Gen. Math. Notes, Vol. 14, No. 2, February 2013, pp. 23-36
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Monic Chebyshev Approximations for Solving Optimal Control Problem with Volterra Integro Differential Equations

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(Received: 29-11-12/ Accepted: 17-1-13)

Abstract

In this paper, we formulate and analyze a new model for solving optimal control problems governed by Volterra integro-differential equations. The control and state variables are approximated by using monic Chebyshev series. The optimal control problem is reduced to a constrained optimization problem. Numerical examples are solved to show good ability and accuracy of the present approach.

Keywords: *Optimal control problems, Monic Chebyshev approximation, Volterra Integro – differential equations.*

1 Introduction

Solutions for optimal control problems (OCP) are usually carried out numerically.

Therefore numerical methods and algorithms for solving optimal control problems have evolved significantly. An overview of numerical methods for solving optimal control problems described by ODE and integral equations can be found in [6, 7, 8]. In various branches of applications, the optimal control problems may be governed by integral or integro-differential equations [10]. Integral equations occur amongst others, in neutron transfer theory, in quantum theory and in optimal control theory [9]. The most important methods for solving these types of problem can be found in [2, 3].

In this paper, we present a numerical solution of the optimal control problems governed by Volterra integro-differential equations. In fact, it seems that, with the exception of the simplest physical problems, practically every situation that can be modeled by ordinary differential equations can be extended to a model with Volterra integro equations. For example a general ODE system of interacting biological populations of the form:

$$E\left(x,t,y(t),y^{'}(t),u(t)\right)=0,-1\leq t,x\leq 1,$$

can be extended to Volterra integro-differential equations:

$$y^{'}(x) = F\big(x,y(x),u(x)\big)+S(x) + \int_{-1}^{x} k\big(x,t,y(t),y'(t),u(t)\big)dt. \tag{1.1}$$

Indeed, some related extensions have already been considered in [1] and in other works. Problems in mathematical economics also lead to Volterra integral equations. The relationships among different quantities, for example between capital and investment, include memory effects and the simplest way to describe such memory effects is through Volterra integro operators [10,11].

Let $y(t)\in Y\in\mathcal{R}^N$ is a state vector and $u(t)\in U\in\mathcal{R}^M$ is the control vector where U is a compact set. Let $S(x)\in\mathcal{R}^N$ is a real valued continuous functions on $I=[-1,1]$ and F is continuous function on I.

Suppose that the kernel k is assumed be known and continuous with respect to all variables and control function $u(t)\in U$ is also continuous. In equation (1.1), the we have to find an allowable control $u(t)$ such that:
Minimize the cost functional:

$$J = \int_{-1}^{1} G\big(t,y(t),u(t)\big)dt, \tag{1.2}$$

with initial constraints,

$$L(y(-1), y'(-1)) = 0, \tag{1.3}$$

and the terminal constraints

$$N(y(1), y'(1)) = 0, \tag{1.4}$$

The main idea of this paper is to present a numerical method for solving optimal control problems with Volterra integro differential equations which may lead to present executable numerical approaches for obtaining near optimal solutions of the considered problem. This paper intends to actualize this idea by combining the method of monic Chebyshev approximation and the method of the resulted optimization problem, for providing a numerical scheme to find the optimal solutions.

The remainder of this paper is organized as follows: In section 2, some fundamental information of monic Chebyshev approximations are presented. An approximation of optimal control problem is introduced in section 3. Optimality conditions and the error estimation and associated theorems have been proved in section 4. In section 5, numerical results which demonstrate the efficiency of the new approach are introduced. We end the paper with few concluding remarks in section 6.

