Gen. Math. Notes, Vol. 33, No. 1, March 2016, pp. 17-25
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

The Variational Principle and Complexity of Group Actions

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(Received: 19-1-16 / Accepted: 16-3-16)

Abstract

The complexity of a finite object was introduced by A. Kolmogorow and V. Tihomirov in [1] and it was conjectured that for Z actions the complexity coincides with topological entropy [2]. In the present paper we introduce complexity for Z^n actions and prove the Kolmogorow assertion for continuous actions of Z. Then we study the variational principle and complexity for group actions.

Keywords: *Variational Principle, Complexity, Dynamical System, Topological Entropy.*

1 Introduction

In dynamical systems and ergodic theory, the topological entropy describes the complexity of a system. Topological entropy is an invariant for equivalent homeomorphisms [3, 4].

Recently Lewis Bowen introduced a collection of entropy invariants for measure-preserving actions of countable sofic group a standard probality space admitting a generating rotation with finite entropy [5]. Given Bowen's work, it is natural to ask whether there exist analogous invariants for continuous actions of a countable sofic group on a compact metrizable space, and if so whether they are connected to Bowen's measure entropy via a variational principle [6].

It is well known that topological entropy is an invariant of topological conjugacy. If the topological entropy is positive, the system is complexity and chaotic. If the entropy is zero, the system is rather simple. However from the theory and application, there still exists relatively complex and chaotic behavior. Therefore, for more general research on complexity of a system. This idea was firstly introduced in the research of ergodic theory and then in symbolic dynamical systems [7, 8] by Frerenczi.

The entropy theory of dynamical systems originated in the papers of A.N. Kolmogorov in the fifties. Topological entropy which is the analog of metric entropy in topological dynamics was introduced by Adler, Konheim and McAndrew [6]. The conjecture was made there that the topological entropy coincides with the least upper bound of the metric entropies over the set of all invariant Borel probability measure. This assertion which has been called the variational principle (VP) was proved by Dinaburg for homeomorphisms of finite dimensional compact [9].

In recent years, there have been a number of papers about the combinatorial notion of symbolic complexity: this is the function counting the number of factors of length n for a sequence. The complexity is an indication of the degree of randomness of the sequence: a periodic sequence has a bounded complexity, the expansion of a normal number has an exponential complexity. For a given sequence, the complexity function is generally not of easy access, and it is a rich and instructive work to compute it; a survey of this kind of results can be found in [10].

2 The Complexity for Z^n Actions

In [11], Tagi-zade and Fayziev are defined the notion of complexity of configuration w from the space Ω, i.e. the minimal account of information necessary for the restoration (decoding) of this configuration. The notion of complexity of finite object was introduced by A. Kolmogorov [12]. By this definition the complexity of finite object x from the set of finite objects X related to the algorithm A defined on the set of finite $0-1$ words p and taking values in X is the quantity

$$C_A(\mathrm{x})=\begin{cases} \inf l(p) & if\ \{p: A(p)=\mathrm{x}\}\neq\varnothing \\ \infty & \text{otherwise}\end{cases}$$

Where $l(p)$ is the length of finite $0-1$ words p.

The notion of trajectorial complexity for an action of Z basing on the symbolic dynamical ideas from one and a notion of Kolmogorov complexity from other was given works of A. Levin and A. Brudno [13, 14]. In [15] A. Tagizade gave an approach for construction the complexity notion in case of non-abelian groups actions and in [16] this approach was generalized to the countable and continuous amenable groups actions.

Let us introduce the notions we need. Let $A = \{a_1, a_2, \dots, a_k\}$ be a finite set of symbols, (alphabet);

$$\Omega = A^{Z^n} = w = \{(w_g): w_g \in A, g \in Z\}$$

be the space of configurations with Tychcnoff topology, σ be the shift in this configuration space:

$$(\sigma^g w)_h = w_{g^{-1}h} g \, , h \in Z^n .$$

Definition 2.1: *A Dynamical system (X, T) is a symbolic on Z^n, if X is the σ-invariant closed subset of Ω and T is the restriction of σ to X.*

Now we define the complexity of the configuration spaces of the symbolic dynamical system (X, T).

Definition 2.2: *For an arbitrary finite subset F of Z^n we denote by A^F the set of stamps (configuration) on F. Every point $w^F = (w_g, g \in F) \epsilon A^F$. On this set A^F is called a configuration stamp.*

Let P be an algorithm defined on some subset of a space of all finite $\{0,1\}$ —words and taking values in the set of all finite words of Ω. By $l(p)$ we denote the number of elements in the finite word in the {0,1}—alphabet. Following the definition of the Kolmogorow asymptotic complexity [12].

