Gen. Math. Notes, Vol. 33, No. 2, April 2016, pp. 24-39

ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

A Special Motion on Dual Hyperbolic Unit Sphere $\tilde{\mathbb{H}}_0^2$

Mustafa Kazaz[1] and Hasan Hüseyin Uğurlu[2]

[1]Celal Bayar University, Faculty of Art and Science
Department of Mathematics, Muradiye Campus, 45140, Manisa, Turkey
E-mail: mustafa.kazaz@cbu.edu.tr
[2]Gazi University, Faculty of Education, Department of Secondary Education, Science and Mathematics Teaching, Mathematics Teaching Program
06500, Ankara, Turkey
E-mail: hugurlu@gazi.edu.tr

(Received: 29-12-15 / Accepted: 12-4-16)

Abstract

In this paper, we study a special motion, called conchoidal motion, on a dual hyperbolic unit sphere $\tilde{\mathbb{H}}_0^2$ in the dual Lorentzian space $\mathbb{D}_1^3$ with dual signature $(+, +, -)$. Then, the results are carried over to the Lorentzian line space IR_1^3 by E. Study's mapping. We also obtain the Study maps of the orbits drawn on the fixed dual hyperbolic unit sphere by unit dual vectors of an orthonormal base.

Keywords: *Conchoidal motion, Dual hyperbolic unit sphere, Study's mapping.*

1 Introduction

W.K. Clifford (1845-1879) introduced dual numbers in the form of $\tilde{\lambda} = \lambda + \varepsilon\lambda^*$ with $\varepsilon^2 = 0$ for studying the non-Euclidean geometry [3]. However, its first applications to mechanics are due to E. Study (1860-1930) which defined dual numbers as dual angles to specify the relations between two lines in the Euclidean space $\mathbb{R}^3$. Then, he used dual numbers and dual vectors in his research on the geometry of lines and kinematics, and defined the mapping which is called after his name (E. Study's mapping): The set of oriented straight lines in the Euclidean 3-space $\mathbb{R}^3$ is one-to-one correspondence with the dual points on the surface of a dual unit sphere $\tilde{\mathbb{S}}^2$ in the dual space $\mathbb{D}^3$ of triples of dual numbers [5]. Hence, a differentiable curve on a dual unit sphere $\tilde{\mathbb{S}}^2$ corresponds to a ruled surface in the line space IR^3 [7]. Ruled surfaces have been widely applied in surface design and simulation of rigid bodies [10].

It is known that dual vectors, dual angles, dual orthogonal matrices, the E. Study mapping, etc. are the most important notions for applications of dual geometry to engineering. For example, the dual angle $\tilde{\psi} = \psi + \varepsilon\psi^*$ between two dual unit vectors is formed with real angle ψ between corresponding two directed lines in the line space IR^3 and the shortest distance ψ^* between these directed lines. These notions lay the foundations for the study of spherical and spatial motions. Dual Lorentzian correspondences of these notions were introduced and also, several important theorems and results related to geometry of this space were given by the authors [1] [2] [6].

E. Study's mapping plays a fundamental role between the real and dual Lorentzian spaces [10]. By this mapping, a curve on a dual hyperbolic unit sphere $\tilde{\mathbb{H}}_0^2$ corresponds to a timelike ruled surface in the Lorentzian line space IR_1^3, that is, there exists a one-to-one correspondence between the geometry of curves on $\tilde{\mathbb{H}}_0^2$ and the geometry of timelike ruled surfaces in IR_1^3. Similarly, a timelike (spacelike) curve on a dual Lorentzian unit sphere $\tilde{\mathbb{S}}_1^2$ corresponds to a spacelike (timelike) ruled surface in the Lorentzian line space IR_1^3, that is, there exists a one-to-one correspondence between the geometry of timelike (spacelike) curves on $\tilde{\mathbb{S}}_1^2$ and the geometry of spacelike (timelike) ruled surfaces in IR_1^3 [9].

Since the dual Lorentzian metric is indefinite, the angle concept in this space is very interesting. For instance, the dual hyperbolic angle $\tilde{\psi} = \psi + \varepsilon\psi^*$ between two dual timelike unit vectors is a dual value formed with the (real) hyperbolic angle ψ between corresponding two directed timelike lines in the Lorentzian line space IR_1^3 and the shortest Lorentzian distance between these directed timelike lines.

Planar and spherical conchoidal motions have been introduced by Karger and Novac [4]. Yapar [8] defined the dual spherical conchoidal motion and gave the geometrical interpretations in the Euclidean line space IR^3. In this paper, we define the dual hyperbolic analogy of planar, spherical and dual spherical conchoidal motions.

The Lorentzian motions in the Minkowski 3-space IR_1^3 are represented in the dual Lorentzian 3-space $\mathbb{D}_1^3$ by dual Lorentzian orthogonal 3×3 matrices $M=\left(a_{ij}\right)$, $M^{-1}=\gamma M^T\gamma$, where a_{ij} are dual functions of one variable $t\in IR$ and $\gamma=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ is a signature matrix [9]. This means that when a Lorentzian motion is given in IR_1^3, we can find a corresponding dual Lorentzian orthogonal 3×3 matrix M.

2 Preliminaries

In this section, we give a brief summary of the theory of dual numbers and dual Lorentzian vectors. Let IR_1^3 denote the 3-dimensional Minkowski space over the field of real numbers IR with the Lorentzian inner product $<,>$ given by

$$<\boldsymbol{a},\boldsymbol{b}>=a_1b_1+a_2b_2-a_3b_3,$$

where $\boldsymbol{a}=(a_1,a_2,a_3)$ and $\boldsymbol{b}=(b_1,b_2,b_3)\in IR^3$. A vector $\boldsymbol{a}=(a_1,a_2,a_3)$ of IR_1^3 is said to be timelike if $<\boldsymbol{a},\boldsymbol{a}><0$, spacelike if $<\boldsymbol{a},\boldsymbol{a}>>0$ or $\boldsymbol{a}=\boldsymbol{0}$, and lightlike (null) if $<\boldsymbol{a},\boldsymbol{a}>=0$ and $\boldsymbol{a}\neq\boldsymbol{0}$. Similarly, a curve α is called timelike (spacelike) if $<\alpha',\alpha'><0$ ($<\alpha',\alpha'>>0$), and lightlike (null) if $<\alpha',\alpha'>=0$, where α' is the derivative of α.

