Gen. Math. Notes, Vol. 31, No. 2, December 2015, pp.16-28
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Product of Statistical Manifolds with Doubly Warped Product

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(Received: 16-10-15 / Accepted: 27-11-15)

Abstract

In this paper, we generalize the dualistic structures on warped product manifolds to the dualistic structures on doubly warped product. We extend some results related to the dualistic structure on doubly warped product studied in [8]. We also demonstrate that a dualistic structure on a doubly warped product manifold $(M_1 \times M_1, g_{f_1 f_2})$ induces dualistic structures on the manifolds M_1 and M_2 and conversely, in this case doubly warped product manifold $(M_1 \times M_1, g_{f_1 f_2})$ is a statistical manifold if and only if (M_1, g_1) and (M_1, g_1) are.

Keywords: *Conjugate, doubly warped products, dual connection, product manifold.*

1 Introduction

The warped product provides a way to construct new pseudo-riemannian manifolds from the given ones, see [13],[12] and [11]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows:

Definition 1.1 *Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds and let $f_1 : M_1 \longrightarrow \mathcal{R}^*$ be a positive smooth function on M_1, the warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped* *with the metric tensor* $g_{f_1} := \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$, *where* π_1 *and* π_2 *are the projections of* $M_1 \times M_2$ *onto* M_1 *and* M_2 *respectively.*

The manifold M_1 is called the base of $(M_1 \times M_2, g_{f_1})$ and M_2 is called the fiber. The function f_1 is called the warping function.
The doubly warped product construction in the class of pseudo-Riemannian manifolds generalized the warped product and the direct product. It is obtained by homothetically distorting the geometry of each base $M_1 \times \{q\}$ and each fiber $\{p\} \times M_2$ to get a new "doubly warped" metric tensor on the product manifold and defined as follows:
For $i \in \{1, 2\}$, let M_i be a pseudo-Riemannian manifold equipped with metric g_i, and $f_i : M_i \to \mathcal{R}^*$ be a positive smooth function on M_i. The well-know notion of doubly warped product manifold $M_1 \times_{f_1 f_2} M_2$ is defined as the product manifold $M = M_1 \times M_2$ equipped with pseudo-Riemannian metric which is denoted by $g_{f_1 f_2}$, given by

$$g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2 .$$

In the cases $f_1 = 1$ or $f_2 = 1$ we obtain a warped product or a direct product.

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections on statistical manifolds, compatible with a pseudo-Riemannian metric [1]. Their importance in statistical physics was underlined by many authors; see [3],[4],[5] etc.
Let M be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric g and let ∇, ∇^* be the affine connections on M. We say that a pair of affine connections ∇ and ∇^* are compatible (or conjugate) with respect to g if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \quad for\ all\ X, Y, Z \in \Gamma(TM), \qquad (1)$$

where $\Gamma(TM)$ is the set of all tangent vector fields on M. Then the triplet (g, ∇, ∇^*) is called the dualistic structure on M.
We note that the notion of "conjugate connection " has been attributed to A.P. Norden in affine differential geometry literature (Simon, 2000) and was independently introduced by (Nagaoka and Amari, 1982) in information geometry, where it was called " dual connection" (Lauritzen, 1987). The triplet (M, ∇, g) is called a statistical manifold if it admits another torsion-free connection ∇^* satisfying the equation (1). We call ∇ and ∇^* dual of each other with respect to g.

In the notions of terms on statistical manifolds, for a torsion-free affine connection ∇ and a pseudo-Riemannian metric g on a manifold M, the triple (M, ∇, g) is called a statistical manifold if ∇g is symmetric. If the curvature tensor R of ∇ vanishes, (M, ∇, g) is said to be flat.

This paper extends the study of dualistic structures on warped product and double warped product manifolds in the papers [7] and [8].

The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, the notion of conjugate, horizontal and vertical lifts. In section 3, we define the co-metric $\tilde{g}_{f_1 f_2}$ of $g_{f_1 f_2}$ the metric of the doubly warped products by using the musical isomorphisms, we calculate the gradient of the lift of f_1 (resp. f_2), it has been shown that its gradient is horizontal (resp. vertical) and π_1 related to gradient of f_1 on M_1 (resp. π_2 related to gradient of f_2 on M_2), we show that dualistic structures on manifolds (M_1, g_1) and (M_2, g_2) induce the dualistic structure on the doubly warped product manifold $(M_1 \times M_2, g_{f_1 f_2})$ and conversely. Moreover (M_1, g_1) and (M_2, g_2) are statistical manifolds if and only if $(M_1 \times M_2, g_{f_1 f_2})$ is a statistical manifold.