Now we define the complexity $C_P(w^F)$ of the stamp w^F relatively to the program P:

$$C_P(w^F) = \begin{cases} inf\,\{l(p): P(p) = w^F\} & if \quad \inf\{l(p): P(p) = w^F\} \neq \varnothing \\ \infty & if \quad \{l(p): P(p) = w^F\} = \varnothing \end{cases} .$$

Now we define the complexity $C_p(w)$ for the configuration $w \in X$ relatively to the program:

$$C_p(w) = \lim_{k\to\infty} \sup \frac{1}{|I_k|} c_p(w|I_k) \, ,$$

where $I_k = \{(i_1, i_2, \dots, i_n) \epsilon Z^n \; : \; -k \leq i_j \leq k \, , j = 1,2, \dots, n\}$, $|I_k| = (2k+1)^n$.

Now let $C_p(X)$ define the complexity of the configuration space X relatively to the program P as:

$$C_p(X) = \lim_{k\to\infty} sup_{w\in X} \frac{1}{|I_k|} C_p\,(w|I_k)\ .$$

Let P be such a program that for an arbitrary program P' we have a constant $C(P,P')$ such that for every stamp W^F the inequality

$$C_p(w^F) \ \leq\ C_{P'}(w^F) + C(P+P')$$

holds.

We call this program P the asymptotical optimal program. The existence of such a program P was proved.

Proposition 2.3: *For every symbolic* (X,T) *and arbitrary optimal programs* P_1 *and* P_2 *,*

C_{P_1}(X) =C_{P_2}(X) [17].

3 Varational Principle

We introduce the concept and definitions needed. Let T be a continuous action of the G group with lattice L. We denote $K(G)$ the collection of all compact subsets of the G-group with lattice L.

Let A be the set of all finite open covering of the topological space Y. For an action T of the G -group with lattice L on the compacts metric space. We denote by $\Re(X,T)$ the set of all T-invariant Borel normalized measure X. Further let $P(X)$ be the set of all finite measure partitions of space (X, B).

Definition 3.1: *This is map* $T: XxG \to X$ *which is continuous and such that for* $\forall g1, g2 \in G$ *and* $\forall x \in X$ *the equality* $T^{g_1 g2}x = T^{g1}(T^{g_2}x)$. *Let* $\{\ell_n\}_{n=1}^{\infty}$ *be sequence of monoton growing to* L. *For* $n \in N$ *let*

$$F_n = \{(G/L)\ell_n\}.$$

For a finite subset K of the group with lattice L and $A\ =\left(C_A^1,\ldots C_A^n\right)\in P(X)$ we set

$$A^K = \left\{\bigcap_{g\in K} T^{g-1}C_A^{i(g)} \neq \phi \in A, g\in K\right\}.$$

We have that $A^K \in P(X)$.

For $c \in G$ we denote by $K(G,c)$ the set of all sequences $K = \{K_n\}_{n=1}^{\infty}$ from $K(G)$ satisfying the following condition $Card Kn \leq |Fn|$ for $n = 1,2,\ldots$

We have

$$h_\mu^K(T) = \lim_{n\to\infty}\sup |F_n|^{-1} H_\mu(A^{K_n}),$$

Here

$$H_\mu(A) = -\sum \mu(a)\log\mu(a)$$

is the entropy function;

$$h_\mu(A) = \sup_{K\in K(G)} h_\mu^K(T,A)$$

$$h_\mu^K(A) = \sup_{K\in P(X)} h_\mu^K(T,A).$$

Definition 3.2: *The conditional entropy function for* $\mu \in \Re(X,T)$ *and partitions* $A_1, A_2 \in P(X)$ *is*

$$H_\mu(A_1 \mid A_2) = -\sum_{i\in A_1}\sum_{j\in A_2}\mu(C_A^i | C_A^j)\log\mu(C_A^i | C_A^j).$$

Definition 3.3: *The following equivalent we will vibrational principle (VP) the for topological entropy*

$$h_{top}(T) = \sup_{\mu\in(X,T)} h_\mu(T).$$

In this section we prove the relationship between topological entropy and the complexity for group actions i.e.:

$$C_p(X) = h_{top}(T) = sup\{h_\mu(T): \mu \in M(X,T)\}.$$

The proof for the following theorem is inspired on Misiurewicz proof of the variational principle, presented for example in heorem 8.6 of [18].

Lemma 3.4: *Let* (X,T) *be a sembolic daynamical system,* $T \to T$ *a homeomorphism,* $E_n \subset X$ *an* (n,ε) *−separated set. Then*

$$\lim_{n\to\infty} sup\frac{1}{n}log\#(E_n) \leq h_\mu(T)[19].$$

Theorem 3.5: *Let (X,T) be a symbolic system on Z^n. Then,*

$C_p(X) = h_{top}(T) = sup\{h_\mu(T): \mu \in M(X,T)\}$.