2 Preliminaries

The monic Chebyshev approximations of a given function $f(x) \epsilon\, C^{\infty}[-1,1]$ using $(N+1)$ Chebyshev Gauss-Lobatto (CGL) points $x_i = -cos\left(\frac{i\pi}{N}\right), i = 0,1, \dots . N$, are:

$$f(x) \cong \sum_{n=0}^{N} c_n\ a_n \varphi_n\ (x), \tag{2.1}$$

where $\varphi_n(x)$ is the monic Chebyshev polynomials,

$c_n = 1, n = 0,1, \dots, N-1, c_N = \frac{1}{2}$,
And

$$a_n = \begin{cases} \frac{1}{N}\sum_{j=0}^{N} \theta_j \; f(x_j)\,, & n = 0, \\ \frac{2}{N}\sum_{j=0}^{N} \theta_j \; f(x_j)\, x_j, & n = 1, \\ \frac{1}{2^{1-2n}N}\sum_{j=0}^{N} \theta_j \; f(x_j)\, \varphi_n(x_j), & n = 2, \dots, N, \end{cases} \tag{2.2}$$

where $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_j = 1$ for $j = 1,2, \dots, N-1$.
The integration of $f(x)$ is approximated by [4]:

$$\int_{-1}^{x_i} f(x)d(x) \cong \sum_{j=0}^{N} b_{i,j}^{(1)} f(x_j)\,,\; i = 0,1, \dots N,$$

where the entries of $b_{i,j}^{(1)}$ are the elements of the matrix B as given in [4]:

$$b_{i,j}^{(1)} = \frac{\theta_j}{N}(x_i + 1) + \frac{\theta_j}{N} x_j (x_i^2 - 1) + \frac{\theta_j}{N}\sum_{\mathrm{n}=2}^{\mathrm{N}} c_n 2^n \varphi_{\mathrm{n}}(x_j)\left(\frac{T_{n+1}(x_i)}{2(n+1)} - \frac{T_{n-1}(x_i)}{2(n-1)} + \frac{(-1)^{n+1}}{n^2-1}\right),$$
$$i,j = 0,1, \dots, N. \tag{2.3}$$

The operation matrix of the successive integration is given by:

$$\int_{-1}^{x_i}\int_{-1}^{t_{n-1}} \dots \int_{-1}^{t_2}\int_{-1}^{t_1} f(t_0)\, dt_0\, dt_1 \dots dt_{n-2}\, dt_{n-1} = B^{(n)}[f],$$

where,

$B^{(n)} = \left[b_{i,j}^{(n)}\right], i,j = 0,1, \dots, N$, and;

$$b_{i,j}^{(n)} = \frac{(x_{i-}\, x_j)^{n-1}}{(n-1)!} b_{i,j}^{(1)}\,, i,j = 0,1, \dots, N.$$

3 Monic Chebyshev Approach for Solving OCP

Monic Chebyshev approximation is adopted here to approximate the solution of the problem. We start with monic Chebyshev approximation for the highest-order derivative, $y^{(n)}$, and generate approximations to the lowest-order derivatives through successive integrations of the approximation of the highest-order derivative, as follows:

Let

$$y'(t) = \psi(t)\,, \tag{3.1}$$

where $\psi(t_i), i = 0,1,2, \dots N$ are unknowns. This will lead us to

$$y(t_i) = \sum_{j=0}^{N} b_{i,j}^{(1)}\, \psi(t_j) + c_0, \qquad i = 0,1, \dots, N$$

where the constant c_0 may be defined from the given condition.
Consider the following approximation of the control variable:

$$u(t_j) = \sum_{n=0}^{M} a_n \varphi_n(t_j)\,, j = 0,1, \dots M. \tag{3.2}$$

The optimal control problem (1.1)-(1.2) is now replaced by the following constrained optimization problems.

Minimize

$$J = \sum_{j=0}^{N} b_{N,j}^{(1)}\; G\left(t_j, \sum_{s=0}^{N} b_{j,s}^{(1)}\, \psi(t_s) + c_0\,, \sum_{n=0}^{M} a_n\, \varphi_n(t_j) \right). \tag{3.3}$$

Subject to:

$$\psi(x_i) = F\left(x_i, \sum_{j=0}^{N} b_{i,j}^{(1)}\, \psi(x_j) + c_0\,, \sum_{n=0}^{M} a_n\, \varphi_n(x_i) \right) + H(x_i) + \sum_{j=0}^{N} b_{i,j}^{(1)}\; k\left(x_i, t_j, \sum_{s=0}^{N} b_{j,s}^{(1)}\, \psi(t_s) + c_0, \psi(t_j), \sum_{n=0}^{M} a_n\, \varphi_n(t_j) \right), \quad i = 0,1, \dots N. \tag{3.4}$$

The resulted NLP problem (3.3) and (3.4) can be solved by using well-known solvers.