2 Preliminaries

2.1 Statistical Manifolds

We recall some standard facts about Levi-Civita connections and the dual statistical manifold. Many fundamental definitions and results about dualistic structure can be found in Amari's monograph ([1],[2]).

Let (M, g) be a pseudo-Riemannian manifold. The metric g defines the musical isomorphisms

$$\begin{array}{rccc} \sharp_g : & \Gamma(T^*M) & \rightarrow & \Gamma(TM) \\ & \alpha & \mapsto & \sharp_g(\alpha) \end{array}$$

such that $g(\sharp_g(\alpha), Y) = \alpha(Y)$, and its inverse $\flat_g$. We can thus define the co-metric $\widetilde{g}$ of the metric g by :

$$\widetilde{g}(\alpha, \beta) = g(\sharp_g(\alpha), \sharp_g(\beta)). \quad (2)$$

A fundamental theorem of pseudo-Riemannian geometry states that given a pseudo-Riemannian metric g on the tangent bundle TM, there is a unique connection (among the class of torsion-free connection) that "preserves" the metric; as long as the following condition is satisfied:

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z) \quad for\ X, Y, Z \in \Gamma(TM) \quad (3)$$

Such a connection, denoted as $\hat{\nabla}$, is known as the Levi-Civita connection. Its component forms, called Christoffel symbols, are determined by the components of pseudo-metric tensor as ("Christoffel symbols of the second Kink")

$$\hat{\Gamma}_{ij}^{k} = \sum_{l} \frac{1}{2} g^{kl} (\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l})$$

and ("Christoffel symbols of the first Kink")

$$\hat{\Gamma}_{ij,k} = \frac{1}{2} (\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}).$$

The Levi-Civita connection is compatible with the pseudo metric, in the sense that it treats tangent vectors of the shortest curves on a manifold as being parallel.

It turns out that one can define a kind of "Compatibility" relation more generally than expressed by the (3), by introducing the notion of "Conjugate" (denoted by *) between two affine connections.

Let (M, g) be a pseudo-Riemannian manifold and let ∇, ∇^* be affine connections on M. A connection ∇^* is said to be "conjugate" to ∇ with respect to g if

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \quad for\ X, Y, Z \in \Gamma(TM). \tag{4}$$

Clearly,

$$(\nabla^*)^* = \nabla.$$

Otherwise, $\hat{\nabla}$, which satisfies the (3), is special in the sense that it is self-conjugate

$$(\hat{\nabla})^* = \hat{\nabla}.$$

Because pseudo-metric tensor g provides a one-to-one mapping between vectors in the tangent space and co-vectors in the cotangent space, the equation (1) can also be seen as characterizing how co-vector fields are to be parallel-transported in order to preserve their dual pairing $< \cdot, \cdot >$ with vector fields.
Writing out the equation (1) explicitly,

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki,j} + \Gamma_{kj,i}^*, \tag{5}$$

where

$$\nabla_{\partial_i}^* \partial_j = \sum_l \Gamma_{ij}^{*l} \partial_l$$

so that

$$\Gamma_{kj,i}^* = g(\nabla_{\partial_j}^* \partial_k, \partial_i) = \sum_l g_{il} \Gamma_{kj}^{*l}.$$

In the following, a manifold M with a pseudo-metric g and a pair of conjugate connections ∇, ∇^* with respect to g is called a " pseudo-Riemannian manifold with dualistic structure " and denoted by (M, g, ∇, ∇^*).
Obviously, ∇ and ∇^* (or equivalently, Γ and Γ^*) satisfy the relation

$$\hat{\nabla} = \frac{1}{2}(\nabla + \nabla^*) \quad (\textit{or equivalently}, \hat{\Gamma} = \frac{1}{2}(\Gamma + \Gamma^*)).$$

