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Relations and Applications on Proximity Structures

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Abstract

The purpose of this work is to construct a type of proximity relations in ideally proximity spaces based on an ideal $\mathcal{I}$ and three types of the proximity δ. Examples are given for the constructed relation. The constructed γ -relation is a pre- basic (resp., Ef, Lo) - proximity if δ is the basic (resp., Ef, Lo)- proximity. Properties and characterization of γ - relation are obtained. Connections between types of ideal and properties of γ -relation are obtained. The suggested relation can help in the fields of uncertainty processing in the context of rough set data analysis which are related to a lot of real life applications.

Keywords: *Topological ideal, proximity space, compactification, rough set theory.*

1 Introduction

The modern view for topological spaces is a pair of a nonempty universal set of objects with a structure based on smallest conditions suitable to define neighborhood and continuity. The concept of relation is a simple mathematical tool that can be easily used by non mathematicians like engineers, biologists. Our goal in this work is to use the concept of relations and ideals to enrich proximity spaces with new tools that help in applications.

The process of constructing topological structure on a universe is not only useful as a mathematical procedure but also it is considered as a modeling process for mathematizing quantitative and qualitative data [13-16]. For example, the information table 1 is about six patients and their corresponding symptoms and noticed disease.

Table 1: An information system about six patients and their conditional symptoms C and decision disease D.

	conditional symptoms C				D
	Temperature	Headache	Cough	Muscle-Pain	Flu
P_1	high	no	severe	yes	yes
P_2	high	yes	severe	no	yes
P_3	very high	yes	severe	yes	yes
P_4	normal	no	mild	yes	no
P_5	high	yes	mild	no	no
P_6	very high	no	severe	yes	yes

We can construct a topology τ on the set $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ using the set of symptoms {T, H, C, MP} to measure the accuracy of decision (Flu).

To the best of our knowledge, proximity structures have not been used in the field of data analysis via rough set theory. Our approach initiate examples for using proximity structures and relation in the context of rough set data analysis RSDA .

2 Preliminaries

We start by recalling that a sub-collection $\mathcal{I}$ of the power $\mathcal{P}(X)$ of a set X is called an ideal [9] on X if

(i) $A\in\mathcal{I}$ and $B\subset A \Longrightarrow B\in\mathcal{I}$ and

(ii) $A\in\mathcal{I}$ and $B\in\mathcal{I} \Longrightarrow A\cup B\in\mathcal{I}$.

Definition 2.1. An ideal $\mathcal{I}$ is said to be:

(a) [9] Free ideal, if $\cup\{E\subset X \mid E\in\mathcal{I}\}=X$. Equivalently, every point in X is in $\mathcal{I}$.

(b) [8, 9] Ideal of finite subsets, if $\mathcal{I}= \mathcal{I}_f = \{E\subset X \mid E \text{ finite}\}$.

(c) [9, 11] Regular ideal w.r.t. any proximity δ, if for every B$\in\mathcal{I}$ $\exists$ E$\in\mathcal{I}$ s.t. B$\overline{\delta}$(X-E) where ($\overline{\delta}$ means not - δ).

Remark 2.2. The ideal $\mathcal{I}_f$ of finite subsets is a free ideal and every free ideal contains an ideal $\mathcal{I}_f$ of finite subsets.

Proposition 2.3. **[5, 8]** (i) The intersection of two ideals on a non-empty

set X is an ideal, but the union of two ideals is not an ideal in general.

(ii) The sum $\mathcal{I}\vee\mathcal{J}$ of two ideals $\mathcal{I}$ and $\mathcal{J}$ on a non-empty set X is the ideal $\{$E$\cup$H $\mid$ E$\in\mathcal{I}$ and H$\in\mathcal{J}\}$.

(iii) The restriction of an ideal $\mathcal{I}$ on a set X to a subset A$\subset$X, denoted by $\mathcal{I}|_A$, is an ideal defined by $\mathcal{I}|_A$=$\{$B$\in\mathcal{I}$ $\mid$ B$\subset$A$\}$=$\{$B$\cap$A $\mid$ B$\in\mathcal{I}\}$, moreover $\mathcal{I}|_A\subset\mathcal{I}$.

Definition 2.4. A binary relation δ on $\mathcal{P}$(X) is said to be a pre-basic proximity, if δ satisfies the following axioms:

(i) AδB $\Longrightarrow$ BδA.

(ii) (A$\cup$B)δC$\Longleftrightarrow$AδC or BδC.

(iii) AδB$\Longrightarrow$A$\neq\emptyset$ and B$\neq\emptyset$.

A pre-basic proximity δ on $\mathcal{P}$(X) is said to be basic [17], if it satisfies the following condition:

(iv) A$\cap$B$\neq\emptyset$ $\Longrightarrow$ AδB.

A relation δ on $\mathcal{P}$(X) is said to be an Ef [4, 11]-(resp., a pre-Ef) proximity, if δ is a basic (resp., pre-basic) proximity with additional condition:

(v) A$\overline{\delta}$B$\Longrightarrow$ $\exists$E$\subset$ X s.t. A $\overline{\delta}$ E and (X-E)$\overline{\delta}$ B.

A basic (resp., pre-basic) proximity δ on $\mathcal{P}$(X) is said to be a Lo [10, 11]-(resp., pre-Lo) proximity, if δ satisfies the following condition:

(vi) AδB and $\{$b$\}\delta$C $\forall$ b$\in$B implies AδC.

A relation δ is said to be separated [12], if it satisfies:

(vii) xδy$\Longrightarrow$x=y.

Leader [10] has introduced the concept of local proximity space.