The norm of a vector $\boldsymbol{a}$ is defined by $|\boldsymbol{a}|=\sqrt{|<\boldsymbol{a},\boldsymbol{a}>|}$. Now, let $\boldsymbol{a}=(a_1,a_2,a_3)$ and $\boldsymbol{b}=(b_1,b_2,b_3)$ be two vectors in IR_1^3, then the Lorentzian cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a}\times\boldsymbol{b}=(a_3b_2-a_2b_3,\ a_1b_3-a_3b_1,\ a_1b_2-a_2b_1).$$

A dual number has the form $\tilde{\lambda}=\lambda+\varepsilon\lambda^*$, where λ and λ^* are real numbers and ε stands for the dual unit which is subject to the rules:

$$\varepsilon\neq0,\ \varepsilon^2=0,\ 0\varepsilon=\varepsilon0=0,\ 1\varepsilon=\varepsilon1=\varepsilon.$$

Like a real number which can be considered as an angle, in differential geometry and motion analysis of spatial mechanisms, a dual number is also commonly referred as a dual angle $\tilde{\lambda}=\lambda+\varepsilon\lambda^*$ between two lines in the space. The real part λ of the dual angle is the projected angle between the lines, and the dual part λ^* is the length along the common normal of the lines. We denote the set of all dual numbers by $\mathbb{D}$:

$$\mathbb{D}=\left\{ \tilde{\lambda}=\lambda+\varepsilon\lambda^* \,\middle|\, \lambda,\, \lambda^* \in IR,\, \varepsilon^2=0 \right\}.$$

Equality, addition and multiplication are defined in $\mathbb{D}$ by

$$\lambda+\varepsilon\lambda^*=\beta+\varepsilon\beta^* \text{ iff } \lambda=\beta \text{ and } \lambda^*=\beta^*,$$

$$(\lambda+\varepsilon\lambda^*)+(\beta+\varepsilon\beta^*)=(\lambda+\beta)+\varepsilon(\lambda^*+\beta^*),$$

And

$$(\lambda+\varepsilon\lambda^*)(\beta+\varepsilon\beta^*)=\lambda\beta+\varepsilon(\lambda\beta^*+\lambda^*\beta),$$

respectively. Then it is easy to show that $(\mathbb{D},+,.)$ is a commutative ring with unity. The numbers $\varepsilon\lambda^*$ $(\lambda^*\in IR)$ are divisors of 0. We note that if λ and β are two nonzero elements of a ring R such that $\lambda\cdot\beta=0$, then either λ or β is a divisor of 0 (or zero-divisor). Moreover, if $\tilde{\lambda}=\lambda+\varepsilon\lambda^*$, $\tilde{\beta}=\beta+\varepsilon\beta^*\in\mathbb{D}$ with $\beta\neq 0$ then the division is given by

$$\frac{\tilde{\lambda}}{\tilde{\beta}}=\frac{\lambda+\varepsilon\lambda^*}{\beta+\varepsilon\beta^*}=\frac{\lambda}{\beta}+\varepsilon\left(\frac{\lambda^*}{\beta}-\frac{\lambda\beta^*}{\beta^2}\right).$$

Now, let f be a differentiable function with dual variable $\tilde{x}=x+\varepsilon x^*$. Then the Maclaurin series generated by f is

$$f(\tilde{x})=f(x+\varepsilon x^*)=f(x)+\varepsilon x^* f'(x),$$

where $f'(x)$ is the derivative of $f(x)$. Then it is easy to see that

$$\sin(\tilde{x})=\sin(x+\varepsilon x^*)=\sin x+\varepsilon x^*\cos x,$$
$$\cos(\tilde{x})=\cos(x+\varepsilon x^*)=\cos x-\varepsilon x^*\sin x,$$
$$\sinh(\tilde{x})=\sinh(x+\varepsilon x^*)=\sinh x+\varepsilon x^*\cosh x,$$
$$\cosh(\tilde{x})=\cosh(x+\varepsilon x^*)=\cosh x+\varepsilon x^*\sinh x,$$

$$\sqrt{\tilde{x}} = \sqrt{x+\varepsilon x^*} = \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}}, \quad (x>0).$$

Let $\mathbb{D}^3$ be the set of all triples of dual numbers, i.e.

$$\mathbb{D}^3 = \left\{ \tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \,\middle|\, \tilde{a}_i = a_i + \varepsilon a_i^* \in \mathbb{D},\ 1 \le i \le 3 \right\}.$$

The elements of $\mathbb{D}^3$ are called as dual vectors. A dual vector $\tilde{\boldsymbol{a}}$ may be expressed in the form $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \boldsymbol{a} + \varepsilon \boldsymbol{a}^* = (a_1, a_2, a_3) + \varepsilon (a_1^*, a_2^*, a_3^*)$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{a}^* = (a_1^*, a_2^*, a_3^*)$ are the vectors of IR^3. Now, let $\tilde{\boldsymbol{a}} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$, $\tilde{\boldsymbol{b}} = \boldsymbol{b} + \varepsilon \boldsymbol{b}^* \in \mathbb{D}^3$ and $\tilde{\lambda} = \lambda + \varepsilon \lambda^* \in \mathbb{D}$. Then we define

$$\tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}} = \boldsymbol{a} + \boldsymbol{b} + \varepsilon (\boldsymbol{a}^* + \boldsymbol{b}^*),$$
$$\tilde{\lambda}\tilde{\boldsymbol{a}} = (\tilde{\lambda}\tilde{a}_1, \tilde{\lambda}\tilde{a}_2, \tilde{\lambda}\tilde{a}_3) = \lambda \boldsymbol{a} + \varepsilon (\lambda \boldsymbol{a}^* + \lambda^* \boldsymbol{a}).$$