4 Optimality Conditions and Error Analysis

From the calculus of variations, the previous problem can be expressed as that of minimizing the augmented functional [5]:

$$J = \int_{-1}^{1} (\mu\, y' + H)dt \qquad (4.1)$$

Subject to (1.1), (1.3) and (1.4) where the Hamiltonian is defined by:

$$H = G - \mu \quad y', \qquad (4.2)$$

In equations (4.1) and (4.2), $\mu(t)$ is Lagrange multiplier variable. The state $y(t)$, the control $u(t)$ and the Lagrange multipliers $\mu(t)$ that solve the problem must satisfy the constraints (1.1)-(1.4) and the following optimality conditions:

$$H_x - \mu' = 0.$$

$$H_u = 0.$$

The performance index is then approximated as follows:

$$J_N = \sum_{i=0}^{N} b_{N,i}^{(1)}\, G\big(t_i, y(t_i), u(t_i)\big)$$

The constrained optimization problem takes the final form:
Minimize

$$J = J[\alpha, \beta]. \qquad (4.3)$$

Subject to

$$W[\alpha, \beta] = 0, \qquad (4.4)$$

where, $\alpha = [\psi(t_0), \psi(t_1), \dots, \psi(t_N)], \beta = [a_0, a_1, \dots, a_M]$.
For the stopping criteria, we used:

$$|J(\alpha_{N+1}, \beta_{N+1}) - J(\alpha_N, \beta_N)| < \epsilon$$

To decide whether the computed solution in close enough to the optimal solution.

Now, the error estimation in the dynamic system and objective functions are considered. First, the following assumptions are presented to obtain error estimation.

i. The function k is Lipchitz with respect to y and y' with Lipchitz constant $L_{k_{y,y'}}$.

$$\left\|k(x,t,y_1,y'_1,u)-k(x,t,y_2,y'_2,u)\right\|$$

$$\leq L_{k_{y,y'}}\{\|y_1-y_2\|+\|y'_1-y'_2\|\}, \quad (4.5)$$

for all $y_1, y_2 \in Y, u \in U$ and for all $x, t \in I$.

ii. The function $u(t): I \to U$ is satisfy Lipschitz condition with constant L_u.

$\|u(t_1)-u(t_2)\| \leq L_u|t_1-t_2|$, for all $t_1, t_2 \in I$. (4.6)

iii. The function k is jointly Lipschitz with respect to y', y and u with constant $L_{k_{y,y',u}}$.

$$\left\|k(x,t,y_1,y'_1,u_1)-k(x,t,y_2,y'_2,u_2)\right\| \leq L_{k_{y,y',u}}\{\|y_1-y_2\|+\|y'_1-y'_2\|$$

$$+\|u_1-u_2\|\} \quad (4.7)$$

for all $y_1, y_2 \in Y, u_1, u_2 \in U$ and for all $x, t \in I$.

iv. Let max

$$\left\{\left\|\frac{\partial k\left(x,t,y(t),y'(t),u(t)\right)}{\partial t}\right\|, y \in Y, u \in U, x, t \in I\right\} = K_t. \quad (4.8)$$

Lemma 1 *If* $y(t): I \to Y \subset \mathcal{R}^n$ *is differentiable function, then* $\|y'\| \leq M_{y'}$ *and* $\|y''\| \leq M_{y''}$

Proof.