Definition 2.5. A basic proximity δ on $\mathcal{P}$(X) is called a local proximity w.r.t. an ideal $\mathcal{I}$ on X, if it satisfies the following conditions:

(i) If AδB, then Aδ(B$\cap$C) for some C$\in\mathcal{I}$.

(ii) Given A and B such that B$\in\mathcal{I}$, if A$\overline{\delta}$B then $\exists$E$\in\mathcal{I}$ s.t. A$\overline{\delta}$E and (X-E)$\overline{\delta}$B.

In the following example, the Rough sets technique is applied on the information system of students enrollment qualification.

Example 2.6. Let X={S_1, S_2, S_3, S_4, S_5, S_6} be a universe, using the set of qualification (finite set of attributes) {IT, Programming, English} to measure the accuracy of decision, the domain of attribute IT ={medium, good, very good}, the domain of attribute Programming ={medium, good} and the domain of attribute English ={medium, good, very good}, $\Re$ = {(S_1, S_1), (S_2, S_2), (S_3, S_3), (S_4, S_4), (S_5, S_5), (S_6,S_6), (S_4, S_6), (S_6,S_4)} is an equivalence relation on X where [S_1]={S_1}, [S_2]={S_2}, [S_3]={S_3}, [S_4]=[S_6]={S_4, S_6}, [S_5]={S_5} are equivalence classes. The upper approximation of A$\subseteq$X is $\overline{R}$A=$\cup${[x] | [x]$\cap$A$\neq\phi$}. Let A={S_1, S_4}, B={S_5, S_6}, then $\overline{R}$A={[S_1], [S_4]} and $\overline{R}$B={[S_4], [S_5]}. If AδB$\Leftrightarrow$ $\overline{R}$A$\cap\overline{R}$B$\neq\phi$, then δ is Ef-proximity on X. Also, it is possible to construct types of ideals in information system.

Table 2: An information system $\mathcal{IS}$ of students enrollment qualification.

	IT	Programming	English	Decision
S_1	good	good	good	yes
S_2	medium	good	good	no
S_3	very good	good	good	yes
S_4	medium	medium	good	no
S_5	good	medium	very good	yes
S_6	medium	medium	good	no

Definition 2.7.[1] A relation $\delta\subseteq$X$\times\mathcal{P}$(X) is said to be a $\mathcal{K}$-proximity on a given set X if it satisfies the following conditions for every subsets A and B of X and x$\in$X:

(i) xδA$\cup$B$\Longleftrightarrow$xδA or xδ B.

(ii) x$\overline{\delta}\emptyset$ $\forall$x$\in$X.

(iii) x∈A $\Longrightarrow$ xδA.

(iv) x$\bar{\delta}$A $\Longrightarrow$ ∃E⊆X s.t. x$\bar{\delta}$E and y$\bar{\delta}$A ∀y∈(X-E).

Table 3: A simple decision system of age and LEMS and the corresponding decision attribute (Walking ability).

	conditional C		D
	Age	LEMS	Walk
O_1	16-30	1-25	yes
O_2	31-45	26-49	yes
O_3	16-30	1-25	no
O_4	31-45	26-49	yes

Example 2.8.[18] Let us consider a decision system shown in Table 3. The universe set X consists of four objects X={O_1, O_2, O_3, O_4}, and the set of attributes includes two attributes conditions age and LEMS (Lower Extremity Motor Score). The domain of attribute age={16-30, 31-45}, the domain of attribute LEMS={1-25, 26-49} and the domain of decision attribute (Walk) with two values {Yes, no}. Let $\Re$={(O_1, O_1), (O_2, O_2), (O_3, O_3), (O_4, O_4), (O_1, O_3), (O_2, O_4), (O_3, O_1), (O_4, O_2)} is an equivalence relation on X where [O_1]=[O_3]={O_1, O_3}, [O_2]=[O_4]={O_2, O_4} are equivalence classes. Let A={O_1, O_3, O_4}⊆ X, it is clear that x δ A ⇔ [x] ∩ A $\neq \phi$, then δ is $\mathcal{K}$-proximity on X.

Definition 2.9.(i)[10, 11, 12] A subset B of local (resp., a Lo- or an Ef-) proximity space (X, δ) is a δ-neighborhood of A (in symbols A≺B), if A$\bar{\delta}$(X-B). The family$\mathcal{N}$(δ, A)={B⊂X | A≺B} is a δ-nbd. system of a subset A of X.

(ii)[1] A subset B of $\mathcal{K}$-proximity space (X, δ) is a δ-neighborhood of x (in symbols x≺B), if x$\bar{\delta}$(X-B). The family $\mathcal{N}$(δ, {x})={B⊂X | x≺B} is a δ-nbd. system of x.

(iii) [1, 10, 11, 12] The intersection of two δ-nbds is a δ-nbd.

(iv)[1, 10, 11, 12] For a local (resp., a Lo, an Ef or $\mathcal{K}$)-proximity relation δ; $\mathcal{N}$(δ, x)⊂$\mathcal{N}$(τ_δ, x) where $\mathcal{N}$(τ_δ, x) is the nbd. system of x w.r.t. τ_δ.

For a local (resp., a Lo, an Ef or $\mathcal{K}$)-proximity space (X, δ) the operator Cl_δ(A)={x∈X | xδA} ∀ A⊂X is a Kuratowski closure operator which produce the topology τ_δ generated by δ. Note that τ_δ is a completely regular, if δ is an Ef-proximity or a local proximity. Also, if (X, τ) is a completely regular space, then there exists proximity δ on X which compatible with τ i.e., τ_δ=τ.