Then $\mathbb{D}^3$ becomes a unitary $\mathbb{D}$-module with these operations. It is called $\mathbb{D}$-module or dual space. The dual Lorentzian inner product of two dual vectors $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$, $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = \boldsymbol{b} + \varepsilon \boldsymbol{b}^*$ is defined by

$$<\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}> = \tilde{a}_1\tilde{b}_1 + \tilde{a}_2\tilde{b}_2 - \tilde{a}_3\tilde{b}_3 = <\boldsymbol{a}, \boldsymbol{b}> + \varepsilon (<\boldsymbol{a}, \boldsymbol{b}^*> + <\boldsymbol{a}^*, \boldsymbol{b}>),$$

where $<\boldsymbol{a}, \boldsymbol{b}>$ is the Lorentzian inner product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the Minkowski 3-space IR_1^3. Then a dual vector $\tilde{\boldsymbol{a}} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^*$ is said to be timelike if $\boldsymbol{a}$ is timelike, spacelike if $\boldsymbol{a}$ is spacelike or $\boldsymbol{a} = \boldsymbol{0}$, and lightlike (null) if $\boldsymbol{a}$ is lightlike (null) and $\boldsymbol{a} \neq \boldsymbol{0}$. The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by $\mathbb{D}_1^3$:

$$\mathbb{D}_1^3 = \left\{ \tilde{\boldsymbol{a}} = \boldsymbol{a} + \varepsilon \boldsymbol{a}^* \,\middle|\, \boldsymbol{a},\ \boldsymbol{a}^* \in IR_1^3 \right\}.$$

The dual Lorentzian cross product of two dual vectors $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}} \in \mathbb{D}_1^3$ is defined by

$$\tilde{\boldsymbol{a}} \times \tilde{\boldsymbol{b}} = (\tilde{a}_3\tilde{b}_2 - \tilde{a}_2\tilde{b}_3, \tilde{a}_1\tilde{b}_3 - \tilde{a}_3\tilde{b}_1, \tilde{a}_1\tilde{b}_2 - \tilde{a}_2\tilde{b}_1) = \boldsymbol{a} \times \boldsymbol{b} + \varepsilon (\boldsymbol{a}^* \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}^*),$$

where $\boldsymbol{a} \times \boldsymbol{b}$ is the Lorentzian cross product in IR_1^3.

Let $\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^{*}\in\mathbb{D}_1^3$. Then $\tilde{\boldsymbol{a}}$ is said to be unit dual timelike vector (resp., unit dual spacelike vector) if the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^{*}$ satisfy the following equations:

$<\boldsymbol{a},\boldsymbol{a}>=-1$(resp., $<\boldsymbol{a},\boldsymbol{a}>=1$), $<\boldsymbol{a},\boldsymbol{a}^{*}>=0$.

The set of all unit dual timelike vectors (resp., all unit dual spacelike vectors) is called the dual hyperbolic unit sphere (resp. dual Lorentzian unit sphere), and is denoted by $\tilde{\mathbb{H}}_0^2$ (resp.,$\tilde{\mathbb{S}}_1^2$) [9].

A ruled surface is a surface generated by the motion of a straight line in IR^3. This line is the generator of the surface. A ruled surface is said to be timelike if the induced metric on the surface is a Lorentzian metric (i.e., the normal vector of the ruled surface at every point is a spacelike vector), and spacelike if the induced metric on the surface is a positive defined Riemannian metric (i.e., the normal vector of the ruled surface at every point is a timelike vector) [9].

Theorem 2.1: (E. Study's Mapping): *The unit dual timelike vectors of the dual hyperbolic unit sphere $\tilde{\mathbb{H}}_0^2$ are in one-to-one correspondence with the directed timelike lines of the Minkowski 3-space IR_1^3 (Fig. 1). [9]*

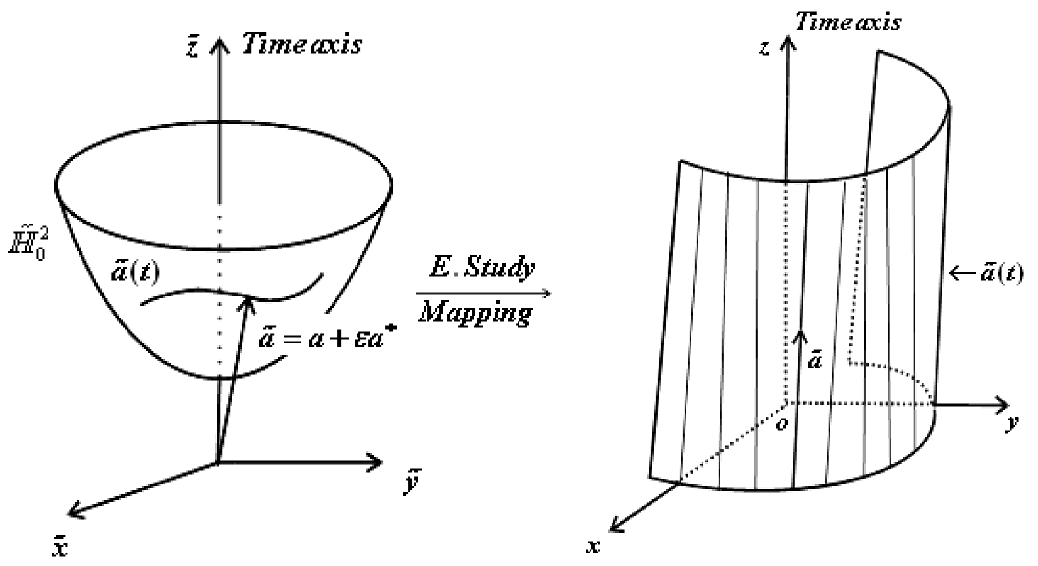

Fig. 1: The curve $\tilde{a}(t)$ on $\tilde{\mathbb{H}}_0^2$ and the corresponding timelike ruled surface in $\mathbb{R}_1^3$