Suppose the function $k\big(x,t,y(t),y'(t),u(t)\big)$ is bounded and from equation (1.1), we have

$$\|y'(x)\| \leq \left\|F\big(x,y(x),u(x)\big)\right\| + \|S(x)\| + \left\|\int_{-1}^{x} k\big(x,t,y(t),y'(t),u(t)\big)dt\right\|$$

$$\leq M_{y'} \quad (4.9)$$

where,

$M_{y'} = M_F + M_H + M_k$ for all $t \in I$ and $max\left\{\left\|k\big(x,t,y(t),y'(t),u(t)\big)\right\|, u \in U, y \in Y\right\} = M_k$.

Now suppose the function

$$\frac{\partial k\left(x,t,y(t),y'(t),u(t)\right)}{\partial x}$$

is bounded, by using the second time-derivative of $y'(x)$ in equation (1.1), we get

$$y''(x) = F'\big(x,y(x),u(x)\big) + H'(x) + k\big(x,x,y(x),y'(x),u(x)\big) + \int_{-1}^{x} \frac{\partial k\left(x,t,y(t),y'(t),u(t)\right)}{\partial x}\,dt, \tag{4.10}$$

then

$$\|y''(x)\| \le \big\|F'\big(x,y(x),u(x)\big)\big\| + \|S'(x)\| + \big\|k\big(x,x,y(x),y'(x),u(x)\big)\big\| + \int_{-1}^{x} \left\|\frac{\partial k\left(x,t,y(t),y'(t),u(t)\right)}{\partial x}\right\| dt \le M_{y''} \tag{4.11}$$

where

$M_{y''} = M_{F'} + M_{H'} + M_k + K_x$ and

$\big\|F'\big(x,y(x),u(x)\big)\big\| \le M_{F'}, \|S'(x)\| \le M_{H'}$.

Assume $(C[I],\|.\|)$ to be the space of all continuous functions with norm $\|y'(x)\| = max_{\forall x\in I}|y'(x)|$. We denote the error of $y(x)$ by $e_1 = \|y - y_m\|$. By using Lemma 1 and under the assumption in this section, we have the following Theorems.

Errors in Dynamic System

Theorem 1 *For every $u(t) \in U$, the error estimation in equation (1.1) satisfies*

$$\|y - y_m\| \le \varepsilon + \breve{k}, \tag{4.12}$$

where,

$$\breve{k} = \left(K_s + L_{k_{y,y',u}}\left(M_{y'} + M_{y''} + L_u\right)\right)$$

$$\varepsilon = \delta_1 + \delta_2 + \lambda + \gamma + M_{k_m}$$

And

$$\lambda = \big\|F\big(x,y(x),u(x)\big) - F_m\big(x,y(x),u(x)\big)\big\|,$$

$$\gamma = \|S(x) - S_m(x)\|,$$

$$\delta_1 = \big\|y_m(x) - y'_m(x)\big\|,$$

$$\delta_2 = \|y(x) - y'(x)\|$$

Proof.

Let $t_j = -cos\frac{j\pi}{N}$ and $x_i = -cos\frac{i\pi}{N}$ for $i, j = 0,1,2, N-1$ and it is obvious that $y_m(t_j) = y(-cos\frac{j\pi}{N})$ and
$y_m(x_i) = y\left(-cos\frac{i\pi}{N}\right)$.
If $x = x_i$, then $y_m(x_i) = y(x_i)$, let $x \neq x_i$,we have