Every Ef-proximity space is a Lo-proximity space. Every Lo-proximity space induces a topology τ_δ which is R_0; Conversely, every R_0 space (X, τ) has a compatible Lo-proximity.

Definition 2.10. [12] Let (X, δ) be any proximity space and A⊂X then:

(i) A is δ-open, if (x$\overline{\delta}$$\mathrm{A}^c$ ∀ x∈A).

(ii) A is δ-closed, if (xδ A ⇒ x∈A).

Lemma 2.11.[12] For subsets A and B of any proximity space (X, δ), if AδB, A⊂C and B⊂D, then CδD.

Proposition 2.12.(i)[9, 10, 11] Let (X, δ) be a local (resp., an Ef, a Lo)

- proximity space then for subsets A and B of X, AδB iff Cl_δA$\delta$$\mathrm{Cl}_\delta$B, where the closure is taken w.r.t. τ_δ.

(ii)[1] Let (X, δ) be $\mathcal{K}$-proximity space, then for subsets B of X and x∈X, {x}δB $\Longrightarrow$ Cl_δ{x}$\delta$$\mathrm{Cl}_\delta$B, where the closure is taken w.r.t. τ_δ.

Proposition 2.13.[12] Let (X, δ, $\mathcal{I}$) be a local proximity space, then:

(i) If X∈$\mathcal{I}$, then (X, δ) is an Ef-proximity space.

(ii) Every singleton is in $\mathcal{I}$.

(iii) Given A∈$\mathcal{I}$ there exists B∈$\mathcal{I}$ such that A$\overline{\delta}$$\mathrm{B}^c$.

If (X, δ) is an Ef-proximity space and $\mathcal{I}$ is the power set of X, then (X, δ, $\mathcal{I}$) is a local proximity space.

Definition 2.14.(i)[11, 12] A function f from an Ef (resp., a Lo)-proximity space (X, δ_1) to an Ef (resp., a Lo)- proximity space (Y, δ_2) is called a proximally continuous, if Aδ_1B implies f(A)$\delta_2$$f$(B). Equivalently, f is a proximally continuous, if C$\overline{\delta_2}$D implies f^{-1}(C)$\overline{\delta_1}$$f^{-1}$(D). If X =Y, then we say that, δ_1 is finer than δ_2 (in symbols $\delta_2 < \delta_1$) that is Aδ_1B implies Aδ_2B.

(ii)[11, 12] A bijective function f from an Ef (resp., a Lo)-proximity space (X, δ_1) to an Ef (resp., a Lo)- proximity space (Y, δ_2) is called a proximally isomorphic (or δ-homeomorphic), if both f and f^{-1} are proximally continuous functions.

(iii)[9] A function f from a local proximity space (X, δ_1, $\mathcal{I}$) to a local proximity space (Y, δ_2, $\mathcal{J}$) is said to be a proximally continuous, if Aδ_1B implies f(A)$\delta_2 f$(B) and A$\in\mathcal{I}$ implies f(A)$\in\mathcal{J}$. If X=Y, then δ_1 is finer than δ_2.

(iv)[1] A function f from an $\mathcal{K}$-proximity space (X, δ_1) to an $\mathcal{K}$-proximity space (Y, δ_2) is called a proximally continuous, if xδ_1B implies f(x)$\delta_2 f$(B). Equivalently, f is a proximally continuous if x$\overline{\delta_1} f^{-1}$(B) implies y$\overline{\delta_2}$B for every x$\in f^{-1}$(y). If X=Y and the identity map is a proximally continuous we say that, δ_1 is finer than δ_2 (in symbols $\delta_2 < \delta_1$), if xδ_1B implies xδ_2B .

Definition 2.15.[12] If (X, δ) is any proximity space and Y$\subset$X. For subsets A and B of Y we define Aδ_YB iff AδB.

Definition 2.16.[12] Given a function f:X$\longrightarrow$(Y, δ_2), the coarsest proximity δ_0 which may be assigned to X in order that f proximally continuous is defined by: A$\overline{\delta_0}$B iff $\exists$C$\subset$Y s.t. f(A)$\overline{\delta_2}$(Y-C) and f^{-1}(C)$\subset$(X-B).

An important concept to construct the compactification of Ef-proximity and local proximity is a cluster and for a Lo-proximity a bunch is defined.

Definition 2.17.[12, 19] A collection σ of subsets of a basic (resp., a local, an Ef-, a Lo-) proximity space (X, δ) is called a δ-clan if the following condition is satisfied: A$\in\sigma$ and B$\in\sigma$ $\Longrightarrow$AδB.

A δ-clan σ of subsets of a local (resp., an Ef-, a Lo-) proximity space (X, δ) is called a δ-cluster, if the following conditions are satisfied:

(i) AδB $\forall$B$\in\sigma$ $\Longrightarrow$A$\in\sigma$.

(ii) A$\cup$B$\in\sigma$ $\Longrightarrow$A$\in\sigma$ or B$\in\sigma$.

A δ-clan σ of subsets of a Lo-proximity space (X, δ) is called a δ-bunch, if the following conditions are satisfied:

(i) A$\in\sigma$$\Longleftrightarrow$$Cl(A)$$\in\sigma$.

(ii) A$\cup$B$\in\sigma$$\Longrightarrow$ A$\in\sigma$ or B$\in\sigma$.

Remark 2.18. In a Lo-proximity every δ-cluster is a δ-bunch.

Remark 2.19. For each x$\in$X, the collection σ_x=$\{$A $\mid$ Aδx$\}$ is a cluster which is called a point cluster. Also, the family of all δ-clusters w.r.t. Ef-proximity is denoted by $\mathcal{H}$.