Thus an affine connection ∇ on (M, g) is metric if and only if $\nabla^* = \nabla$ (that it is self-conjugate).
For a torsion-free affine connection ∇ and a pseudo-Riemannian metric g on a manifold M, the triplet (M, ∇, g) is called a statistical manifold if ∇g is symmetric. If the curvature tensor $\mathcal{R}$ of ∇ vanishes, (M, ∇, g) is said to be flat.
For a statistical manifold (M, ∇, g), the conjugate connection ∇^* with respect to g is torsion-free and $\nabla^* g$ symmetric. Then the triplet (M, ∇^*, g) is called the dual statistical manifold of (M, ∇, g) and (∇, ∇^*, g) the dualistic structure on M. The curvature tensor of ∇ vanishes if and only if that of ∇^* does and in such a case, (∇, ∇^*, g) is called the dually flat structure [2].
It can be shown that for a pair of conjugate connections ∇, ∇^*, their curvature tensors $\mathcal{R}$, $\mathcal{R}^*$ satisfy

$$g(\mathcal{R}(X, Y)Z, W) + g(Z, \mathcal{R}^*(X, Y)W) = 0. \tag{6}$$

If the curvature tensor $\mathcal{R}$ of ∇ vanishes, ∇ is said to be flat.
So, ∇ is flat if and only if ∇^* is flat. In this case, (M, g, ∇, ∇^*) is said to be dually flat.

2.2 Horizontal and Vertical Lifts

Throughout this paper M_1 and M_2 will be respectively m_1 and m_2 dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and

$$\pi_1 : M_1 \times M_2 \to M_1 \quad , \quad \pi_2 : M_1 \times M_2 \to M_2$$

the usual projection maps. We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of M_1 and M_2 separately. For details see [13].

Let φ_1 in $C^\infty(M_1)$. The horizontal lift of φ_1 to $M_1 \times M_2$ is $\varphi_1^h = \varphi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p \in M_1$ and $X_p \in T_pM_1$. For any $q \in M_2$ the horizontal lift of X_p to $T_{(p,q)}(M_1 \times M_2)$ is the unique tangent vector $X^h_{(p,q)}$ in $T_{(p,q)}(M_1 \times \{q\})$ such that

$$\begin{cases} d_{(p,q)}\pi_1(X^h_{(p,q)}) = X_p, \\ d_{(p,q)}\pi_2(X^h_{(p,q)}) = 0. \end{cases}$$

We can also define the horizontal lifts of vector fields as follows. Let $X_1 \in \Gamma(TM_1)$. The horizontal lift of X_1 to $\Gamma(T(M_1 \times M_2))$ is the vector field $X_1^h \in \Gamma(T(M_1 \times M_2))$ whose value at each (p,q) is the horizontal lift of the tangent vector $(X_1)p$ to $T_{(p,q)}(M_1 \times M_2)$. For $(p,q) \in M_1 \times M_2$, we will denote the set of the horizontal lifts to $T_{(p,q)}(M_1 \times M_2)$ of all the tangent vectors of M_1 at p by $\mathcal{L}^h_{(p,q)}(M_1)$. We will denote the set of the horizontal lifts of all vector fields on M_1 by $\mathcal{L}^h(M_1)$.

The vertical lift φ_2^v of a function $\varphi_2 \in C^\infty(M_2)$ to $M_1 \times M_2$ and the vertical lift X_2^v of a vector field $X_2 \in \Gamma(TM_2)$ to $\Gamma(T(M_1 \times M_2))$ are defined in the same way using the projection π_2. Note that the spaces $\mathcal{L}^h(M_1)$ of the horizontal lifts and $\mathcal{L}^v(M_2)$ of the vertical lifts are vector subspaces of $\Gamma(T(M_1 \times M_2))$ but neither is invariant under multiplication by arbitrary functions $\varphi \in C^\infty(M_1 \times M_2)$.

We define the horizontal lift of a covariant tensor ω_1 on M_1 to be its pullback ω_1^h to $M_1 \times M_2$ by the means of the projection map π_1, i.e. $\omega_1^h := \pi_1^*(\omega_1)$. In particular, for a 1-form α_1 on M_1 and a vector field X on $M_1 \times M_2$, we have

$$(\alpha_1^h)(X) = \alpha_1(d\pi_1(X)).$$

Explicitly, if u is a tangent vector to $M_1 \times M_2$ at (p,q), then

$$(\alpha_1^h)_{(p,q)}(u) = (\alpha_1)_p(d_{(p,q)}\pi_1(u)).$$

Similarly, we define the vertical lift of a covariant tensor w_2 on M_2 to be its pullback ω_2^v to $M_1 \times M_2$ by the means of the projection map π_2.