$$\begin{aligned}
\|y - y_m\| &\leq \delta_1 + \delta_2 + max_{\forall\, x\in I}\left|y'(x) - y'_m(x)\right| \\
&\leq \delta_1 + \delta_2 + \lambda + \gamma \\
&\quad + max_{\forall\, x\in I}\left(\int_{-1}^{x}\left\|k\big(x,t,y(t),y'(t),u(t)\big)\right.\right. \\
&\quad \left.\left. - k\left(x,t,y_m(t),y'_m(t),u_m(t)\right)\right\| dt\right) \\
&\leq \delta_1 + \delta_2 + \lambda + \gamma \\
&\quad + max_{\forall\, x\in I}\left(\int_{-1}^{x}\left\|k\big(x,t,y(t),y'(t),u(t)\big)\right.\right. \\
&\quad \left. - k\left(x,t_j,y(t),y'(t),u(t)\right)\right\| dt \\
&\quad + \int_{-1}^{x}\left\|k\left(x,t_j,y(t),y'(t),u(t)\right)\right. \\
&\quad \left. - k\left(x,t_j,y(t_j),y'(t_j),u(t_j)\right)\right\| dt \\
&\quad + \int_{-1}^{x}\left\|k\left(x,t_j,y_m(t_j),y'_m(t_j),u_m(t_j)\right)\right. \\
&\quad \left.\left. - k\left(x,t,y_m(t),y'_m(t),u_m(t)\right)\right\| dt\right) \\
&\leq \delta_1 + \delta_2 + \lambda + \gamma + M_{k_m} \\
&\quad + \left(K_s + L_{k_{y,y',u}}\left(M_{y'} + M_{y''} + L_u\right)\right) \\
&\leq \varepsilon + \breve{k},
\end{aligned} \tag{4.13}$$

then for $y_m \in Y, u_m \in U$, $x, t,\ t_2 \in I$
$\max\left\{\left\|k\left(x,t_1,y_m(t_1),y'_m(t_1),u_m(t_1)\right) - k\left(x,t,y_m(t),y'_m(t),u_m(t)\right)\right\|\right\} = M_{k_m}$ for sufficiently small ε and the proof is complete .

Errors in Objective Functional

Now for the purpose of error estimation in equation (1.2) consider the following assumptions:

i. let

$$\left\{\max\left\|\frac{\partial G(t,y,u)}{\partial t}\right\| ,y \in Y,u \in U,t \in I \right\} = G_\tau. \qquad (4.14)$$

and let

$$\max \quad \{\|G(t_1,y_m,u_m) - G(t_2,y_m,u_m)\| ,y_m \in Y,u_m \in U,t_1,\ t_2 \in I\} = M_{G_m}. \quad (4.15)$$

ii. the function G jointly Liptschitz with respect to y and u, with Liptschitz constant $L_{G_{y,u}}$,

$$\|G(t,y_1,u_1) - G(t,y_2,u_2)\| \leq L_{G_{y,u}}\{\|y_1 - y_2\| + \|u_1 - u_2\|\}, \qquad (4.16)$$

for all $y_1,\ y_2 \in Y, u_1,\ u_2 \in U$ and for all $t \in I$.

Theorem 2 *If the assumptions equations (4.14) – (5.16) are satisfied, for every $u(t) \in U$ the error estimation in (1.2) satisfies*
$\|J - J_m\| \leq \delta + \widetilde{G}$,
where
$\delta = M_{G_m}$, $\tilde{G} = \left(G_t + L_{G_{y,u}}\left(M_{y'} + L_u\right)\right)$.

Proof. Let $t_i = -cos\frac{i\pi}{N}$ $i = 0,1,2,N-1$, and it is obvious that $y_m(t_i) = y(-cos\frac{i\pi}{N})$ and $u_m(t_i) = u\left(-cos\frac{i\pi}{N}\right)$,

then

$$\begin{aligned}
\|J - J_m\| &= max_{\forall\ t\in I}|J(y(t),u(t)) - J_m(y(t),u(t))| \\
&\leq max_{\forall\ t\in I}\left(\int_{-1}^{1}\left\|G\big(t,y(t),u(t)\big) - G\big(t,y_m(t),u_m(t)\big)\right\|dt\right) \\
&\leq max_{\forall\ t\in I}\left(\int_{-1}^{1}\left\|G\big(t,y(t),u(t)\big) - G\big(t_i,y(t),u(t)\big)\right\|dt + \right. \\
&\int_{-1}^{1}\left\|G\big(t_i,y(t),u(t)\big) - G\big(t_i,y(t_i),u(t_i)\big)\right\|dt \\
&\left. + \int_{-1}^{1}\left\|G\big(t_i,y_m(t_i),u_m(t_i)\big) - G\big(t,y_m(t),u_m(t)\big)\right\|dt\right)
\end{aligned}$$

$$\leq TM_{G_m} + \left(G_t + L_{G_{y,u}}\left(M_{y'} + L_u\right)\right) \leq \delta + \tilde{G}. \quad (4.17)$$

For sufficiently small δ and the proof is complete.

