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Fourier Transform in $L^p(R)$ Spaces, $p \geq 1$

Devendra Kumar and Dimple Singh

Department of Mathematics
Research and Post Graduate Studies
M.M.H.College, Model Town, Ghaziabad-201001, U.P.India
E-mail:d_kumar001@rediffmail.com

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Abstract

A method for restricting the Fourier transform of $f \in L^p(R), 1 \leq p \leq \infty$, spaces have been discussed by using the approximate identities.

Keywords: *Approximate identities, convolution operator, Schwartz space and atomic measure.*

1 Introduction

Let $f \in L^1(R)$.The Fourier transform of $f(x)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_R f(x) e^{-i\xi x} dx, \xi \in R. \tag{1}$$

If $f \in L^1(R)$ and $\hat{f} \in L^1(R)$, then the inverse Fourier transform of $\hat{f}$ is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_R \hat{f}(\xi) e^{i\xi x} d\xi \tag{2}$$

for a.e. $x \in R$. If f is continuous, then(1.2) holds for every x.

It is known that several elementary functions, such as constant function, $\sin wt$, $\cos wt$, do not belongs to $L^1(R)$ and hence they do not have Fourier transforms. But when these functions are multiplied by characteristic function, the resulting functions belongs to $L^1(R)$ and have Fourier transforms. Many applications, including the analysis of stationary signals and real time signal processing, make an effective use of Fourier transform in time and frequency domains.

The remarkable success of the Fourier transform analysis is due to the fact that, under certain conditions, the signal can be reconstructed by the Fourier inversion formula. Thus the Fourier transform theory has been very useful for analyzing harmonic signals or signals for which there is no need for local information. On the other hand, Fourier transform analysis has also been very useful in many other areas, including quantum mechanics, wave motion and turbulence.

By Lebesgue lemma we have if $f \in L^1(R)$ then $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$, it follows that Fourier transform is a continuous linear operator from $L^1(R)$ into $C_o(R)$, the space of all continuous functions on R which decay at infinity, that is, $f(x) \to 0$ as $|x| \to \infty$. Roughly we say that if $f \in L^1(R)$, it does not necessarily imply that $\hat{f}$ also belongs to $L^1(R)$.

Bellow [1] and Reinhold - Larsson [2] constructed examples of sequence of natural numbers along which the individual ergodic theorem holds in some L^p spaces (good behavior) and not in others (bad behaviour). In particular, well behaved sequences were perturbed in such a way that good behavior persists only in certain spaces.

In the present work we provide a method for restricting the Fourier transform of $f \in L^p(R)$ spaces using the pointwise convergence of convolution operators for approximate identities.

Definition 1.1. Let $\varphi \in L^1(R)$ such that $\hat{\varphi}(0) = 1$. Then $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$ is called an approximate identity if
(i) $\int_R \varphi_\varepsilon(x)dx = 1$
(ii) $\sup_{\varepsilon>0} \int_R |\varphi_\varepsilon(x)|dx < +\infty$,
(iii) $\lim_{\varepsilon\to 0} \int_{|x|>\delta} |\varphi_\varepsilon(x)|dx = 0$,for all $\delta > 0$.

Proof. Properties (i) and (ii) can be proved by observing

$$\int_R \varphi_\varepsilon(x)dx = \int_R \varepsilon^{-1}\varphi(x/\varepsilon)dx = \int_R \varphi(x/\varepsilon)d(x/\varepsilon) = 1.$$

For (iii), we have

$$\int_{|x|>\delta} \varphi_\varepsilon(x)dx = \int_{|x|>\delta} \frac{1}{\varepsilon}\varphi(x/\varepsilon)dx = \int_\delta^\infty \frac{1}{\varepsilon}\varphi(x/\varepsilon)dx + \int_{-\infty}^{-\delta} \frac{1}{\varepsilon}\varphi(x/\varepsilon)dx.$$

Substituting $y = x/\varepsilon$, we get

$$\lim_{\varepsilon\to 0}\int_{\delta/\varepsilon}^{\infty}\varphi(y)dy+\int_{-\infty}^{-\delta/\varepsilon}\varphi(y)dy=0.$$

Definition 1.2. A sequence of functions $\{\phi_n\}_{n\in N}$ such that $\phi_n(x) = n\phi(nx)$ where $n = \frac{1}{\varepsilon}$, $n \to \infty, \varepsilon \longrightarrow 0$ is called an approximate identity if
(i)$\int_R \phi_n(x)dx = 1$ for all n,
(ii)$\sup_n \int_R |\phi_n(x)|dx < +\infty$,
(iii)$\lim_{n\to\infty}\int_{|x|>\delta}|\phi_n(x)|dx = 0$ for every $\delta > 0$.