Observe that if $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m_1}}\}$ is the local basis of the vector fields (resp. $\{dx^1, \ldots, dx^{m_1}\}$ is the local basis of 1-forms) relative to a chart (U, Φ) of M_1 and $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{m_2}}\}$ is the local basis of the vector fields (resp. $\{dy^1, \ldots, dy^{m_2}\}$ the local basis of the 1-forms) relative to a chart (V, Ψ) of M_2, then $\{(\frac{\partial}{\partial x^1})^h, \ldots, (\frac{\partial}{\partial x^{m_1}})^h, (\frac{\partial}{\partial y^1})^v, \ldots, (\frac{\partial}{\partial y^{m_2}})^v\}$ is the local basis of the vector fields (resp. $\{(dx_1)^h, \ldots, (dx_{m_1})^h, (dy_1)^v, \ldots, (dy_{m_2})^v\}$ is the local basis of the 1-forms) relative to the chart $(U \times V, \Phi \times \Psi)$ of $M_1 \times M_2$.

The following lemma will be useful later for our computations.

Lemma 2.1 *[10]*

1. *Let $\varphi_i \in C^\infty(M_i)$, $X_i, Y_i \in \Gamma(TM_i)$, $\alpha_i \in \Gamma(T^*M_i)$, $i = 1, 2$, let $\varphi = \varphi_1^h + \varphi_2^v$, $X = X_1^h + X_2^v$ and $\alpha, \beta \in \Gamma(T^*(M_1 \times M_2))$. Then*

   i/ For all $(i, I) \in \{(1,h), (2,v)\}$ we have

$$X_i^I(\varphi) = X_i(\varphi_i)^I, \quad [X, Y_i^I] = [X_i, Y_i]^I \quad \text{and} \quad \alpha_i^I(X) = \alpha_i(X_i)^I.$$

ii/ If for all $(i,I)\in\{(1,h),(2,v)\}$ we have $\alpha(X_i^I)=\beta(X_i^I)$, then $\alpha=\beta$.

2. *Let ω_i and η_i be r-forms on M_i, $i=1,2$, and set $\omega=\omega_1^h+\omega_2^v$ and $\eta=\eta_1^h+\eta_2^v$. Then we have*

$$d\omega=(d\omega_1)^h+(d\omega_2)^v \quad \text{and} \quad \omega\wedge\eta=(\omega_1\wedge\eta_1)^h+(\omega_2\wedge\eta_2)^v.$$

Remark 2.2 *Let X be a vector field on $M_1\times M_2$, such that $d\pi_1(X)=\varphi(X_1\circ\pi_1)$ and $d\pi_2(X)=\phi(X_2\circ\pi_2)$. Then $X=\varphi X_1^h+\phi X_2^v$.*

3 About Doubly Warped Products

3.1 The Doubly Warped Product

let $\psi:M\to N$ be a smooth map between smooth manifolds and g be a metric on k-vector bundle (F,P_F) over N. The metric $g^\psi:\Gamma(\psi^{-1}F)\times\Gamma(\psi^{-1}F)\to C^\infty(M)$ on the pull-back $(\psi^{-1}F,P_{\psi^{-1}F})$ over M is defined by

$$g^\psi(U,V)(p)=g_{\psi(p)}(U_p,V_p),\quad \forall\ U,V\in\Gamma(\psi^{-1}F),p\in M.$$

Given a linear connection ∇^N on k-vector bundle (F,P_F) over N, the pull-back connection $\overset{\psi}{\nabla}$ is the unique linear connection on the pull-back $(\psi^{-1}F,P_{\psi^{-1}F})$ over M such that, for each $W\in\Gamma(F)$, $X\in\Gamma(TM)$

$$\overset{\psi}{\nabla}_X(W\circ\psi)=\nabla^N_{d\psi(X)}W. \tag{7}$$

Further, let $U\in\psi^{-1}F$, $p\in M$ and $X\in\Gamma(TM)$. Then

$$(\overset{\psi}{\nabla}_XU)(p)=(\nabla^N_{d_p\psi(X_p)}\widetilde{U})(\psi(p)), \tag{8}$$

where $\widetilde{U}\in\Gamma(F)$ with $\widetilde{U}\circ\psi=U$.
Now, let π_i, i=1,2, be the usual projection of $M_1\times M_2$ onto M_i, given a linear connection $\overset{i}{\nabla}$ on vector bundle $\Gamma(TM_i)$, the pull-back connection $\overset{\pi_i}{\nabla}$ is the unique linear connection on the pull-back $M_1\times M_2\to\pi_i^{-1}(TM_i)$, such that, for each $Y_i\in\Gamma(TM_i)$, $X\in\Gamma(TM_1\times M_2)$

$$\overset{\pi_i}{\nabla}_XY_i\circ\pi_i=\overset{i}{\nabla}_{d\pi_i(X)}Y_i. \tag{9}$$