5 Numerical Examples

Consider the following problems to show the effectiveness of our technique.

Example 1 Consider the following optimal control problem [1]

$$J = \frac{1}{2}\int_{-1}^{1}\left[\left(\frac{t+1}{2}\right)y(t) - u(t)\right]^2 dt,$$

governed by the Volterra integro differential equation:

$$2y'(t) = 1 - \frac{7}{12}\left(\frac{t+1}{2}\right)^4 + \int_{-1}^{t}\left[\left(\frac{x+1}{2}\right)^2\left(\frac{t+1}{2}\right) + \left(\frac{x+1}{2}\right)u(x)\right]y'(x)\,dx, \quad -1 \leq t \leq 1.$$

$$y(-1) = 0, y(1) = 1.$$

The exact optimal trajectory and control functions are $y(t) = \frac{t+1}{2}$ and $u(t) = \left(\frac{t+1}{2}\right)^2$ respectively. The optimal solution of the objective J^* and the maximum absolute errors $e_1 = \|y - y_m\|$ are presented in Table (1). Whereas, the comparison of the exact and approximate optimal control and trajectory may be seen in Fig. (1).

Table 1: The results of applying proposed method in Example 1

N=M	J^*	e_1
6	0.384399264D-02	0.1363D-01
8	0.122373752D-02	0.7713D-02
10	0.502645876D-03	0.4967D-02
12	0.242771958D-03	0.3456D-02
14	0.131162919D-03	0.2542D-02
16	0.769704276D-04	0.1948D-02

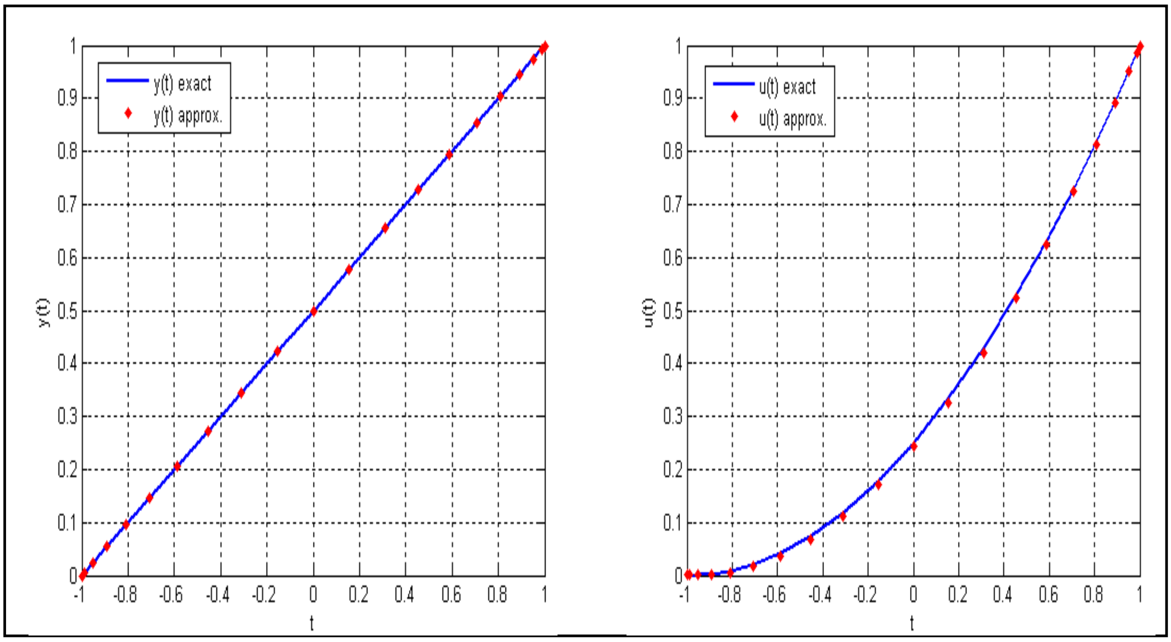

Fig.(1): The exact and approximate optimal control and state functions

Example 2 Consider the following optimal control problem

$$J = \frac{1}{2}\int_{-1}^{1}\left[\left(y(t) - sin\left(\frac{t+1}{2}\right)\right)^2 \left(u(t) - \frac{t+1}{2}\right)^2\right]dt$$