In the consequence of above Definition 1.2, we can easily prove the following proposition.

Proposition 1.1. A sequence of functions$\{\phi_n\}_{n\in N}$ with $\phi_n \geq 0, \hat{\phi}_n(0) = 1$ is an approximate identity if for every $\varepsilon > 0$ there exists $n_o \in N$ so that for all $n \geq n_o$ we have $\int_{-\varepsilon}^{\varepsilon}\phi_n > 1 - \varepsilon$.

Let us consider the class $S(R)$ of rapidly decreasing $C^\infty-$functions on R i.e., Schwartz class such that

$$S(R) = \{f : R \to R, \sup_{x\in R}(x^n\frac{d^m}{dx^m}f)(x) < \infty\}n, m \in N \cup (0).$$

It is well known that if $f \in S(R)$ then $\hat{f} \in S(R)$ and $S(R) \subset L^p(R)$.To prove the denseness of $S(R) \in L^p(R)$, we have

$$\rho \in S(R) \Rightarrow |\rho(x)| \leq \frac{c}{1+|x|^n}.$$

For $1 \leq p < \infty$,

$$\int_R |\rho(x)|^p dx \leq \int_R \frac{c^p}{(1+|x|^n)^p} < \infty$$

which gives $\rho \in L^p(R)$. Define a sequence $\{\rho_N\}$ such that

$$\rho_N(x) = \begin{cases} f(x), & \text{if } -N \leq x \leq N; \\ 0, & \text{otherwise ;} \end{cases}$$

$\Rightarrow \exists \rho_N \in S(R)$, $f \in L^p(R)$ such that

$$\int_R |\rho_N - f|^p dx \to 0$$

as $N \to \infty$. Hence $S(R)$ is dense in $L^p(R)$.

Remark 1.1. If $0 \leq \phi(x) \in S(R)$ and $\hat{\phi}(0) = 1$. Then $\phi_n(x) = n\phi(nx)$ is an approximate identity.

Proposition 1.2. If $f \in L^1(R)$ and $\phi \in S(R)$ then $\phi * f \in S(R)$.

Proof. We have

$$\phi * f = \int_R \phi(y) f(x-y) dy$$

$$\frac{d^n}{dx^n}(\phi * f) = \int_R \phi(y) \frac{d^n}{dx^n} f(x-y) dy$$

or

$$|x|^n \frac{d^n}{dx^n}(\phi * f) = |x|^n \int_R f(x-y) \frac{d^n}{dy^n}\phi(y) dy,$$

substituting $x - y = z$, we obtain,

$$= \int_R f(y)|x|^n \frac{d^n}{dx^n}\phi(x-y) dy$$

using $|x-y| \leq |x| + |y| \leq \frac{3|x|}{2}$, we get

$$= \int_{|y| > \frac{|x|}{2}} f(y)|x|^n \frac{d^n}{dx^n}\phi(x-y) dy + \int_{|y| \leq \frac{|x|}{2}} f(y)|x|^n \frac{d^n}{dx^n}\phi(x-y) dy \to 0.$$

Proposition 1.3. If $\phi_n(x)$ is an approximate identity and $f \in L^p(R)$ then

$$\phi_n * f \to f \in L^p(R).$$

Proof. Consider

$$\begin{aligned} [\int_R |(\phi_n * f)(x) - f(x)|^p dx]^{1/p} &= [\int_R dx| \int_R \phi_n(x-y) f(y) dy - f(x)|^p]^{1/p} \\ &= [\int_R dx| \int_R \phi_n(y) f(x-y) dy - f(x)|^p]^{1/p} \end{aligned}$$

using $f(x)=\int_R f(x)\phi_n(y)dy$ in above we obtain

$[\int_R dx|\int_R \phi_n(y)(f(x-y)-f(x))dy|^p]^{1/p}$

$$\begin{aligned}
&\leq \; [\int_R dx \int_{|y|>\delta} |\phi_n(y)|^p|f(x-y)-f(x)|^p dy]^{1/p} \\
&+ \; [\int_R dx \int_{|y|\leq\delta} |\phi_n(y)|^p.|f(x-y)-f(x)|^p]^{1/p} \qquad (3)\\
&\leq \; \int_{|y|>\delta} dy|\phi_n(y)|[\int_R dx|f(x-y)-f(x)|^p]^{1/p} \\
&+ \; \int_{|y|\leq\delta} dy|\phi_n(y)|[\int_R |f(x-y)-f(x)|^p dx]^{1/p} \qquad (4)\\
&\leq \; \int_{|y|>\delta} dy|\phi_n(y)|(2\parallel f\parallel_p)+\int_{|y|\leq\delta} dy|\phi_n(y)|\sup_{|y|\leq\delta}[\int_R |f(x-y)-f(x)|^p dx]^{1/p}.
\end{aligned}$$

Proceeding limits as $n\to\infty$, the right hand side tends to zero since

$$\sup_{|y|<\delta}[\int_R |f(x-y)-f(x)|^p dx]^{1/p}\to 0.$$

Hence the proof is completed.

