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On the Rotation Entropy of Additive Cellular Automata f_∞

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Abstract

In this paper, we first introduce the definition of rotational sets for cellular automata f_∞ using Misiurewicz's rotational sets. After that, we define rotational entropy functions for cellular automata f_∞ from Bowen's definiton of rotational entropy. Finally, we compare rotational entropy function with topologic entropy function for cellular automata f_∞.

Keywords: *Rotation Set, Rotation Entropy, Topological Entropy, Cellular Automata.*

1 Introduction and Background

In this paper, we study rotational entropy of cellular automata f_∞. We define rotational entropy for cellular automata f_∞ and prove that it is a topological invariant closely related to the rotation set Poincaré associated to each orientation preserving homeomorphism of the circle a number, designated rotation number that quantifies the asymptotic behaviour of different orbits. Furthermore there are orbits whose rotational behaviour is so chaotic that one cannot associate a single number to its wrapping. In general, the rotation of each orbit is captured by its rotational interval. The unions of all rotation intervals of a map is designated the rotation set of the given mapping ([1], [2], [3]).

Frank in ([4],[5]) proved that any orbit of an annulus homeomorphism isotopic to the identity, with finitely many periods has rotation number. Therefore examples with chaotic rotations are a topological invariant that roughly tells how many different orbits a map has. However we may have a positive entropy homeomorphism with trivial rotation set. Bowen's definition of topological entropy suggests a natural way of measuring the chaotic rotation of a given homeomorphism.

We give an introduction to additive cellular automata theory then discuss and the rotation entropy of additive cellular automata f_∞ . Cellular automata were introduced by Ulam and Von Neumann, have been systematically studied by Hedlund from a purely mathematical point of view ([5], [6]). The study of such dynamics called cellular automata has received remarkable attention in the last few years ([4], [6], [7]). For a definition and some properties of additive one dimensional cellular automata we refer to [8].

The Notion of the rotation number of an orientation preserving homeomorphism of a circle was introduced by Poincar$\acute{e}$ [9], and since then it has proved to be very useful. It was generalized to the case of continuous maps of a circle of degree one by Newhous, Palis and Takens in 1979 [10]. In this case one gets a rotational interval. This concept also is very useful. This idea appears in the papers of Kim, MacKay and Guckenheimer [11]. In section this we present definition of the rotation set. The straightforward generalization of the definition of the rotation set would be the following.

Consider the circle $\mathrm{T} = {}^{\mathrm{R}}/_{\mathrm{Z}}$ with the natural projection $\pi\colon \mathrm{R} \to \mathrm{T}$. If $\mathrm{f}\colon \mathrm{T} \to \mathrm{T}$ is a continuous map, then there is a continuous $\mathrm{F}\colon \mathrm{R} \to \mathrm{R}$ such that the diagram

Definition 1.1: *(see [1]): Let $A = \{a_1, a_2, \dots, a_k\}$ be a finite set of symbols and $\Omega = A^{Z^n} = w = \{w_g : w_g \in A, g \in Z\}$ be the space of configurations with Tychonoff topology, σ be the shift in this configuration space:*

$$(\sigma^g w)_h = w_{g^{-1}h} \quad , g, h \in Z^n.$$

Assume that a function $f(w_{-g}, \dots, w_g)$ with values in A is given. This function generated a cellular automata f_∞ of by the formula:

$$f_\infty(w) = (y_n)_{g=-\infty}^{\infty} \ , y_n = f(w_{-g}, \dots, w_g)$$

Lemma 1.2: *Cellular automata f_∞ is continuous and commutes with left shift [4].*

Definition 1.3: *f is additive if and only if it can be written as*

$$f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} \lambda_i . w_{n+1} (modr)$$

where $\lambda_i \in A$.

Let us consider particular case when

$f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} \lambda_i . w_{n+1} (modr)$.

Given integer $s \leq t$ and a block

$$I_j = (i_1, i_2, \dots, i_n) \in Z^n : -g_i \leq i_j \leq g, j: 1,2, \dots, n, \left|I_g\right| = 2(2g+1)^n.$$

Let $\xi(s,t)$ denote the partition of ? into the cylinder sets of the form $I_j = (i_1, i_2, \dots, i_n) \in Z^n$.