Further, let $U\in\Gamma(\pi_i^{-1}(TM_i))$, $(p,q)\in M_1\times M_2$ and $X\in\Gamma(T(M_1\times M_2))$. Then

$$(\overset{\pi_i}{\nabla}_XU)(p,q)=(\overset{i}{\nabla}_{d_{(p,q)}\pi_i(X_{(p,q)})}\widetilde{U})\pi_i(p,q), \tag{10}$$

Definition 3.1 *Let (M_1, g_1) and (M_2, g_2) be pseudo-Riemannian manifolds and let $f_1 : M_1 \to \mathcal{R}^*$ and $f_2 : M_2 \to \mathcal{R}^*$ be a positive smooth functions. The Doubly warped product is the product manifold $M_1 \times M_2$ furnished with the metric tensor $g_{f_1 f_2}$ defined by*

$$g_{f_1 f_2} = (f_2^v)^2 \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2. \tag{11}$$

Explicitly, if $X, Y \in \Gamma(TM_1 \times M_2)$, then

$$g_{f_1 f_2}(X,Y) = (f_2^v)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)).$$

By analogy with [6] we will denote this structure by $M_1 \times_{f_1 f_2} M_2$. The function $f_i : M_i \to \mathcal{R}^+ - \{0\}$ $(i \in \{1,2\})$ is called the warping function.
If (M_1, g_1) and (M_2, g_2) are both Riemannian manifolds, then $M_1 \times_{f_1 f_2} M_2$ is also a Riemannian manifold. We call $M_1 \times_{f_1 f_2} M_2$ as a Lorentzian doubly warped product if (M_2, g_2) is Riemannian and either (M_1, g_1) is Lorentzian or else (M_1, g_1) is a one-dimensional manifold with a negative definite metric $-dt^2$.

Proposition 3.2 *With the notation above, let $X_i, Y_i \in \Gamma(TM_i)$, $i = 1, 2$. Then the equation (11) is equivalent to*

$$\left\{ \begin{array}{l} g_{f_1 f_2}(X_1^h, Y_1^h) = (f_2^v)^2 g_1(X_1, Y_1)^h; \\ g_{f_1 f_2}(X_1^h, Y_2^v) = g_{f_1 f_2}(X_2^v, Y_1^h) = 0; \\ g_{f_1 f_2}(X_2^v, Y_2^v) = (f_1^h)^2 g_2(X_2, Y_2)^v. \end{array} \right. \tag{12}$$

Proof: By definition of the doubly warped metric,

$$g_{f_1 f_2}(X_1^h, Y_1^h)(p,q) = f_2^2(q) g_1(X_1, Y_1)(p) \quad \textit{and} \quad g_{f_1 f_2}(X_1^h, Y_2^v)(p,q) = 0.$$

Writing f_1^h for $f_1 \circ \pi_1$ and f_2^v for $f_2 \circ \pi_2$. Then it is easily seen that Equation (12) hold.

A direct computation using Proposition 3.2 and the definition of the musical isomorphism gives the following proposition.

Proposition 3.3 *([9]) Let (M_i, g_i)be a pseudo-Riemannian manifold and let $f_i : M_i \to \mathcal{R}_+^*$, be a positive smooth function, $i = 1, 2$. The co-metric $\widetilde{g}_{f_1, f_2}$ of g_{f_1, f_2} is characterized by the following identities*

$$\left\{ \begin{array}{l} \widetilde{g}_{f_1 f_2}(\alpha_1^h, \beta_1^h) = \frac{1}{(f_2^v)^2} \widetilde{g}_1(\alpha_1, \beta_1)^h; \\ \widetilde{g}_{f_1 f_2}(\alpha_1^h, \beta_2^v) = \widetilde{g}_{f_1 f_2}(\alpha_2^v, \beta_1^h) = 0; \\ \widetilde{g}_{f_1 f_2}(\alpha_2^v, \beta_2^v) = \frac{1}{(f_1^h)^2} \widetilde{g}_2(\alpha_2, \beta_2)^v. \end{array} \right. \tag{13}$$

for any $\alpha_i, \beta_i \in \Gamma(T^* M_i)$, $i = 1, 2$. Where $\widetilde{g}_i$ is the co-metric of g_i.