3 Conchoidal Motion on the Dual Hyperbolic Unit Sphere $\tilde{\mathbb{H}}_0^2$

We will define conchoidal motion on the dual hyperbolic unit sphere $\tilde{\mathbb{H}}_0^2$. Let us consider a fixed dual orthonormal frame $R = \{0; \tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3\,(\text{timelike})\}$ and denote this frame by the dual hyperbolic unit sphere H'. Let H_0^1 be a great hyperbolic circle (a geodesic) on H' and C be a point not lying on H_0^1. The frame $\{0; \tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3\}$ is chosen as shown in Fig. 2, where $\tilde{\boldsymbol{u}}_2$ and $\tilde{\boldsymbol{u}}_3$ lie in the timelike plane of the great hyperbolic circle H_0^1, and the timelike plane $\tilde{\boldsymbol{u}}_1\tilde{\boldsymbol{u}}_3$ contains the chosen point C. Let us consider an orthonormal dual frame $\{0; \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3\,(\text{timelike})\}$ as shown in Fig. 2.

The frame $\{0; \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3\}$ moves now in such a way that the timelike vector $\tilde{\boldsymbol{v}}_3$ rotates in the great hyperbolic circle H_0^1 while the timelike plane $\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{v}}_3$ passes through the point C all the time. As the parameter of the motion we choose the dual hyperbolic angle $\tilde{\psi} = \psi + \varepsilon\psi^*$ of the timelike vectors $\tilde{\boldsymbol{v}}_3$ and $\tilde{\boldsymbol{u}}_3$, where $\tilde{\boldsymbol{v}}_3 = \tilde{\boldsymbol{u}}_3 \cosh\tilde{\psi} + \tilde{\boldsymbol{u}}_2 \sinh\tilde{\psi}$.

Further, we can write

$$\tilde{\boldsymbol{v}}_1 = \tilde{A}_1\tilde{\boldsymbol{u}}_1 + \tilde{A}_2\tilde{\boldsymbol{u}}_2 + \tilde{A}_3\tilde{\boldsymbol{u}}_3, \quad \langle \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_1 \rangle = \tilde{A}_1^{\,2} + \tilde{A}_2^{\,2} - \tilde{A}_3^{\,2} = 1, \tag{1}$$

where $\tilde{A}_i$ $(1 \le i \le 3)$ are dual numbers. By orthonormality,

we have $< \tilde{\boldsymbol{v}}_3, \tilde{\boldsymbol{v}}_1 >= 0$, i.e.

$$< \tilde{\boldsymbol{u}}_3\cosh\tilde{\psi} + \tilde{\boldsymbol{u}}_2\sinh\tilde{\psi},\ \tilde{A}_1\tilde{\boldsymbol{u}}_1 + \tilde{A}_2\tilde{\boldsymbol{u}}_2 + \tilde{A}_3\tilde{\boldsymbol{u}}_3 >= 0,$$

or

$$-\tilde{A}_3\cosh\tilde{\psi} + \tilde{A}_2\sinh\tilde{\psi} = 0. \tag{2}$$

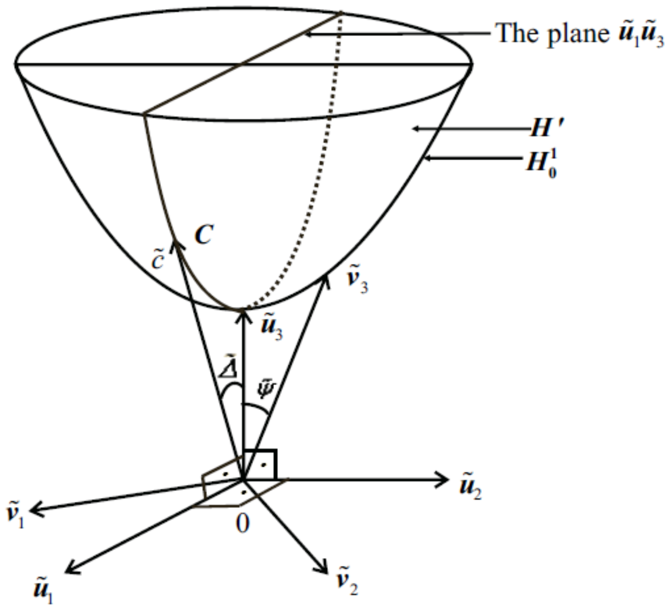

Fig. 2: Dual hyperbolic conchoidal motion ($\tilde{\boldsymbol{c}}$, $\tilde{\boldsymbol{u}}_3$ and $\tilde{\boldsymbol{v}}_3$ are unit dual timelike vectors and the others are unit dual spacelike vectors)

Further, we may write $\tilde{\boldsymbol{c}}$ as follows:

$$\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{u}}_3 \cosh\tilde{\Delta} + \tilde{\boldsymbol{u}}_1 \sinh\tilde{\Delta},$$

where $\tilde{\Delta} = \sigma + \varepsilon\sigma^*$ is dual hyperbolic angle between the timelike vectors $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{u}}_3$. Since the timelike plane $\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{v}}_3$ has to pass through the point C all the time, the vectors $\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_3$ and $\tilde{\boldsymbol{c}}$ must be co-planar, that is, $\det(\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_3, \tilde{\boldsymbol{c}}) = 0$. Thus we get the equation.