Lemma 1.4: *Suppose that* $f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} w_{n+1} (modr)$ *and* $\xi(-g,g)$ *is a partition of* Ω *where* $g \geq 2$*, then the partition* $\xi(-g,g)$ *is a generator for additive automata* f_∞[6].

2 Rotations Sets and Rotational Entropy

The Notion of the rotation number of an orientation preserving homeomorphism of a circle was introduced by Poincaré[7]., and since then it has proved to be very useful. It was generalized to the case of continuous maps of a circle of degree one by Newhous, Palis and Takens in 1979 [14]. In this case one gets a rotational interval. This concept also is very useful. This idea appears in the papers of Kim, MacKay and Guckenheimer [13]. In section this we present definition of the rotation set. The straightforward generalization of the definition of the rotation set would be the following.

Consider the circle $\mathrm{T} = {}^{\mathrm{R}}/_{\mathrm{Z}}$ with the natural projection $\pi: \mathrm{R} \to \mathrm{T}$. If $f: \mathrm{T} \to \mathrm{T}$ is a continuous map, then there is a continuous $\mathrm{F}: \mathrm{R} \to \mathrm{R}$ such that the diagram

$$\begin{tikzcd} \mathrm{R} \arrow[r, "\mathrm{F}"] \arrow[d] & \mathrm{R} \arrow[d] \\ \mathrm{T} \arrow[r, "\mathrm{f}"] & \mathrm{T} \end{tikzcd}$$

commutes. Such F is called a lifting of f. It is unique up to a translation by an integer $(T(x) = T(x) + k,\ k \in Z)$. There is an integer d such that $T(x+1) = T(x) + d$ for all $x \in R$. It is called the degree of the choice of lifting. Denote by $\mathbb{C}$

the family all lifting of continuous degree one maps of$T$ into itself.

Let $T \in \mathbb{C}$. If $k \in Z$, then$F(x) = F(x) + k$. All iterates of F also belong to $\mathbb{C}$,so $T^n(x + k) = F^n(x) + k$. We define upper and lower rotation numbers of $x \in R$ for $F \in \mathbb{C}$ as

$$\overline{\rho(F,x)} = \lim_{n\to\infty} sup \frac{F^n(x) - x}{n},$$

$$\underline{\rho(F,x)} = \lim_{n\to\infty} inf \frac{F^n(x) - x}{n}.$$

If

$$\overline{\rho(F,x)} = \lim_{n\to\infty} sup \frac{F^n(x) - x}{n}$$

$$\underline{\rho(F,x)} = \lim_{n\to\infty} inf \frac{F^n(x) - x}{n},$$

$$\overline{\rho(F,x)} = \underline{\rho(F,x)}.$$

we write$\rho(F,x)$ and call it rotation number of$x$ for F.

Definition 2.1: *The set of rotation vectors at all points where they exist is called the rotation set of* f.

Theorem 2.2: *If* $F \in \mathbb{C}$*is a lifting of a circle map*f *then* $\rho(F,x)$ *exists for all* $x \in R$ *and is independent of* x. *Moreover, it sit rotational if and if* f *has a periodic point [2].*

In [1] Botelho, the definition of rotation entropy based on Bowen's interpretation of the topological entropy is introduced. Let f be an annulus endomorphism and F be one of its lifts. By using the definitions of rotation entropy and Misiurewicz's rotation set, spectrum of rotation entropy functions will be defined.

Definition 2.3: *A subset* E *of* A *is called* (n,ε)*-rotational spanning iff for all* $x \in A$ *there exists* $y \in E$ *so that*

$$|x - y| < \varepsilon \text{ and} \left|\frac{F^n(x) - F^n(y)}{n}\right| < \varepsilon.$$

Proposition 2.4: *There exists minimal* (n,ε)*-rotational spanning sets and is finite [10].*

We denote by $E^r_{n,\varepsilon}$ a minimal (n,ε) -rotational spanning and by $\# E^r_{n,\varepsilon}$ its cardinality. The following two facts are straightforward:

If $n \le m$, then $\#E^r_{n,\varepsilon} \le \#E^r_{m,\varepsilon}$.