Definition 2.20.[11, 12] Let $\mathcal{P}\subset\mathcal{H}$. A subset A of X is called absorbs $\mathcal{P}$, if A$\in\sigma$ for every $\sigma\in\mathcal{P}$.

Lemma 2.21.[11, 12] The binary relation $\widetilde{\delta}$ on the power set of $\mathcal{H}$ defined by $\mathcal{P}\widetilde{\delta}\mathcal{Q}$ iff A absorbs $\mathcal{P}$ and B absorbs $\mathcal{Q}$ implies AδB, is a separated Ef-proximity on $\mathcal{H}$.

Smirnov [17] proved the following result: (X, δ) is Ef-proximity space iff there exists a compact Hausdorff space $\mathcal{H}$ in which X can be topologically embedded so that: AδB in X iff Cl(A)$\cap$Cl(B)$\neq\emptyset$.

Theorem 2.22.[11, 12] Every proximally mapping f of (X, δ_1) onto (Y, δ_2) has a unique extension to a continuous mapping $\overline{f}$ which maps the compactification of X onto the compactification of Y.

Definition 2.23.[10, 11] A subset X of a topological space (Y, τ) is regularly dense in Y, if given U$\in\tau$ and p$\in$U, then $\exists$A$\subseteq$X such that p$\in$Cl(A)$\subset$U.

A characterization of Lo-proximity spaces using clusters given by the following theorem.

Theorem 2.24.[10, 11] Let δ be binary relation on given set X, then the following are equivalent:

(I) There exists T_1-topological space Y and a mapping f of X into Y such that f(X) is regularly dense in Y and AδB in X iff Clf(A)$\cap$Clf(B)$\neq\emptyset$ in Y.

(II) δ is a separated Lo-proximity satisfying the additional axiom: given AδB in X there exists a cluster σ to which both A and B belong.

Theorem 2.25.[3] Let (X, δ) be a separated Ef (resp., Lo) proximity and Σ be the family of all clusters (resp., bunches), then the function Θ: (X, δ) $\longrightarrow$ (Σ, $\widetilde{\delta}$) is a proximally isomorphic to Θ(X) with the subspace proximity induced by $\widetilde{\delta}$ and Θ(X) is a dense in Σ.

Definition 2.26.[9, 11] A topological space (X, τ) is said to be locally compact, if $\forall$x$\in$X $\exists$ nbd. U of x s.t. Cl(U) is compact subspace.

Theorem 2.27. [9] Given a local proximity (X, δ, $\mathcal{I}$) there exists a locally compact, Hausdorff space L and a mapping π of X into L satisfying the following three conditions:

(i) AδB iff Cl(π(A))$\cap$ Cl(π(B))$\neq\phi$in L.

(ii) A$\in\mathcal{I}$ iff Cl(π(A)) is compact in L.

(iii) Cl(π(X))=L.

Such a local compactification is unique. Conversely, if π is a mapping on a set X into a locally compact, Hausdorff space L and

(i) AδB iff Cl(πA))$\cap$Cl(π(B))$\neq\emptyset$ in L .

(ii) A$\in\mathcal{I}$ iff Cl(π(A)) is compact in L.

Then (X, δ, $\mathcal{I}$) is a local proximity space.

3 γ-Proximity Structures

This section is devoted to define a new finer proximity relation γ from a given basic (resp., Ef, Lo)-proximity relation δ and an ideal $\mathcal{I}$ on X and study some of its properties.

Definition 3.1. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space and $\mathcal{I}\neq\{\emptyset\}$. Define the relation γ on P(X) as follows:

AγB iff (A$\cap$E)δ(B$\cap$E) for some E$\in\mathcal{I}$.

Example 3.2.(1) Let δ be the usual metric proximity on the positive real

numbers and let an ideal $\mathcal{I}$ consists of all complements of members of filter generated by filter base consisting of all right rays, then the relation γ is local proximity space.

(2)Let δ be the discrete proximity on X$\neq\phi$ and let an ideal $\mathcal{I}$=$\mathcal{P}$(X)then the relation γ is pre Ef-proximity space.

Example 3.3. Let X=$\{1, 2, 3, 4\}$, A=$\{1, 3, 4\}\subseteq$ X, $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (2, 4), (3, 1), (4, 2)\}$ is an equivalence relation on X where [1]=[3]=$\{1, 3\}$, [2]=[4]=$\{2, 4\}$ are equivalence classes, $\mathcal{I}=\{\{1, 2, 4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1\}, \{2\}, \{4\}, \phi\}$. It is clear that x$\delta$A$\Leftrightarrow$[x]$\cap$ A$\neq\phi$, xγA$\Leftrightarrow$ $\exists$E$\in\mathcal{I}$ such that xδ(E$\cap$A)$\Leftrightarrow$ $\exists$E$\in\mathcal{I}$ such that [x]$\cap$E$\cap$A$\neq\phi$ where E=$\{1, 2, 4\}$, E$\cap$A=$\{1, 4\}$, xγA$\Leftrightarrow$[x]$\cap\{1, 4\}\neq \phi$, then γ and δ are $\mathcal{K}$-proximity on X.