Lemma 3.4 *If $f_i \in C^\infty(M_i)$, $i = 1, 2$. Then the gradient of the lifts f_1^h of f_1 and f_2^v of f_2 to $M_1 \times_{f_1 f_2} M_2$ w.r.t. $g_{f_1 f_2}$ is*

$$grad(f_1^h) = \frac{1}{(f_2^v)^2}(gradf_1)^h \qquad , \qquad grad(f_2^v) = \frac{1}{(f_1^h)^2}(gradf_2)^v \qquad (14)$$

Proof: Let $Z_i \in \Gamma(TM_i)$, $i = 1, 2$. Then, for any $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, we have

$$g_{f_1 f_2}(grad(f_i^I), Z_i^I) = (Z_i(f_i))^I = g_i(gradf_i, Z_i)^I = \tfrac{1}{(f_{3-i}^J)^2} g_{f_1 f_2}((gradf_i)^I, Z_i^I),$$

and

$$g_{f_1 f_2}(grad(f_i^I), Z_{3-i}^J) = 0.$$

Therefore, from Equation (12), we get

$$grad(f_i^I) = \frac{1}{(f_{3-i}^J)^2}(gradf_i)^I.$$

3.2 Dualistic Structure on Doubly Warped Products

Proposition 3.5 *Let $(g_{f_1 f_2}, \nabla, \nabla^*)$ be a dualistic structure on $M_1 \times M_2$. Then there exists an affine connections $\overset{i}{\nabla}$, $\overset{i}{\nabla}{}^*$ on M_i, such that $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is a dualistic structure on M_i, $i = 1, 2$.*

Proof: Taking the affine connections on M_i, $i = 1, 2$.

$$\begin{cases} (\overset{i}{\nabla}_{X_i} Y_i) \circ \pi_i = d\pi_i(\nabla_{X_i^I} Y_i^I), & \forall\ X_i, Y_i \in \Gamma(TM_1) \\ (\overset{i}{\nabla}{}^*_{X_i} Y_i) \circ \pi_i = d\pi_i(\nabla^*_{X_i^I} Y_i^I). & \forall\ (i, I) \in \{(1, h), (2, v)\} \end{cases}$$

Therefore, we have for all $X_i, Y_i, Z_i \in \Gamma(TM_i)$.

$$X_i^I(g_{f_1 f_2}(Y_i^I, Z_i^I)) = g_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) + g_{f_1 f_2}(Y_i^I, \nabla^*_{X_i^I} Z_i^I). \qquad (15)$$

Since, $d\pi_{3-i}(Z_i^I) = 0$, $X_i^I(f_{3-i}^J) = 0$ and $g_{f_1 f_2}(X, Z_i^I) = (f_{3-i}^J)^2 g_i^{\pi_i}(d\pi_i(X), Z_i \circ \pi_i)$, for any $X \in \Gamma(TM_1 \times M_2)$, then the equation (15) is equivalent to

$$(f_{3-i}^J)^2 (X_i(g_i(Y_i, Z_i)))^I = (f_{3-i}^J)^2 \{g_i(\overset{i}{\nabla}_{X_i} Y_i, Z_i) + g_i(Y_i, \overset{i}{\nabla}{}^*_{X_i} Z_i)\}^I.$$

Where $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$. Hence, the pair of affine connections $\overset{i}{\nabla}$ and $\overset{i}{\nabla}{}^*$ are conjugate with respect to g_i.