If $\varepsilon_1 \leq \varepsilon_2$, then $\#E^r_{n,\varepsilon_1} \leq \#E^r_{m,\varepsilon_2}$.

Definition 2.5: *The* (n,ε)*-rotational entropy,*$h_{r,\in}(f)$*, is given by*

$$\lim_{n\to\infty} sup\frac{1}{n}\log\ \#(\ E^r_{n,\varepsilon}).$$

Obviously if $\varepsilon_1 \leq \varepsilon_2$, then $h_{r,\varepsilon_2} \leq h_{r,\varepsilon_1}$.

Definition 2.6: *The* (n,ε)*-rotational entropy of* f*,* $h_r(f)$*, is the limit of* $h_{r,\in}(f)$*as* ε *approach to zero.*

Definition 2.7: *Let* $D \subseteq X$ *be a non-empty set. For* $\varepsilon > 0$*, a set* $D \subseteq X$ *is called an* (n,ε)*-rotational separated set of* D *if* $x,y \in D, x \neq y$ *imlplies*

$$|x-y| > \varepsilon \text{and} \left|\frac{F^n(x)-F^n(y)}{n}\right| > \varepsilon.$$

Let $r_n(D,\varepsilon)$denote the largest cardinality of (n,ε)-rotational separated sets for D.

Definition 2.8: *The* (n,ε)*-rotational entropy,* $h_{r,\in}(f)$*, is given by*

$$lim_{n\to\infty} sup\frac{1}{n}\log\#(D^r_{n,\varepsilon}).$$

Theorem 2.9: *The following limits are equals:*

$$h_r(f) = \lim_{n\to\infty} sup\frac{1}{n}\log\ \#(\ E^r_{n,\varepsilon}) = \lim_{n\to\infty} sup\frac{1}{n}\log\ \#(\ E^r_{n,\varepsilon})\ [10].$$

3 Entropy for Cellular Automata f_∞

Definition 3.1: *Let* $A = \{a_1, a_2, \ldots, a_k\}$ *be a finite set of symbols and*$\Omega = A^{Z^n} = w = \{w_g : w_g \in A, g \in Z\}$ *be the space of configurations with Tychonoff topology,* σ *be the shift in this configuration space:*

$$(\sigma^g w)_h = w_{g^{-1}h} \quad , g,h \in Z^n.$$

Assume that a function$f(w_{-g}, \ldots, w_g)$ *with values in* A *is given. This function generated a cellular automata* f_∞ *of* Ω *by the formula:*

$$f_\infty(w) = (y_n)^{\infty}_{g=-\infty}\ , y_n = f(w_{-g}, \ldots, w_g)\ .$$

Lemma 3.2: *Cellular automata* f_∞ *is continuous and commutes with left shift [5].*

Definition 3.3: f *is additive if and only if it can be written as*

$$f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} \lambda_i . w_{n+1} (modr)$$

where $\lambda_i \in A$.

Let us consider particular case when$f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} \lambda_i . w_{n+1}(modr)$.

Given integer $s \leq t$ and a block

$$I_j = (i_1, i_2, \dots, i_n) \in Z^n : -g_i \leq i_j$$

$$\leq g, j: 1,2, \dots, n, \left|I_g\right| = 2(2g+1)^n.$$

Let $\xi(s,t)$ denote the partition of Ω into the cylinder sets of the form $I_j = (i_1, i_2, \dots, i_n) \in Z^n$.

Lemma 3.4: *Suppose that*$f\left(w_{n-g}, \dots, w_{n+g}\right) = \sum_{i=-g}^{g} w_{n+1}(modr)$ *and* $\xi(-g,g)$ *is a partition of* Ω *where* $g \geq 2$, *then the partition* $\xi(-g,g)$ *is a generator for additive automata* f_∞[6].