Example 3.4. Let X=$\{1, 2, 3, 4\}$, A= $\{1, 3, 4\}\subseteq$ X, $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (2, 4), (3, 1), (4, 2)\}$ is an equivalence relation on X where [1]=[3]=$\{1, 3\}$, [2]=[4]=$\{2, 4\}$ are equivalence classes, $\mathcal{I}=\{\{1, 4\}, \{1\}, \{4\}, \phi\}$. It is clear that x$\delta$A$\Leftrightarrow$[x]$\cap$ A$\neq \phi$, xγA$\Leftrightarrow$ $\exists$E$\in \mathcal{I}$ such that xδ(E$\cap$A)$\Leftrightarrow\exists$E$\in\mathcal{I}$ such that [x]$\cap$ E$\cap$A$\neq\phi$ where E=$\{4\}$, E$\cap$A=$\{4\}$, 1$\bar{\gamma}$A$\Leftrightarrow$[1]$\cap\{4\}=\phi$, then δ is $\mathcal{K}$-proximity on X but γ is not $\mathcal{K}$-proximity on X.

Theorem 3.5. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space. Then γ is a pre-basic (resp., pre-Ef or pre-Lo) proximity, if δ is a basic (resp., an Ef- or a Lo-) proximity.

Proof. Let (X, δ) be a basic proximity space and $\mathcal{I}$ be an ideal on X. Let A and B be subsets of X.

(i) AγB$\Longrightarrow$BγA.

(ii) Aγ(B$\cup$C)$\Longleftrightarrow$ (A$\cap$E)δ((B$\cup$C)$\cap$E) for some E$\in\mathcal{I}$.
$\Longleftrightarrow$(A$\cap$E)δ(B$\cap$E) for some E$\in\mathcal{I}$ or (A$\cap$E)δ(C$\cap$E) for some E$\in\mathcal{I}$.
$\Longleftrightarrow$ AγB or AγC.

(iii) Let A=$\emptyset$ and B$\subset$X, then $\emptyset\bar{\delta}$(B$\cap$E) $\forall$E$\in\mathcal{I}$ and so $\emptyset\bar{\gamma}$B.

Hence γ is a pre-basic proximity on X. By the same manner we shall prove γ is a pre-Ef (resp., pre-Lo)-proximity on X, if δ is an Ef (resp., a Lo)-proximity on X.

Corollary 3.6. Let(X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space. Then γ is finer thanδ.

Proof. Straightforward.

Remark 3.7. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-K proximity space. The proximity relation γ may be defined as follows:

xγB iff xδ(B$\cap$E) for some E$\in\mathcal{I}$.

Remark 3.8. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space. Let Cl$_\delta$A (resp., Cl$_\gamma$A) be the closure of a subset A of X w.r.t. δ (resp., γ), then it is clear that

$\mathrm{Cl}_\gamma A \subset \mathrm{Cl}_\delta A$.

Corollary 3.9. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space. Then:

(i) Every δ-nbd. is a γ nbd.

(ii) Every δ-closed set is a γ-closed.

Proof. Straightforward.

Proposition 3.10. Let (X, δ) be a basic (resp., an Ef-, a Lo-) proximity space with ideals $\mathcal{I}$ and $\mathcal{J}$, then:

(i) $\gamma(\mathcal{I})\subset\gamma(\mathcal{J})$, if $\mathcal{I}\subset\mathcal{J}$.

(ii) $\gamma(\mathcal{I}\cap\mathcal{J})\subset\gamma(\mathcal{I})\cap\gamma(\mathcal{J}) \subset\gamma(\mathcal{I}\vee\mathcal{J})$.

(iii) $\gamma(\mathcal{I})\cup\gamma(\mathcal{J})\subset\gamma(\mathcal{I}\vee\mathcal{J})$.

Proof. As a sample, we give a proof when δ is a basic proximity. Other cases have similar proofs.

(i) Let A and B be subsets of X and A $\gamma(\mathcal{I})$B, then $\exists$E$\in\mathcal{I}$ s.t. (A$\cap$E)δ(B$\cap$E). By hypothesis, $\exists$E$\in\mathcal{J}$ s.t. (A$\cap$E)δ(B$\cap$E) and so A$\gamma(\mathcal{J})$B.

(ii) and (iii) follows directly from (i).

Theorem 3.11. Let δ_1 and δ_2 be two proximities on a non-empty set X and for any ideal $\mathcal{I}$ on X, if $\delta_1<\delta_2$, then $\gamma_1<\gamma_2$.

Proof. Let A and B be subsets of X and $\mathcal{I}$ be any ideal on X and let Aγ_2B, then $\exists$E$\in\mathcal{I}$ s.t. (A$\cap$E)δ_2(B$\cap$E). By hypothesis, $\exists$ E$\in\mathcal{I}$ s.t. (A$\cap$E)δ_1(B$\cap$E) and so Aγ_1B.

Corollary 3.12. Let δ_1 and δ_2 be two proximities on a non-empty set X and $\mathcal{I}$ be an ideal on X. Then:

(i) $\gamma(\delta_1\cap\delta_2, \mathcal{I})=\gamma(\delta_1, \mathcal{I})\cap\gamma(\delta_2, \mathcal{I})$.

(ii) $\gamma(\delta_1\cup\delta_2, \mathcal{I})=\gamma(\delta_1, \mathcal{I})\cup\gamma(\delta_2, \mathcal{I})$.

Proof. Straightforward.

For a subset Y of any $\mathcal{I}$- proximity space (X, δ, $\mathcal{I}$), the space (Y, $\delta|_Y$, $\mathcal{I}|_Y$) is the induced (restricted) $\mathcal{I}|_Y$- proximity space on Y with the restricted ideal

$\mathcal{I}|_Y$ and γ is a generated proximity from δ, $\mathcal{I}$. The next theorem investigate the relation between restricted proximity space $\gamma|_Y$ on Y and the proximity γ_1 generated by $\delta|_Y$ and $\mathcal{I}|_Y$ on Y .

