \to (Y, \sigma)$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to be β-γ-closed if for each $(x, y) \notin G(f)$, there exist a β-γ-open set U containing x and an open set V containing y such that $(U \times V) \cap G(f) = \phi$.*

Lemma 4.5 *The function $f : (X, \tau) \to (Y, \sigma)$ has an β-γ-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $y \neq f(x)$, there exist a β-γ-open set U and an open set V containing x and y respectively, such that $f(U) \cap V = \phi$.*

Proof. It follows readily from the above definition.

Theorem 4.6 *If $f : (X, \tau) \to (Y, \sigma)$ is an injective function with the β-γ-closed graph, then X is β-γ-T_1.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist a β-γ-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \phi$. Therefore $y \notin U$ and it follows that X is β-γ-T_1.

Theorem 4.7 *If $f : (X, \tau) \to (Y, \sigma)$ is an injective β-γ-continuous with a β-γ-closed graph $G(f)$, then X is β-γ-T_2.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since the graph $G(f)$ is β-γ-closed, there exist a β-γ-open set U containing x_1 and open set V containing $f(x_2)$ such that $f(U) \cap V = \phi$. Since f is β-γ-continuous, $f^{-1}(V)$ is a β-γ-open set containing x_2 such that $U \cap f^{-1}(V) = \phi$. Hence X is β-γ-T_2.

Recall that a space X is said to be T_1 if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

Theorem 4.8 *If $f : (X, \tau) \to (Y, \sigma)$ is an surjective function with the β-γ-closed graph, then Y is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By Lemma 4.5, there exist β-γ-open set U and an open set V containing x and y_1 respectively, such that $f(U) \cap V = \phi$. We obtain an open set V containing y_1 which does not contain y_2. It follows that $y_2 \notin V$. Hence, Y is T_1.

Definition 4.9 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be β-γ-W-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a β-γ-open set U in X containing x such that $f(U) \subseteq Cl(V)$.*

Theorem 4.10 *If $f : (X, \tau) \to (Y, \sigma)$ is β-γ-W-continuous and Y is Hausdorff, then $G(f)$ is β-γ-closed.*

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \phi$. It follows that $Cl(W) \cap V = \phi$. Since f is β-γ-W-continuous, there exists a β-γ-open set U containing x such that $f(U) \subseteq Cl(W)$. Hence, we have $f(U) \cap V = \phi$. This means that $G(f)$ is β-γ-closed.

Definition 4.11 *A subset A of a space X is said to be β-γ-compact relative to X if every cover of A by β-γ-open sets of X has a finite subcover.*

Theorem 4.12 *Let $f : (X, \tau) \to (Y, \sigma)$ have a β-γ-closed graph. If K is β-γ-compact relative to X, then $f(K)$ is closed in Y.*

Proof. Suppose that $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By lemma 4.5, there exists a β-γ-open set U_x containing x and an open neighbourhood V_x of y such that $f(U_x) \cap V_x = \phi$. The family $\{U_x : x \in K\}$ is a cover of K by β-γ-open sets of X and there exists a fnite subset K_0 of K such that $K \subseteq \cup\{U_x : x \in K_0\}$. Put $V = \cap\{V_x : x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \phi$. This means that $f(K)$ is closed in Y.

Theorem 4.13 *If $f : (X, \tau) \to (Y, \sigma)$ has a β-γ-closed graph $G(f)$, then for each $x \in X$. $\{f(x)\} = \cap\{Cl(f(A) : A$ is β-γ-open set containing $x\}$.*

Proof. Suppose that $y \neq f(x)$ and $y \in \cap\{Cl(f(A)) : A$ is β-γ-open set containing $x\}$. Then $y \in Cl(f(A))$ for each β-γ-open set A containing x. This implies that for each open set B containing y, $B \cap f(A) \neq \phi$. Since $(x, y) \notin G(f)$ and $G(f)$ is a β-γ-closed graph, this is a contradiction.

Definition 4.14 *A function $f : (X, \tau) \to (Y, \sigma)$ is called a β-γ-open if the image of every β-γ-open set in X is open in Y.*

Theorem 4.15 *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective β-γ-open function with a β-γ-closed graph $G(f)$, then Y is T_2.*

Proof. Let y_1 and y_2 be any two distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. This implies that there exist a β-γ-open set A of X and an open set B of Y such that $(x, y_2) \in (A \times B)$ and $(A \times B) \cap G(f) = \phi$. We have $f(A) \cap B = \phi$. Since f is β-γ-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is T_2.

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