$$\tilde{A}_1 \sinh\tilde{\psi} \cosh\tilde{\Delta} + \tilde{A}_2 \cosh\tilde{\psi} \sinh\tilde{\Delta} - \tilde{A}_3 \sinh\tilde{\psi} \sinh\tilde{\Delta} = 0. \tag{3}$$

Then, we have three equations altogether for the unknowns $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$:

$$\begin{aligned}
&\tilde{A}_1^{\,2} + \tilde{A}_2^{\,2} - \tilde{A}_3^{\,2} = 1, \\
&-\tilde{A}_3 \cosh\tilde{\psi} + \tilde{A}_2 \sinh\tilde{\psi} = 0, \\
&\left(\tilde{A}_2 \cosh\tilde{\psi} - \tilde{A}_3 \sinh\tilde{\psi}\right) \sinh\tilde{\Delta} + \tilde{A}_1 \sinh\tilde{\psi} \cosh\tilde{\Delta} = 0.
\end{aligned}$$

From the second equation, we obtain $\tilde{A}_3 = \tilde{\lambda} \sinh\tilde{\psi}$, $\tilde{A}_2 = \tilde{\lambda} \cosh\tilde{\psi}$ for some $\tilde{\lambda} \in \mathbb{D}$. Substituting this into the third equation, we obtain

$$\tilde{\lambda}\sinh\tilde{\Delta} + \tilde{A}_1\sinh\tilde{\psi}\,\cosh\tilde{\Delta} = 0, \text{ i.e., } \tilde{A}_1\sinh\tilde{\psi} = -\tilde{\lambda}\tanh\tilde{\Delta}.$$

Multiplying the first equation by $\sinh^2\tilde{\psi}$, upon substitution we have

$$\sinh^2\tilde{\psi} = \tilde{\lambda}^2\left(\tanh^2\tilde{\Delta} + \sinh^2\tilde{\psi}\right), \text{ and then } \tilde{\lambda} = \pm\sinh\tilde{\psi}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}.$$

We choose the plus sign. Then consequently we have

$$\tilde{\boldsymbol{v}}_1 = \Big[-\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}\tanh\tilde{\Delta},\ \sinh\tilde{\psi}\,\cosh\tilde{\psi}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2},$$
$$\sinh^2\tilde{\psi}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}\Big],$$

$$\tilde{\boldsymbol{v}}_3 = \left[0, \sinh\tilde{\psi}, \cosh\tilde{\psi}\right],$$

and from $\tilde{\boldsymbol{v}}_2 = \tilde{\boldsymbol{v}}_1 \times \tilde{\boldsymbol{v}}_3$ we obtain

$$\tilde{\boldsymbol{v}}_2 = \Big[-\sinh\tilde{\psi}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2},\ -\cosh\tilde{\psi}\,\tanh\tilde{\Delta}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2},$$
$$-\sinh\tilde{\psi}\,\tanh\tilde{\Delta}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}\Big].$$

Thus a moving orthonormal dual frame $\{0; \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3\}$ is chosen. Let us represent this moving frame by dual hyperbolic unit sphere H. Then, a dual hyperbolic conchoidal motion which is analogous to the real conchoidal motion [10] is obtained. In this case, dual hyperbolic conchoidal motion is represented by H/H'.

Now, let us choose a fixed point X on the trace of H in the plane $\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{v}}_3$ (we should note that the trace of a surface in any plane is simply the intersection of the surface and the plane). During the dual hyperbolic conchoidal motion, the dual point X draws an orbit on H'. We denote the dual hyperbolic angles of $\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}\tilde{\boldsymbol{v}}_3$ by $\tilde{p} = p + \varepsilon p^*$ and $\tilde{q} = q + \varepsilon q^*$, respectively (see Fig. 3).

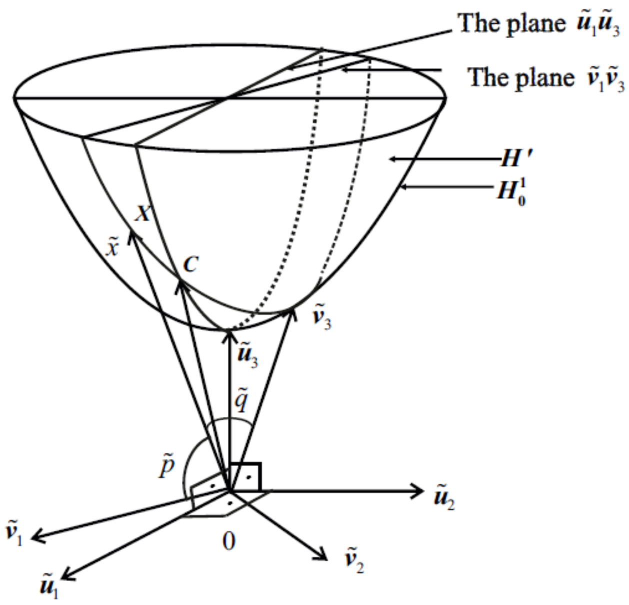

Fig. 3: The timelike vector $\tilde{\boldsymbol{x}}$ is on the plane $\tilde{\boldsymbol{v}}_1\tilde{\boldsymbol{v}}_3$

Then, we may write

$$\tilde{\boldsymbol{x}} = \frac{\tilde{\boldsymbol{v}}_1 \sinh \tilde{p} + \tilde{\boldsymbol{v}}_3 \sinh \tilde{q}}{\sinh(\tilde{p}+\tilde{q})}.$$

Since $p+q=\dfrac{\pi}{2}$ and $\sinh(\pi/2)\neq 0,\ \cosh(\pi/2)\neq 0$, we can write

$$\begin{aligned}\frac{1}{\sinh(\tilde{p}+\tilde{q})} &= \frac{1}{\sinh(p+q)+\varepsilon(p^*+q^*)\cosh(p+q)}\\ &= \frac{1}{\sinh(\pi/2)} + \varepsilon\frac{(p^*+q^*)\cosh(\pi/2)}{\sinh^2(\pi/2)}\\ &= a+\varepsilon a^* = \tilde{a} = \text{constant.}\end{aligned}$$