Theorem 3.13. Let (X, δ, $\mathcal{I}$) be an $\mathcal{I}$-proximity space. Let Y$\subset$X and γ is a generated proximity from δ, $\mathcal{I}$, then $\gamma|_Y$=γ_1.

Proof. Let A and B be subsets of Y

$A\gamma_1 B \Longrightarrow \exists E\in\mathcal{I}|_Y\subset\mathcal{I}$ s.t. $(A\cap E)\delta|_Y(B\cap E)$.

$\Longrightarrow \exists E\in\mathcal{I}$ s.t. $(A\cap E)\delta(B\cap E)$.

$\Longrightarrow A\gamma B \Longrightarrow A\gamma|_Y B$. Hence $\gamma|_Y< \gamma_1$.

$A\gamma|_Y B\Longleftrightarrow A\gamma B \Longleftrightarrow\exists E\in\mathcal{I}$ s.t. $(A\cap E)\delta(B\cap E)$.

$\Longrightarrow \exists E\in\mathcal{I}$ s.t. $(A\cap E\cap Y)\delta|_Y(B\cap E\cap Y)$.

$\Longrightarrow \exists$ $K=(E\cap Y)\in\mathcal{I}|_Y$ s.t. $(A\cap K)\delta|_Y(B\cap K)$.

$\Longrightarrow A\gamma_1 B$. Hence $\gamma_1 <\gamma|_Y$ and so $\gamma|_Y=\gamma_1$.

The following theorems are obvious and the proofs are omitted.

Theorem 3.14. Let (X, δ) be a basic (resp., an Ef a Lo)-proximity space with ideals $\mathcal{I}$ and $\mathcal{J}$ on X, then:

(i) Every γ-clan is a δ-clan.

(ii) Every γ_2-clan is a γ_1-clan ,if δ_2 is finer than δ_1.

(iii) Every $\gamma(\mathcal{I})$-clan is a $\gamma(\mathcal{J})$-clan, if $\mathcal{I}\subset\mathcal{J}$.

Theorem 3.15. Let (X, δ) be an Ef (resp., Lo)-proximity space with ideal $\mathcal{I}$ on X, then:

For every cluster σ_γ of (X, γ) there exists a clusterσ_δ of (X, δ) such that:
$A\in\sigma_\delta\Longleftrightarrow A\delta B$ $\forall B\in\sigma_\gamma$.

Theorem 3.16. Let (X, δ) be a Lo-proximity space, then:

For every bunch σ_γ of (X, γ) there exists a bunch σ_δ of (X, δ) such that:
$A\in\sigma_\delta\Longleftrightarrow Cl_\delta A\in\sigma_\gamma$.

In the following, we determine the type of γ for special types of $\mathcal{I}$ and the following proofs are obvious.

Theorem 3.17. Given (X, δ, $\mathcal{I}$) an Ef (resp., Lo)-proximity space with an ideal $\mathcal{I}$. Then:

(i) γ is an Ef-proximity, if $\mathcal{I}$ is a free ideal, $\mathcal{I}_f$ or P(X).

(ii) γ is a Lo-proximity, if $\mathcal{I}$ is a free ideal, finite ideal or P(X).

(iii)γ is a local proximity, if $\mathcal{I}$ is a free regular ideal.

Theorem 3.18. Given (X, δ, $\mathcal{I}$) a $\mathcal{K}$-proximity space with an ideal $\mathcal{I}$. Then γ is a $\mathcal{K}$-proximity, if $\mathcal{I}$ is a free ideal, $\mathcal{I}_f$ or P(X).

4 Proximity Structures Generated by Special Cases of Ideal

In the sequel we use the following notations. Let X be a non-empty set and (Y, δ, $\mathcal{I}$) be a proximity space with an ideal $\mathcal{I}$. Let f:X$\longrightarrow$Y be an injective function. Let δ_0 be the coarsest proximity on X making f proximally continuous. Let γ_0 be the proximity on X generated by δ_0 and $f^{-1}(\mathcal{I})$. Let γ be the proximity on Y generated by δ and $\mathcal{I}$. Let γ_1 be the proximity on Y generated by γ.

Lemma 4.1. If f:(X, δ_0, $f^{-1}(\mathcal{I})$)$\longrightarrow$(Y, δ, $\mathcal{I}$) is proximally continuous, then the function f:(X, γ_0)$\longrightarrow$(Y, γ) is also proximally continuous. Furthermore, if (Y, δ, $\mathcal{I}$) is an Ef-proximity with a free ideal $\mathcal{I}$, then f:(X, γ_1) $\longrightarrow$(Y, γ) is proximally continuous.

Proof. Firstly, let (Y, δ, $\mathcal{I}$) be any proximity space with an ideal $\mathcal{I}$ and let A and B be subsets of X. Suppose Aγ_0B. Then $\exists$ E$\in$ $f^{-1}(\mathcal{I})$ s.t. A$\cap$Eδ_0B$\cap$E. Since f is one-to-one proximally continuous function, then f(A)$\cap f$(E)δf(B)$\cap f$(E) and so f(A)γf(B). Hence f:(X, γ_0)$\longrightarrow$(Y, γ) is proximally continuous.
Secondly, let (Y, δ, $\mathcal{I}$) be an Ef-proximity with the free ideal and let f(A)$\overline{\gamma}f$(B). Then $\exists$C$\subset$Y s.t. f(A)$\overline{\gamma}$(Y-C) and C$\overline{\gamma}f$(B). Hence $\exists$C$\subset$Y s.t. f(A)$\overline{\gamma}$(Y-C) and C$\cap f$(B)=$\emptyset$. Consequently, $\exists$C$\subset$Y s.t. f(A)$\overline{\gamma}$(Y-C) and f^{-1}(C)$\subset$X-$f^{-1}f$(B)$\subset$(X-B). Hence A$\overline{\gamma_1}$B. It follows that f:(X, γ_1)$\longrightarrow$(Y, γ) is proximally continuous.