Proposition 3.6 *Let $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ be a dualistic structure on M_i, $i = 1, 2$. Then there exists a dualistic structure on $M_1 \times M_2$ with respect to $g_{f_1 f_2}$.*

Proof: Let ∇ and ∇^* be the connections on $M_1 \times M_2$ given by

$$\begin{cases} d\pi_i(\nabla_X Y) & = \overset{\pi_i}{\nabla}_X d\pi_i(Y) + Y(\ln f^J_{3-i}) d\pi_i(X) + X(\ln f^J_{3-i}) d\pi_i(Y) \\ & -(f^J_{3-i})^{-2} f^I_i g^{\pi_{3-i}}_{3-i}(d\pi_{3-i}(X), d\pi_{3-i}(Y))((gradf_i) \circ \pi_i), \\ \\ d\pi_i(\nabla^*_X Y) & = \overset{\pi_i}{\nabla}{}^*_X d\pi_i(Y) + Y(\ln f^J_{3-i}) d\pi_i(X) + X(\ln f^J_{3-i}) d\pi_i(Y) \\ & -(f^J_{3-i})^{-2} f^I_i g^{\pi_{3-i}}_{3-i}(d\pi_{3-i}(X), d\pi_{3-i}(Y))((gradf_i) \circ \pi_i), \end{cases} \tag{16}$$

for any $X, Y \in \Gamma(TM_1 \times M_2)$.
Or, for any $X_i, Y_i \in \Gamma(TM_i)$ we have

$$\begin{cases} \nabla_{X^I_i} Y^I_i = (\overset{i}{\nabla}_{X_i} Y_i)^I - (\frac{g_i(X_i, Y_i)}{2f_i^2})^I (gradf^2_{3-i})^J; \\ \nabla^*_{X^I_i} Y^I_i = (\overset{i}{\nabla}{}^*_{X_i} Y_i)^I - (\frac{g_i(X_i, Y_i)}{2f_i^2})^I (gradf^2_{3-i})^J; \\ \nabla_{X^I_i} Y^J_{3-i} = \nabla^*_{X^I_i} Y^J_{3-i} = (X_i(\ln f_i))^I Y^J_{3-i} + (Y_{3-i}(\ln f_{3-i}))^J X^I_i, \end{cases} \tag{17}$$

where $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$. Let us assume that $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is a dualistic structure on M_i, $i = 1, 2$. Let A be the tensor field of type $(0, 3)$ defined by

$$A(X, Y, Z) = X(g_{f_1 f_2}(Y, Z)) - g_{f_1 f_2}(\nabla_X Y, Z) - g_{f_1 f_2}(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in \Gamma(TM_1 \times M_2)$, if $X_i, Y_i, Z_i \in \Gamma(TM_i)$, $i = 1, 2$. Then we have

$$X^I_i(g_{f_1 f_2}(Y^I_i, Z^I_i)) = X^I_i((f^J_{3-i})^2 g_i(X_i, Y_i)^I).$$

Since $d\pi_{3-i}(X^I_i) = 0$, it follows that $d\pi_{3-i}(X^I_i)(f_{3-i}) = X^I_i(f^J_{3-i}) = 0$, and hence

$$X^I_i(g_{f_1 f_2}(Y^I_i, Z^I_i)) = (f^J_{3-i})^2 (X(g_i(Y_i, Z_i)))^I.$$

As $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is dualistic structure, we have thus

$$X^I_i(g_{f_1 f_2}(Y^I_i, Z^I_i)) = (f^J_{3-i})^2 \{g_i(\overset{i}{\nabla}_{X_i} Y_i, Z_i)^I + g_i(Y_i, \overset{i}{\nabla}{}^*_{X_i} Z_i)^I\}.$$

From Proposition 3.2 and Equations (17), then it's easily seen that the following equation holds

$$A(X^I_i, Y^I_i, Z^I_i) = 0.$$

In the different lifts, we have

$$X^J_{3-i}(g_{f_1 f_2}(Y^I_i, Z^I_i)) = 2g_i(Y_i, Z_i)^I (f^J_{3-i} X_{3-i}(f_{3-i}))^J,$$

$$g_{f_1 f_2}(\nabla_{X^J_{3-i}} Y^I_i, Z^I_i) = g_{f_1 f_2}((X_{3-i}(\ln f_{3-i}))^J Y^I_i, Z^I_i) = (f^2_{3-i} X_{3-i}(f_{3-i}))^J g_i(Y_i, Z_i)^I,$$

and

$$g_{f_1 f_2}(\nabla^*_{X^J_{3-i}} Z^I_i, Y^I_i) = g_{f_1 f_2}(\nabla_{X^J_{3-i}} Z^I_i, Y^I_i) = (f^2_{3-i} X_{3-i}(f_{3-i}))^J g_i(Y_i, Z_i)^I.$$

We add these equations and obtain

$$A(X^J_{3-i}, Y^I_i, Z^I_i) = 0$$

Hence the same applies for $A(X^I_i, Y^I_i, Z^J_{3-i}) = A(X^J_{3-i}, Y^I_i, Z^I_i) = 0$.
This proves that ∇^* is conjugate to ∇ with respect to $g_{f_1 f_2}$.