So, $\tilde{\boldsymbol{x}}$ can be written as follows,

$$\tilde{\boldsymbol{x}} = \tilde{a}(\tilde{\boldsymbol{v}}_1 \sinh \tilde{p} + \tilde{\boldsymbol{v}}_3 \sinh \tilde{q}). \tag{4}$$

Making the necessary calculations for $\tilde{\boldsymbol{x}}$ we have

$$\begin{aligned}\boldsymbol{x} &= a(x_1, x_2, x_3)\\ &= a\Big(-A^{-1/2}\tanh\sigma \sinh p,\ \ A^{-1/2}\sinh\psi\cosh\psi\sinh p + \sinh\psi \sinh q,\\ &\quad A^{-1/2}\sinh^2\psi \sinh p + \cosh\psi \sinh q\Big);\end{aligned} \tag{5}$$

$$
\begin{aligned}
\boldsymbol{x}^* &= a(x_1^*,\, x_2^*,\, x_3^*) + a^*(x_1\,,\, x_2\,,\, x_3\,) \\
&= a\Big(-p^* A^{-1/2}\tanh\sigma\cosh p + \psi^* A^{-3/2}\tanh\sigma\,\sinh\psi\,\cosh\psi\,\sinh p + \sigma^* A^{-3/2}\tanh^2\sigma \\
&\quad \operatorname{sech}^2\sigma\,\sinh p - \sigma^* A^{-1/2}\operatorname{sech}^2\sigma\,\sinh p,\;\; p^* A^{-1/2}\sinh\psi\,\cosh\psi\,\cosh p \\
&\quad + q^*\sinh\psi\,\cosh q + \psi^*\cosh\psi\,\sinh q - \psi^* A^{-3/2}\sinh^2\psi\,\cosh^2\psi\,\sinh p \\
&\quad + \psi^* A^{-1/2}\cosh 2\psi\,\sinh p - \sigma^* A^{-3/2}\tanh\sigma\,\operatorname{sech}^2\sigma\,\sinh\psi\,\cosh\psi\,\sinh p, \\
&\quad p^* A^{-1/2}\sinh^2\psi\,\cosh p + q^*\cosh\psi\,\cosh q + \psi^*\sinh\psi\,\sinh q \\
&\quad - \psi^* A^{-3/2}\sinh^3\psi\,\cosh\psi\,\sinh p + \psi^* A^{-1/2}\sinh 2\psi\sinh p \\
&\quad - \sigma^* A^{-3/2}\tanh\sigma\,\operatorname{sech}^2\sigma\,\sinh^2\psi\,\sinh p\Big) \\
&\quad + a^*\Big(-A^{-1/2}\tanh\sigma\sinh p,\;\; A^{-1/2}\sinh\psi\cosh\psi\sinh p + \sinh\psi\sinh q, \\
&\quad A^{-1/2}\sinh^2\psi\sinh p + \cosh\psi\sinh q\Big),
\end{aligned}
\tag{6}
$$

where $\boldsymbol{x}$ and $\boldsymbol{x}^*$ are the real and dual parts of $\tilde{\boldsymbol{x}}$, respectively, and $A = \sinh^2\psi + \tanh^2\psi$, $\Delta = \sigma + \varepsilon\sigma^*$=const., $\tilde{p}, \tilde{q} =$ const. Equations (5) and (6) depend on only two parameters ψ and ψ^*. Thus, equations (5) and (6) represent a timelike line congruence in IR_1^3 (for details on congruencies, see [2] [6]).

A timelike line congruence may be expressed as follows: Let $\boldsymbol{m} = \boldsymbol{m}(\psi, \psi^*)$ be a position vector of the reference surface of a timelike line congruence, and let $\boldsymbol{x} = \boldsymbol{x}(\psi, \psi^*)$ be a unit vector in direction of a timelike line x of the timelike line congruence. Let $\boldsymbol{y}$ denote the position vector of an arbitrary point $Y = (y_1, y_2, y_3)$ of the fixed timelike line x of the timelike line congruence in IR_1^3. Then we have $\boldsymbol{y} = \boldsymbol{m} + p\boldsymbol{x}$. We know that the moment vector $\boldsymbol{x}^*$ of the vector $\boldsymbol{x}$ with respect to the origin 0 is $\boldsymbol{x}^* = \boldsymbol{m} \times \boldsymbol{x}$ and $\boldsymbol{x} \times \boldsymbol{x}^* = \boldsymbol{m} + < \boldsymbol{m}, \boldsymbol{x} > \boldsymbol{x}$. Then we may write $\lambda = p - \langle \boldsymbol{m}, \boldsymbol{x} \rangle$. Thus, we have

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{x}(\psi,\psi^*) \times \boldsymbol{x}^*(\psi,\psi^*) + \lambda \boldsymbol{x}(\psi,\psi^*) \\
&= a(x_1\,,\, x_2,\, x_3) \times (a(x_1^*,\, x_2^*,\, x_3^*) + a^*(x_1\,,\, x_2,\, x_3)) + \lambda(x_1\,,\, x_2,\, x_3) \\
&= a^2(x_1\,,\, x_2,\, x_3) \times (x_1^*,\, x_2^*,\, x_3^*) + aa^*(x_1\,,\, x_2,\, x_3) \times (x_1\,,\, x_2,\, x_3) + \lambda(x_1\,,\, x_2,\, x_3) \\
&= a^2(x_1\,,\, x_2,\, x_3) \times (x_1^*,\, x_2^*,\, x_3^*) + \lambda(x_1\,,\, x_2,\, x_3).
\end{aligned}
\tag{7}
$$