Proof: Let the complexity $C_p(X)$ of the space X be finite and equal to α. So we have

$$\lim_{n\to\infty} sup \frac{1}{|I_k|} \sup C_p(wI_k) = \alpha.$$

Then let $\varepsilon > 0$ be an arbitrary number. There is some $n_0 \in N$such that $\forall k > n_0$

$$\frac{1}{|I_k|} \sup C_p(wI_k) \leq a+\varepsilon .$$

So we have

$$\sup C_p(wI_k) \leq (a+\varepsilon)|I_k| . \quad (3.1)$$

The inequality shows us that the number of different restrictions of points of Xon the I_k set is not bigger than $2^{(a+\varepsilon)|I_k|+1}$.

To prove this, we can write from the definition,

$$P: \bigcup_{n=1}^{\infty} \{0,1\}^n \to \bigcup_{F\subset Z^n} A^F$$

$CardF < \infty$

for any P program all. Now we will find some set U such that

$U = \cup_{n=1}^{\infty}\{0,1\}^n$ and $P(U) = V$,

where $V = \{\overline{w} = (w_{g,}g \in I_k): \exists\widetilde{w} \in X, \widetilde{w|}I_{k=}\overline{w}\} = A^{I_k} \cap X|I_k$. We have

$$Cardp^{-1} (\{A^{I_k} \cap X|I_k\}) \geq Card\left(\{A^{I_k} \cap X|_{I_k}\}\right) .$$

Let us fix $U = \cup_{n=1}^{\sup(w|I_k)}\{0,1\}^n$. We will show that

$P(U) = A^{I_k} \cap X|_{I_k}$.

Let us take $\forall\overline{w} \notin A^{I_k} \cap X_{|I_k}$. From the definition of $C_p\left(w|_{I_k}\right)$ we have

$C_p(\overline{w}) \leq \sup C_p(w|_{I_k})$.

So there is some finite word, $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n, n \leq \sup C_p(X|_{I_k})$ such that $P(a_1, a_2, \ldots, a_n) = \overline{w}$.

$P(U) = A^{I_k} \cap X|_{I_k}$.

Now we will show that

$CardU \leq 2^{(a+\varepsilon)|I_k+1|}$

Indeed, from (3,1) we have

$U = \bigcup_{n=1}^{\sup C_p(w|I_k)} \{0,1\}^n \subset \bigcup_{n=1}^{(a+e)|I_k|} \{0,1\}^n$.

Thus,

$\mathrm{Card}\left(\bigcup_{n=1}^{(a+\varepsilon)|I_k|} \{0,1\}^n\right) = \sum_{n=1}^{(a+\varepsilon)|I_k|} \{0,1\}^n$

$$= \sum_{n=1}^{(a+\varepsilon)|I_k|} 2^n = 2^{(a+\varepsilon)|I_k+1|}.$$

So we have

$$CardV \leq CardU \leq 2^{(a+e)|I_k|+1}. \quad (3.2)$$

To finish the proof of the theorem we need first some facts about topological entropy.

Theorem 3.6: *Let* (X,T) *be a symbolic dynamical system. Then*

$$h_\mu(\sigma) = \lim_{k\to\infty} sup \frac{1}{|I_k|} \log A_k ,$$

where $A_k = Card|\{\omega|I_k : \omega \in X\}|$ [1].

Proof: From Theorem 3.6, Lemme 3 4 and (3.2) we have

$A_k \leq 2^{(a+e)|I_k+1}$

and then

$$\lim_{k\to\infty} sup \frac{1}{|I_k|}\log A_k \ \leq \lim_{k\to\infty} sup \frac{1}{|I_k|}\log 2^{(a+\varepsilon)|I_k+1|}$$

$h_\mu(T) \leq a + \varepsilon.$

Hence

$h_\mu(T) \leq C_p(X).$

Now we will prove the inverse inequality. Let $h_\mu(T) \leq b$. Then for $\epsilon > 0$ there exists

$n_0 \in N$ such that $\forall k > n_0$ we can write

$$\frac{1}{|I_k|}\log A_k \ \leq b + \varepsilon$$

$$\log_2 A_k \ \leq (b + \varepsilon)|I_{km}|$$

$A_k \leq 2^{(b+\varepsilon)|I_k|}.$

Now let us fix some $k > k_0$. For this k we can define some finite program *P* such that it is defined on the finite word

$$a\epsilon\{0,1\}^{(b+\varepsilon)|I_k|+2}$$

and can give us all the finite restriction the space *X* on I_k. Now will continue the program P in the following way:

One will divide the big cube I_{km} into $|I_{km}|/|I_k|$ domains every part of which is equal to I_k and now the program *P* on each domain of the big cube. Certainly this program P will be defined on the {0,1} words with length not bigger than

$$(b + \varepsilon)|I_k|\frac{|I_{km}|}{|I_k|} = (b + \varepsilon)|I_k| \qquad .$$

Thus, the complexity of the space *X* relatively to this program *P* is not bigger than $(b + \varepsilon)$. Because of that complexity for an arbitrary asymptotically program. *P* will not than be bigger than b.

Acknowledgements: The research was supported by Necmettin Erbakan University Scientific Research Project: 131210004.

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