Proposition 1.4. Let $\phi_n=\alpha_n\varphi_n+(1-\alpha_n)\sigma_n$, where $\{\varphi_n\}_{n\in N}$, $\{\sigma_n\}_{n\in N}$ are approximate identities and $0\leq\alpha_n\leq 1$.

(a) For $1\leq p<+\infty$ and every $f\in L^p(R)$, $\lim_{n\to\infty}(\phi_n-\varphi_n)*f\to 0$ and $\lim_{n\to\infty}(\phi_n-\sigma_n)*f\to 0$.

(b)For every $f\in L^\infty(R)$, $\lim_{n\to\infty}(\phi_n-\varphi_n)*f\to 0$ a.e. .

(c) For $1\leq p<\infty$, if $\sum_n(1-\alpha_n)^p<+\infty$, then for every $f\in L^p(R)$, $\lim_{n\to\infty}(\phi_n-\varphi_n)*f\to 0$ a.e. .

Proof.(a) Set $1\leq p\leq\infty$, and $f\in L^p(R)$. In view of Minkowski's inequality

$$\parallel(\phi_n-\varphi_n)*f\parallel_p\leq(1-\alpha_n)(\parallel\sigma_n*f-f\parallel_p+\parallel\varphi_n*f-f\parallel_p)$$

and using Proposition 1.3 we obtain $\parallel(\phi_n-\sigma_n)*f\parallel_p\to 0$.

(b)For $f\in L^\infty(R)$, $|(\phi_n-\varphi_n)*f|\leq\parallel(\phi_n-\varphi_n)*f\parallel\to 0$ by part(a).

(c)For $f \in L^p(R)$

$$\begin{aligned}\int_R \sum_n (1-\alpha_n)^p |\sigma_n * f(x)|^p dx &= \sum_n \| (1-\alpha_n)\sigma_n * f \|_p^p \\ &\leq \sum_n (1-\sigma_n)^p \| f \|_p^p < +\infty.\end{aligned}$$

Then $(1-\alpha_n)\sigma_n * f \to 0$ a.e. . Similarly $(\alpha_n - 1)\varphi_n * f \to 0$ a.e. .

Definition 1.3. An approximate identity $\{\phi_n\}$ is called L^p- good if $\phi_n * f \to f$ a.e. for all $f \in L^p(R)$, and it is called good if it is L^p-good for every $1 \leq p \leq +\infty$. An approximate identity $\{\phi_n\}$ is called L^p-bad if there exists $f \in L^p(R)$ such that $\phi_n * f \nrightarrow f$ on a set of positive measure.

Definition 1.4. Let $\{\varphi_n\}_{n\in N}$ and $\{\sigma_n\}_{n\in N}$ be approximate identities, α_n be a sequence of real numbers with $0 \leq \alpha_n \leq 1$ and $\alpha_n \to 1$. We call perturbed approximate identities any approximate identity $\{\phi_n\}_{n\in N}$ of the form $\phi_n\varphi_n + (1-\alpha_n)\sigma_n$.

2 Main Results

Theorem 2.1.

(i)Given any good approximate identity $\{\varphi_n\}_{n\in N}$ there exists a perturbed approximate identity $\{\phi_n\}_{n\in N}$ such that $f \in L^q(R)$

$$(\phi_n \hat{*} f)(\xi) = \hat{\phi}_n(\xi)\hat{f}(\xi)$$

$$\left(\hat{\phi}_n(\xi)\hat{f}(\xi)\right)\hat{} \to f(x)$$

for $q \geq p, p \in [1,\infty)$ and

$$(\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} \nrightarrow f(x)$$

for $1 \leq q < p$.

(ii)$(\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} \to f(x)$ for $q > p$ and

$(\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} \nrightarrow f(x)$ for $1 \leq q \leq p$.

(iii)$(\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} \to f(x)$ for $q = \infty$

$(\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} \nrightarrow f(x)$ for $1 \leq q < \infty$.