We recall that the connection ∇ on $M_1 \times M_2$ induced by $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ on M_1 and M_2 respectively, is given by the equations (17).

Proposition 3.7 *$(M_1, \overset{1}{\nabla}, g_1)$ and $(M_2, \overset{2}{\nabla}, g_2)$ are statistical manifolds if and only if $(M_1 \times M_2, g_{f_1 f_2}, \nabla)$ is a statistical manifold.*

Proof: Let us assume that $(M_i, \overset{i}{\nabla}, g_i)$, $(i = 1, 2)$ is a statistical manifold.
Firstly, we show that ∇ is torsion-free. Indeed; by Equation (16), we have for any $X, Y \in \Gamma(TM_1 \times M_2)$

$$d\pi_i(T(X,Y)) = \overset{\pi_i}{\nabla}_X d\pi_i(Y) - \overset{\pi_i}{\nabla}_Y d\pi_i(X) - d\pi_i([X,Y])$$

Since for $i = 1, 2$, $\overset{i}{\nabla}$ is torsion-free, then

$$\overset{\pi_i}{\nabla}_X d\pi_i(Y) - \overset{\pi_i}{\nabla}_Y d\pi_i(X) = d\pi_i([X,Y])$$

Therefore, from Remark 2.2, the connection ∇ is torsion-free.
Secondly, we show that $\nabla g_{f_1, f_2}$ is symmetric. In fact; for $i = 1, 2$,

$$(\nabla g_{f_1 f_2})(X^I_i, Y^I_i, Z^J_i) = X^I_i(g_{f_1 f_2}(Y^I_i, Z^I_i)) - g_{f_1 f_2}(\nabla_{X^I_i} Y^I_i, Z^I_i) - g_{f_1 f_2}(Y^I_i, \nabla_{X^I_i} Z^I_i)$$

by Equations (12), (17) and since $(\overset{i}{\nabla} g_i)$, $i = 1, 2$, is symmetric, we have

$$\begin{aligned}
(\nabla g_{f_1 f_2})(X^I_i, Y^I_I, Z^I_i) &= (f^J_{3-i})^2((\overset{i}{\nabla} g_i)(X_i, Y_i, Z_i))^I \\
&= (f^J_{3-i})^2((\overset{i}{\nabla} g_i)(Y_i, X_i, Z_i))^h \\
&= (\nabla g_{f_1 f_2})(Y^I_i, X^I_I, Z^I_i).
\end{aligned}$$

In the different lifts, we have

$$(\nabla g_{f_1 f_2})(X^I_i, Y^I_i, Z^J_{3-i}) = (\nabla g_{f_1 f_2})(X^J_{3-i}, Y^I_i, Z^I_i) = (\nabla g_{f_1 f_2})(X^I_i, Y^I_{3-i}, Z^I_i) = 0,$$

Therefore, $(\nabla g_{f_1 f_2})$ is symmetric. Thus $(M_1 \times M_2, g_{f_1 f_2}, \nabla)$ is a statistical manifold.

Conversely, if $(M_1 \times M_2, g_{f_1 f_2}, \nabla)$ is statistical manifold, then $(\nabla g_{f_1 f_2})$ is symmetric and ∇ is torsion-free, particularly, when $X_i, Y_i, Z_i \in \Gamma(TM_i)$, we have

$$\begin{cases} (\nabla g_{f_1 f_2})(X^I_i, Y^I_I, Z^I_i) = (\nabla g_{f_1 f_2})(Y^I_i, X^I_I, Z^I_i), \\ T(X^I_i, Y^I_i) = 0. \end{cases} \qquad \forall\ i = 1, 2,$$

Then, by Equations (12) and (17), we obtained, for $i = 1, 2$, $\overset{i}{\nabla} g_i$, is symmetric and $\overset{i}{\nabla}$, is torsion-free. Therefore, $(M_i, \overset{i}{\nabla}, g_i)$, $(i = 1, 2)$ is statistical manifold.

Acknowledgements: A big part of this work was done at The Raphael Salem Laboratory of mathematics, University of Rouen (France), Djelloul Djebbouri will like to thank Raynaud de Fitte Paul, Simon Raoult for very useful discussions and the Mathematic section for their hospitality.

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