Theorem 4.2. $\gamma_1 < \gamma_0$ and the equality hold, if $\mathcal{I}$ is a free ideal .

Proof. Let (Y, δ, $\mathcal{I}$) be any proximity with an ideal $\mathcal{I}$. For subsets A and B of X.

A$\overline{\gamma_1}$B$\Longrightarrow$ $\exists$C$\subset$Y s.t. f(A)$\overline{\gamma}$(Y-C) and f^{-1}(C)$\subset$(X-B).

$\Longrightarrow$ $\exists$C$\subset$Y s.t. $\forall$E$\in\mathcal{I}$; f(A)$\cap$E$\overline{\delta}$(Y-C)$\cap$E and f^{-1}(C)$\subset$(X-B).

$\Longrightarrow$ $\exists$C$\subset$Y s.t. $\forall$E$\in\mathcal{I}$; f(A$\cap f^{-1}$ E))$\overline{\delta}$Y-(C$\cup$E^c) and f^{-1}(C$\cup$E^c)$\subset$X-(B$\cap f^{-1}$(E)).

$\Longrightarrow$ $\forall f^{-1}$(E)$\in f^{-1}(\mathcal{I})$; A$\cap f^{-1}$(E)$\overline{\delta_0}$B$\cap f^{-1}$(E)$\Longrightarrow$A$\overline{\gamma_0}$B i.e. $\gamma_1 < \gamma_0$.

Let $\mathcal{I}$ is a free ideal, then A$\overline{\gamma_0}$B$\Longrightarrow$ $\forall$E$\in$ $f^{-1}(\mathcal{I})$; (A$\cap$E)$\overline{\delta_0}$(B$\cap$E).

$\Longrightarrow$ $\forall$E$\in f^{-1}$ $\mathcal{I}$); $\exists$C$\subset$Y s.t. f(A$\cap$E) $\overline{\delta}$(Y-C) and f^{-1}(C)$\subset$ X-(E$\cap$B).

$\Longrightarrow$ $\forall$E$\in f^{-1}(\mathcal{I})$; f(A)$\cap$ f(E)$\overline{\delta}$(Y-C)$\cap f$(E) and (B$\cap$E)$\subset f^{-1}$(Y-C).

$\Longrightarrow$ B$\subset$X-f^{-1}(C) and so f^{-1}(C)$\subset$(X-B)$\Longrightarrow$ A$\overline{\gamma_1}$B$\Longrightarrow$$\gamma_0 < \gamma_1$.

Proposition 4.3. Let (X, δ, $\mathcal{I}$) be a separated Ef-proximity space with an ideal $\mathcal{I}_f$ and γ be the separated Ef-proximity space generated by δ and $\mathcal{I}_f$ on X. Assume that Σ(resp., $\mathcal{H}$) be the set of clusters w.r.t. γ (resp., δ) on X. For $\sigma\in\Sigma$; $\phi(\sigma)$=$\{$A$\subset$X$|$ AδK $\forall$K$\in\sigma\}$, then ϕ is a map of Σ into $\mathcal{H}$ which is fixed on the set of cluster points. Furthermore $\sigma\subset\phi(\sigma)$, for every $\sigma\in\Sigma$.

Proof. Let $\sigma\in\Sigma$, we shall prove that $\sigma\subset\phi(\sigma)$. Suppose A$\subset$X and A$\notin$ $\phi(\sigma)$, then $\exists$K$\in$ σ s.t. A$\overline{\delta}$K. Since δ<γ, then $\exists$K$\in\sigma$ s.t. A$\overline{\gamma}$K and so A$\notin\sigma$. Hence $\sigma\subset\phi(\sigma)$. Now we shall prove $\phi(\sigma)$ is a cluster w.r.t. δ:

(i) Let A_1 and A_2 be subsets of X and $A_1\overline{\delta}A_2$,then $\exists$K$\subset$X s.t. $A_1\overline{\delta}$K and

(X-K)$\overline{\delta}A_2$. Since X$\in\sigma$, then $\forall$K$\subset$X , K$\in\sigma$ or (X-K)$\in\sigma$. Hence $\exists$K$\in$ σ s.t. $A_1\overline{\delta}$K or $\exists$(X-K)$\in\sigma$ s.t. (X-K)$\overline{\delta}A_2$ and so A_1 $\notin\phi(\sigma)$ or A_2 $\notin\phi(\sigma)$.

(ii) A$\notin\phi(\sigma)$, then $\exists$K$\in\sigma$ s.t. A$\overline{\delta}$K. Since $\sigma\subset\phi(\sigma)$, then $\exists$K$\in\phi(\sigma)$ s.t. A$\overline{\delta}$K.

(iii) Let (A$\cup$B)$\in\phi(\sigma)$, then (A$\cup$B)δK $\forall$K$\in\sigma$. Let K$\in\sigma$, then K=(K-H)$\cup$(K$\cap$H) $\forall$H$\subset$X and so (K-H)$\in\sigma$ or (K$\cap$H)$\in\sigma$ $\forall$H$\subset$X. Since (A$\cup$B)δK, then (A$\cup$B)δ(K-H) or (A$\cup$B)δ(K$\cap$H) $\forall$H$\subset$X. It follows that $\forall$H$\subset$X; Aδ(K-H) or Bδ(K-H) or Aδ(K$\cap$H) or Bδ(K$\cap$H). Hence $\forall$H$\subset$X; Aδ(K-H)$\cup$(K$\cap$H) or Bδ(K-H)$\cup$(K$\cap$H). Which implies that A$\in\phi(\sigma)$ or B$\in\phi(\sigma)$. Consequently, $\phi(\sigma)$ is a cluster w.r.t. δ.