Definition 3.5: *Let* $\rho(f_\infty, x)$ *be the set of all limits of convergent subsequences of the sequence*

$$\left(\frac{f_\infty^n(x) - x}{n}\right)_{n=1}^{\infty}$$

However, we shall mainly use another definition. The motivation is a follows. The aim of the rotation set is to measure the average movement of many point. This average movement is measured by finite parts of orbits, then passing with the lenghts to infinity. Therefore we shall not do it and we shall taken the limits of all convergent sequences

$$\left(\frac{f_\infty^n(x_i) - x_i}{n_i}\right)_{n=1}^{\infty} x_i \in R^m, n_i \to \infty.$$

The set obtained in such away we shall the rotation set of cellular automata for $f_\infty$and $\rho(f_\infty)$.

Now, we introduce the definition of the rotational entropy of cellular automata iff for all f_∞ ., based on Bowen's and Botelho's [11] interpretation of the topological entropy.

Definition 3.6: *A subset* E *of* Ω *is called* $(n, \varepsilon, f_\infty)$ *rotational spanning for cellular automata iff for all* $x \in \Omega$*there exist* $y \in E$ *so that*

$$|x-y|<\varepsilon\text{and}\left|\frac{f_{\infty}^{n_i}(x)-f_{\infty}^{n_i}(y)}{n_i}\right|<\varepsilon i=1,2,\dots,n-1.$$

Proposition 3.7: *There exists a minimal (n,ε,f_∞)rotational spanning for cellular automataf_∞ and it is finite.*

Proof: For fixed $n$and ε, given $x\in w$ there exists $0\le\delta\le\varepsilon$ so that

$$|x-y|<\varepsilon\text{and }\left|\frac{f_{\infty}^{n_i}(x)-f_{\infty}^{n_i}(y)}{n_i}\right|<\varepsilon i=1,2,\dots,n-1.$$

and $y\in\beta(x,\delta)$.

Since A is compact, there are k balls with centers $x_1,x_2,\dots,x_g$ that cover the whole annulus. The set $\{x_1,x_2,\dots,x_g\}$ is (n,ε,f_∞) rotational spannig.

We denote by E a minimal (n,ε,f_∞) rotational spannig for cellular automata and by $\neq E_{n,\varepsilon}^{c,a}$ its cardinality. The follows two cellular automata are straight-forward:

a) If $n\le m$ then $\neq E_{n,\varepsilon}^{c,a}\le\neq E_{m,\varepsilon}^{c,a}$.
b) If $\varepsilon_1\le\varepsilon_2$ then $\neq E_{n,\varepsilon_2}^{c,a}\le\neq E_{m,\varepsilon_1}^{c,a}$.

Definition 3.8: *The(n,ε,f_∞) rotation entropy for cellular automata f_∞ , $h_{r,\varepsilon}^{c,a}$, is given by*

$$\lim_{n\to\infty} sup\frac{1}{n}log\neq E_{n,\varepsilon}^{c,a}.$$

Obviously if $\varepsilon_1\le\varepsilon_2$then $0<h_{r,\varepsilon_2}^{c,a}\le h_{r,\varepsilon_1}^{c,a}$[3].

Definition 3.9: *The rotational entropy for cellular automata f_∞, $h_r^{c,a}(f_\infty)$, is limit of $h_{r,\varepsilon}^{c,a}$ as ε approaches zero.*

Lemma 3.10: *It can be written as*

$$h_{r,n_0}^{c,a}(f_\infty)=h_r^{c,a}(f_\infty).$$

Proof: If the set $E_{n,\varepsilon}^{c,a}$ is a minimal (n,ε,f_∞)rotational spanning set then it is $(n\ge n_0,\varepsilon,f_\infty)$ rotational spanning, therefore$\neq E_{n,\varepsilon}^{c,a}\ge\neq E_{n\ge n_0,\varepsilon}^{c,a}$. This implies that

$h_r^{c,a}(f_\infty)\ge h_{r,n_0}^{c,a}(f_\infty)$.