Since (y_1, y_2, y_3) are the coordinates of Y we have

$$
\begin{aligned}
y_1 &= a^2\Big(\; p^* A^{-1/2}\sinh\psi\cosh p\sinh q - q^* A^{-1/2}\sinh\psi\sinh p\cosh q + \psi^*\sinh^2 q \\
&\quad - \psi^* A^{-3/2}\sinh^2\psi\cosh\psi\sinh p\sinh q - \psi^* A^{-1}\sinh^2\psi\sinh^2 p \\
&\quad + \psi^* A^{-1/2}\cosh\psi\sinh p\sinh q - \sigma^* A^{-3/2}\tanh\sigma\operatorname{sech}^2\sigma\sinh\psi\sinh p\sinh q\;\Big) \\
&\quad - a\lambda A^{-1/2}\tanh\sigma\sinh p;
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
y_2 = a^2 \Big(& p^* A^{-1/2} \tanh\sigma \cosh\psi \sinh q \cosh p - q^* A^{-1/2} \tanh\sigma \cosh\psi \sinh p \cosh q \\
& -\psi^* A^{-3/2} \tanh\sigma \sinh\psi \cosh^2\psi \sinh p \sinh q - \psi^* A^{-1} \tanh\sigma \sinh 2\psi \sinh^2 p \\
& -\psi^* A^{-1/2} \tanh\sigma \sinh\psi \sinh p \sinh q \\
& +\sigma^* A^{-1} \operatorname{sech}^2\sigma \sinh^2\psi \sinh^2 p - \sigma^* A^{-3/2} \tanh^2\sigma \operatorname{sech}^2\sigma \cosh\psi \sinh p \sinh q \\
& +\sigma^* A^{-1/2} \operatorname{sech}^2\sigma \cosh\psi \sinh p \sinh q \Big) + a\lambda(A^{-1/2} \sinh\psi \cosh\psi \sinh p + \sinh\psi \sinh q);
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
y_3 = a^2 \Big(& p^* A^{-1/2} \tanh\sigma \sinh\psi \sinh q \cosh p - q^* A^{-1/2} \tanh\sigma \sinh\psi \sinh p \cosh q \\
& -\psi^* A^{-3/2} \tanh\sigma \sinh^2\psi \cosh\psi \sinh p \sinh q - \psi^* A^{-1} \tanh\sigma \cosh 2\psi \sinh^2 p \\
& -\psi^* A^{-1/2} \tanh\sigma \cosh\psi \sinh q \sinh p - \sigma^* A^{-3/2} \tanh^2\sigma \operatorname{sech}^2\sigma \sinh\psi \sinh p \sinh q \\
& +\sigma^* A^{-1} \operatorname{sech}^2\sigma \sinh\psi \cosh\psi \sinh^2 p + \sigma^* A^{-1/2} \operatorname{sech}^2\sigma \sinh\psi \sinh p \sinh q \Big) \\
& +a\lambda(A^{-1/2} \sinh^2\psi \sinh p + \cosh\psi \sinh q).
\end{aligned} \tag{10}
$$

If we take $\tilde{q} = q + \varepsilon q^* = 0$, then the condition (4) gives us $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{v}}_1$. Thus, from equations (8), (9) and (10) we have

$$
y_1 = -\psi^* \sinh^2\psi(\sinh^2\psi + \tanh^2\sigma)^{-1} - \lambda \tanh\sigma(\sinh^2\psi + \tanh^2\sigma)^{-1/2}, \tag{11}
$$

$$
\begin{aligned}
y_2 = & (-\psi^* \tanh\sigma \sinh 2\psi + \sigma^* \operatorname{sech}^2\sigma \sinh^2\psi)(\sinh^2\psi + \tanh^2\sigma)^{-1} \\
& +\lambda \sinh\psi \cosh\psi(\sinh^2\psi + \tanh^2\sigma)^{-1/2},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
y_3 = & \left(-\psi^* \tanh\sigma \cosh 2\psi + \sigma^* \operatorname{sech}^2\sigma \sinh\psi \cosh\psi\right)(\sinh^2\psi + \tanh^2\sigma)^{-1} \\
& +\lambda \sinh^2\psi (\sinh^2\psi + \tanh^2\sigma)^{-1/2}.
\end{aligned} \tag{13}
$$

If we put $\sigma = 0,\ \sigma^* \neq 0$ in equations (11), (12) and (13), then we get

$$
y_1 = -\psi^*, \tag{14}
$$

$$
y_2 = \sigma^* + \lambda \cosh\psi, \tag{15}
$$

$$
y_3 = \sigma^* \frac{\cosh\psi}{\sinh\psi} + \lambda \sinh\psi. \tag{16}
$$

Equations (14), (15) and (16) give a two-parameter family (linear congruence) of the timelike straight lines which are the intersection of the planes $y_1 = -\psi^*$ and the timelike ruled surfaces given by

$$
\cosh^2\psi \left(y_2 - \frac{\lambda}{\cosh\psi} \right)^2 - y_3^2 \sinh^2\psi = 0. \tag{17}
$$

Thus we give the following theorem.

Theorem 3.1: *During the dual hyperbolic conchoidal motion, in the case of* $\sigma = 0,\ \sigma^* \neq 0$*, the Study map in* IR_1^3 *of the orbit which is drawn on the* H' *by* $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{v}}_1$ *are the straight lines which are the intersections of the planes* $y_1 = -\psi^*$ *and the timelike ruled surfaces given by*

$$\cosh^2\psi\left(y_2 - \frac{\lambda}{\cosh\psi}\right)^2 - y_3^2\sinh^2\psi = 0\,.$$

Now, let us take $\tilde{p} = p + \varepsilon p^* = 0$ in equation (4). In this case, $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{v}}_3$. Thus from equations (8), (9) and (10)

$$y_1 = \psi^*\,, \tag{18}$$
$$y_2 = \lambda\sinh\psi\,, \tag{19}$$
$$y_3 = \lambda\cosh\psi\,, \tag{20}$$

are obtained. From equations (18), (19) and (20) we have

$$y_3^2 - y_2^2 = \lambda^2,\quad y_1 = \psi^*. \tag{21}$$

Thus we have the following theorem.