Now we shall prove $\phi(\sigma_x)=\sigma_x$. It is clear that $\sigma_x \subset\phi(\sigma_x)$. Let $A\in\phi(\sigma_x)$, then $A\delta K\ \forall K\in\sigma_x$. Which implies that $A\gamma K\ \forall K\in\sigma_x$ and so $A\in\sigma_x$. Hence ϕ is one-to-one map on the set of cluster points.

Theorem 4.4. Let $(X, \delta, \mathcal{I}_f)$ be a separated Ef-proximity space with ideal $\mathcal{I}_f$ and γ be a separated Ef-proximity relation generated by δ and $\mathcal{I}_f$ on X. Then the function $\phi:(\Sigma,\widetilde{\gamma}) \longrightarrow(\mathcal{H}, \widetilde{\delta})$ given in the Proposition 4.3. is a proximally continuous mapping.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of Σ.

$\phi(\mathcal{A})\overline{\widetilde{\delta}}\ \phi(\mathcal{B})\Longrightarrow\exists A, B \subset X$ s.t. A absorbs $\phi(\mathcal{A})$, B absorbs $\phi(\mathcal{B})$ and $A\bar{\delta}B$.

$\Longrightarrow\exists A, B \subset X$ s.t. $A\in C\ \forall C\in\phi(\mathcal{A})$, $B\in C\forall C\in\phi(\mathcal{B})$ and $A\bar{\delta}B$.

Since $\phi(\mathcal{A})=\{\phi(\sigma)|\sigma \in\mathcal{A}\}$, utilizing Proposition 4.3., then $A\delta K\ \forall K\in\sigma$, $\forall\sigma\in\mathcal{A}$, $B\delta K\ \forall K\in\sigma$, $\forall\sigma\in\mathcal{B}$ and $A\bar{\delta}B$. Since $\delta < \gamma$, then $\exists A, B \subset X$ s.t. $(A\gamma K\ \forall K\in\sigma)\ \forall\sigma\in\mathcal{A}$, $(B\gamma K\ \forall K\in\sigma)\ \forall\sigma\in\mathcal{B}$ and $A\bar{\gamma}B$.

$\Longrightarrow \exists A, B \subset X$ with $A\in\sigma\ \forall\sigma\in\mathcal{A}$, $B\in\sigma\forall\sigma \in\mathcal{B}$ and $A\bar{\gamma}B$.

$\Longrightarrow\mathcal{A}\ \overline{\widetilde{\gamma}}\ \mathcal{B}\Longrightarrow \phi$ is a proximally continuous mapping.

The following proofs are obvious.

Proposition 4.5. Let $(X, \delta, \mathcal{I})$ be a separated Lo-proximity space with an ideal $\mathcal{I}_f$ and γ be the separated Lo-proximity space generated by δ and $\mathcal{I}_f$ on X. Assume that Σ (resp., $\mathcal{H}$) is the set of clusters w.r.t. γ (resp., δ) on X. For $\sigma\in\Sigma$; $\phi(\sigma)=\{A\subset X|A\delta K\ \forall K\in\sigma\}$, then ϕ is a map of Σ into $\mathcal{H}$ which is fixed on the set of cluster points. Furthermore $\sigma\subset\phi(\sigma)$, for every $\sigma\in\Sigma$.

Theorem 4.6. Let $(X, \delta, \mathcal{I}_f)$ be a separated Lo-proximity space with an ideal $\mathcal{I}_f$ and γ be the separated Lo-proximity generated by δ and $\mathcal{I}_f$ on X. Then the function $\phi:(\Sigma,\widetilde{\gamma})\longrightarrow(\mathcal{H}, \widetilde{\delta})$ given in the Proposition 4.3. is a proximally continuous mapping.

Proposition 4.7. Let $(X, \delta, \mathcal{I})$ be a separated local proximity space with a free regular ideal $\mathcal{I}$ and γ be the separated local proximity space generated by δ and $\mathcal{I}$ on X. Assume that Σ(resp., $\mathcal{H}$) is the set of γ (resp., δ)-clusters on X.For $\sigma\in\Sigma$; $\phi(\sigma)=\{A\subset X|A\delta K\ \forall K\in \sigma\}$, then ϕ is a map of Σ into $\mathcal{H}$ which is fixed on the set of cluster points. Furthermore $\sigma \subset \phi(\sigma)$, for every $\sigma\in\Sigma$.

Theorem 4.8. Let $(X, \delta, \mathcal{I})$ be a separated local-proximity space with a free regular ideal $\mathcal{I}$ and γ be a separated local-proximity relation generated by δ and $\mathcal{I}$ on X such that $\delta < \gamma$, then the function ϕ:$(\Sigma, \widetilde{\gamma}) \longrightarrow (\mathcal{H}, \widetilde{\delta})$ given in the Proposition 4.5. is a proximally continuous mapping.

5 Conclusions

The constructed relation can be applied in the approximation and nearness of concepts in information systems $\mathcal{IS}$. This in turn can help in fields like Artificial intelligence $\mathcal{AI}$. which is widely used in different areas of real life applications such as medical diagnoses, political, social and economic studies.

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