To prove the other inequality, one notice that for $\varepsilon>0$ there exists $\delta\le\varepsilon$ so that

$|x-y|<\delta$and $\left|\frac{f_\infty^{n_i}(x)-f_\infty^{n_i}(y)}{n_i}\right|<\varepsilon i=1,2,\dots,n-1.$

Consequently we have$\neq E_{n,\varepsilon}^{c,a}\geq\neq E_{n\geq n_0,\varepsilon}^{c,a}$ and

$$\lim_{\delta\to 0}\lim_{n\to\infty} sup\frac{1}{n}log\neq E_{n\geq n_0,\delta}^{c,a}$$

$$\geq\lim_{\varepsilon\to 0}\lim_{n\to\infty} sup\frac{1}{n}log\neq E_{n\geq n_0,\varepsilon}^{c,a}$$

proving that $h_{r,n_0}^{c,a}(f_\infty)\geq h_r^{c,a}(f_\infty)$.

Definition 3.11: *Let f_∞ be any uniformly continuous map metric space (Ω,δ), a set $E\subset\Omega$ is said to be (n,ε,f_∞) seperated under f_∞ if for every pair $x\neq y$ in E there is a $i\in\{1,2,\dots,n-1\}$ with the property that*

$$\left|\frac{f_\infty^{n_i}(x)-f_\infty^{n_i}(y)}{n_i}\right|\geq\varepsilon.$$

For each compact set $K\subset\Omega$ let

$$A_K(n,\varepsilon,f_\infty)=max\{E: E\subset K is (n,\varepsilon,f_\infty) separated under f_\infty\},$$

$$h_r^{c,a}(f_\infty,\varepsilon)=\lim_{n\to\infty}\frac{1}{n}log\left(\neq A_{n,\varepsilon}^{c,a}\right)$$

and

$$h_r^{c,a}(f_\infty)=\lim_{\varepsilon\to\infty}h_r^{c,a}(f_\infty,\varepsilon).$$

Proposition 3.12: *The topological entropy f is defined by$h(f)$. If Ω represents a lift of f, Ω is uniformly continuous and the following definition due to Bowen makes sence:*

$$h(\Omega)=\sup_{kcompact} h_r^{c,a}(f_\infty)\,.$$

It is known fact that $h(f)=h(\Omega)$. Therefore $h_r^{c,a}(f_\infty)\leq\ h(f)=h(\Omega)$.

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References

[1] F. Botelho, Rotation entropy for annulus homeomorphisms, *Pacific J. Math.*, 132(1991), 1-19.

[2] M. Misiurewicz and K. Ziemian, Rotations for maps of Tori, *J. London*

Math. Soc, 40(2) (1989), 490-506.

[3] H. Poincaré, *Oeuvres Completes*, Tomeİ (Gauthier-Villars, Paris, (1952), 137-158.

[4] J. Fanks, Recurence and fixed point of surface homeomorphisms, *Ergodiv Theory and Dynamical System*, 8(1988), 99-108.

[5] S. Kim, R.S. Mackay and J. Guckenheimer, Resonance regions for families of the torus maps, *Nonlinearity*, 2(1989), 391-404.

[6] G.A. Hedlund, Endomorphisms and automorphisms of full shift dynamical system, *Math. Syst. Theor*, 3(1969), 320-375.

[7] H. Akın, On the measure entropy of additive cellular automata f_∞, *Entropy*, 5(2003), 233-238.

[8] G. Cattaneo, E. Formenti, G. Manzini and L. Margara, Ergodicity, transitivity and regularity for linear cellular automata over Z^m, *Theoret. Comput. Sci.*, 233(2000), 147-164.

[9] M.A. Shereshevsky, Ergodic properties of certain surjective cellular automata, *Mh. Math.*, 114(1992), 305-316.

[10] F. Botelho, Rotation sets for annulus homeomorphisms, *Pacific J. Math.*, 132(1988), 251-266.

[11] R. Bowen, Invariant measures for Makov maps of the interval, *Comm Math Physics*, 69(1976), 1-17.