governed by the Volterra integro differential equation:

$$2y'(t) = -\frac{t+1}{2} sin\left(\frac{t+1}{2}\right) + cos\left(\frac{t+1}{2}\right) - \frac{3}{2}\left(\frac{t+1}{2}\right)^2 + u(t)\left(y(t) + \frac{t+1}{2}\right)$$

$$+\frac{1}{2}\int_{-1}^{t} u(x)dt\,, -1 \le t \le 1\,, y(-1) = 0.$$

The exact optimal trajectory and control functions are $y(t) = sin\left(\frac{t+1}{2}\right)$ and $u(t) = \frac{t+1}{2}$ respectively. The computed results of applying the proposed method in the previous section, and the maximum absolute errors $e_1 = \|y - y_m\|$ are listed in table (2). Also, one can observe the exact and approximate optimal trajectory and control functions in Fig. (2). This confirms that our method gives almost the same solution as the analytic method.

Table 2: The results of applying proposed method in Example 2

N=M	J^*	e_1
4	0.288811265D-11	0.2474442D-5
6	0.228977288D-17	0.1916419D-8
8	0.423786069D-20	0.2346834D-10
10	0.550233994D-20	0.6524747D-10
12	0.176784083D-18	0.1512400D-9

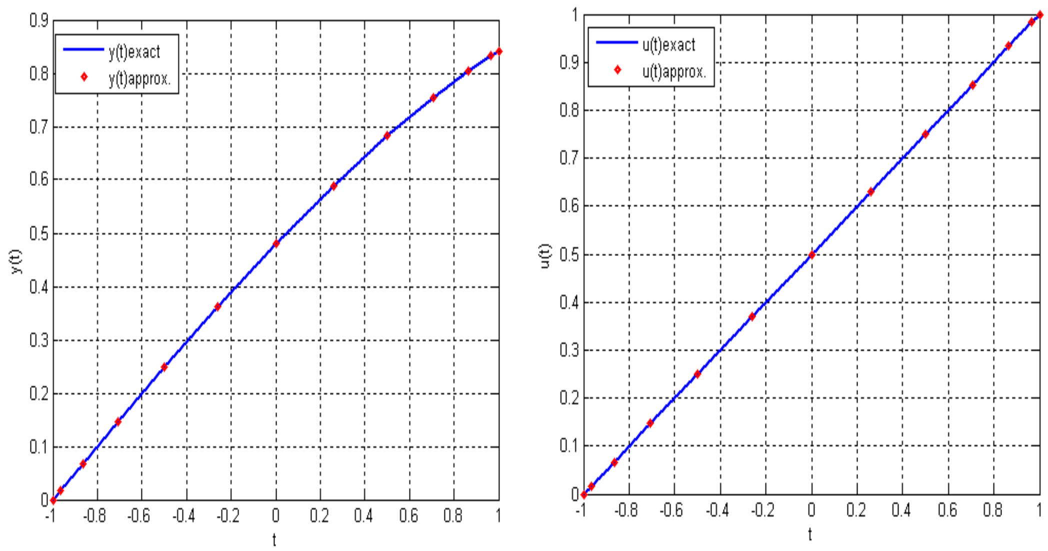

Fig. (2): The exact and approximate optimal control and state functions

6 Conclusion

In this paper, we have proposed a numerical scheme for finding approximate solution of optimal control problems governed by Volterra integro-differential equations. The efficiency of this technique has been shown in the numerical examples. The theorems for obtaining the error estimates for optimal control and the cost functional were conducted and the approximated solutions obtained by FORTRAN codes show the validity and efficiency of the proposed method.

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