Theorem 3.2: *During the dual hyperbolic conchoidal motion* H/H'*, in the case of* $\tilde{p} = p + \varepsilon p^* = 0$ *in equation (4), the Study map of the orbit which is drawn on the* H' *by* $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{v}}_3$ *is the congruence,*

$$y_3^2 - y_2^2 = \lambda^2,\qquad y_1 = \psi^*.$$

Let us now give the analysis of the orbit of $\tilde{\boldsymbol{v}}_2$ during the dual hyperbolic conchoidal motion. We know that

$$\begin{aligned}\tilde{\boldsymbol{v}}_2 = \Big[& -\sinh\tilde{\psi}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}, -\cosh\tilde{\psi}\ \tanh\tilde{\Delta}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}, \\ & -\sinh\tilde{\psi}\ \tanh\tilde{\Delta}\left(\sinh^2\tilde{\psi} + \tanh^2\tilde{\Delta}\right)^{-1/2}\Big].\end{aligned} \tag{22}$$

From equation (22), we obtain

$$\boldsymbol{v}_2 = \left(-A^{-1/2}\sinh\psi,\ \ -A^{-1/2}\tanh\sigma\cosh\psi,\ \ -A^{-1/2}\tanh\sigma\sinh\psi\right) \tag{23}$$

and

$$\begin{aligned}\boldsymbol{v}_2^* = (&\psi^* A^{-3/2}\sinh^2\psi\cosh\psi - \psi^* A^{-1/2}\cosh\psi + \sigma^* A^{-3/2}\tanh\sigma\operatorname{sech}^2\sigma\sinh\psi,\\ &\psi^* A^{-3/2}\tanh\sigma\sinh\psi\cosh^2\psi - \psi^* A^{-1/2}\tanh\sigma\sinh\psi\\ &+\sigma^* A^{-3/2}\tanh^2\sigma\operatorname{sech}^2\sigma\cosh\psi - \sigma^* A^{-1/2}\operatorname{sech}^2\sigma\cosh\psi,\\ &+\psi^* A^{-3/2}\tanh\sigma\sinh^2\psi\cosh\psi - \psi^* A^{-1/2}\tanh\sigma\cosh\psi\\ &+\sigma^* A^{-3/2}\tanh^2\sigma\operatorname{sech}^2\sigma\sinh\psi - \sigma^* A^{-1/2}\operatorname{sech}^2\sigma\sinh\psi)\end{aligned} \tag{24}$$

where $\boldsymbol{v}_2$ and $\boldsymbol{v}_2^*$ are the real and dual parts of $\tilde{\boldsymbol{v}}_2$, respectively, and $A = \sinh^2\psi + \tanh^2\sigma$. Equations (23) and (24) depend on two parameters ψ and ψ^* so equations (23) and (24) represent a timelike line congruence in IR_1^3.

Let $\boldsymbol{g}$ denote the position vector of an arbitrary point $G(g_1, g_2, g_3)$ of a fixed timelike line x of the timelike congruence in IR_1^3. Then, considering equation (7) we have

$$\boldsymbol{g} = \boldsymbol{v}_2(\psi,\psi^*)\times\boldsymbol{v}_2^*(\psi,\psi^*) + u\boldsymbol{v}_2(\psi,\psi^*) . \tag{25}$$

Since (g_1, g_2, g_3) are the coordinates of G we have

$$g_1 = -\psi^* A^{-1}\tanh^2\sigma - uA^{-1/2}\sinh\psi, \tag{26}$$
$$g_2 = \sigma^* A^{-1}\operatorname{sech}^2\sigma\sinh^2\psi - uA^{-1/2}\tanh\sigma\cosh\psi, \tag{27}$$
$$g_3 = -\psi^* A^{-1}\tanh\sigma + \sigma^* A^{-1}\operatorname{sech}^2\sigma\sinh\psi\cosh\psi - uA^{-1/2}\tanh\sigma\sinh\psi , \tag{28}$$

where $A = \sinh^2\psi + \tanh^2\sigma$.

If we take $\sigma = 0$, $\sigma^* \neq 0$ in equations (26), (27) and (28), then

$$g_1 = -u,\ \ g_2 = \sigma^*,\ \ g_3 = \sigma^*\coth\psi . \tag{29}$$

In this case, if we choose $u = -k\psi$ (k constant) in equation (29) we have

$$g_1 = k\tanh^{-1}\left(\frac{g_2}{g_3}\right), \tag{30}$$

which is a Lorentzian helicoid.

If we take $\psi = 0$, $\psi^* \neq 0$ in equations (26), (27) and (28), then

$$g_1 = -\psi^*,\quad g_2 = -u,\quad g_3 = -\psi^*\coth\sigma . \tag{31}$$

If we choose $u = -k\sigma$ (k constant) in equation (31) we have

$$g_2 = k \tanh^{-1}\left(\frac{g_1}{g_3}\right),$$

which is also a Lorentzian helicoid.

4 Conclusions

This paper presents the conchoidal motion on the dual hyperbolic unit sphere $\tilde{\mathbb{H}}_0^2$ in the dual Lorentzian space $\mathbb{D}_1^3$. The orbits drawn on the fixed dual hyperbolic unit sphere by unit dual vectors of an orthonormal base $\{\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3\}$ are obtained. This motion and its results carried to the Lorentzian line space IR_1^3 by means of the Study's mapping. The results may give a way to define new motions and contribute to the study of surface design, manufacturing technology, robotic research, and special and general theory of relativity, and many other areas in three-dimensional Lorentzian space.

Acknowledgements: The authors would like to thank the reviewers and editor for their valuable comments and suggestions to improve the quality of the paper.

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