Proof. (i) Let

$$\begin{aligned} g_n(x) &= \frac{1}{\sqrt{2\pi}}\int_R e^{ix\xi}\hat{\phi}_n(\xi)\hat{f}(\xi)d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_R e^{ix\xi}\frac{\hat{\phi}_n(\xi)}{\sqrt{2\pi}}\int_R e^{-i\xi y}f(y)dyd\xi \\ &= \frac{1}{2\pi}\int_R e^{i(x-y)\xi}\hat{\phi}_n(\xi)\int_R f(y)dy \\ &= \frac{1}{\sqrt{2\pi}}\int_R \phi_n(x-y)f(y)dy \\ or &= (\phi_n * f)(x) \\ (\hat{\phi}_n(\xi)\hat{f}(\xi))\hat{} &= \frac{1}{\sqrt{2\pi}}\int_R e^{ix\xi}\hat{\phi}_n(\xi)\hat{f}(\xi)d\xi = (\phi_n * f)(x). \end{aligned}$$

Fix $q \geq p$ and taking $1-\alpha_n = \frac{1}{(n\log^2 n)^{1/p}}$ Since $\Sigma_n(1-\alpha_n)^q < +\infty$ and φ_n is an L^q-good approximate identity, using Proposition 1.4 we obtain that $\{\phi_n\}$ is also an L^q-good approximate identity.

Hence for $q \geq p$, $(\phi_n * f)(x) \to f(x)$.

Now we have to prove that for each $1 \leq q < p$. there exists $f_q \in L^q(R)$ so that $\limsup_\kappa |x|^\kappa \frac{d^\kappa}{dx^\kappa}(\phi_\kappa * f_q \to \infty)$ on a set of positive measure.

Set

$$f_q(x) = \frac{1}{(x\log^2(x/2))^{1/q}}\chi_{[0,1]}(x) \in L_q(R).$$

Choose

$$r_n = \frac{1}{n^{1+1/p}(\log n)^{2/p}}, a_n = r_n^{\frac{1}{p+1}} = \frac{1}{n^{1/p}(\log n)^{\frac{2}{p(p+1)}}},$$

$$J_n = [a_n - r_n, a_n + r_n]$$

and

$$U_n = [-a_n + r_n, -a_{n+1} + r_{n+1}],$$

for sufficiently large n and for all $\kappa \geq n$, $x \in U_\kappa$,

$$\begin{aligned} \phi_\kappa * f_q(x) &\geq (1-\alpha_\kappa)\sigma_\kappa * f_q(x) \\ &\geq \frac{1}{(\kappa\log^2\kappa)^{1/p}}\int_{-J_\kappa}\sigma_\kappa(y)f_q(x-y)dy. \end{aligned}$$

Now, we get

$$\phi_\kappa * f_q(x) \geq \frac{f_q(C_{r_\kappa}(\log\kappa)^{2/p+1})}{(\kappa\log^2\kappa)^{1/p}} \int_{-J_\kappa} \sigma_\kappa(y)dy$$

or

$$f_q(C_{r_\kappa}(\log\kappa)^{2/p+1}) = \frac{\kappa^{1/q+1/pq}(\log\kappa)^{\frac{2}{pq(p+1)}}}{C^{1/q}(\log(C/2\kappa^{(p+1)/p}(\log\kappa)^{2/p(p+1)}))^{2/q}}.$$

Then

$$\phi_\kappa * f_q(x) \geq C\kappa^{\frac{1}{q}-\frac{1}{p}+\frac{1}{pq}} H_q(\kappa) > \kappa^\delta \geq n^\delta,$$

where

$$H_q(\kappa) = \frac{(\log\kappa)^{\frac{2}{pq(p+1)}-\frac{2}{p}}}{C^{1/q}(\log C/2\kappa^{(p+1)/p}(\log\kappa)^{2/p(p+1)})^{2/q}}$$

and

$$0 < \delta < 1/q - 1/p + 1/pq.$$

So

$$\frac{d^\kappa}{dx^\kappa}(\phi_\kappa * f_q(x)) \geq C\frac{d^\kappa}{dx^\kappa}(\kappa^{1/q-1/p+1/pq}H_q(\kappa))$$

or

$$|x|^n\frac{d^n}{dx^n}(\phi_n * f_q(x)) \geq |x|^n \int_{-J_\kappa} f_q(x-y)\frac{d^n}{dy^n}\sigma_\kappa(y)dy$$

for $\kappa \geq n$

$$\begin{aligned}
|x|^\kappa\frac{d^\kappa}{dx^\kappa}(\phi_\kappa * f_q(x)) \geq |x|^n\frac{d^n}{dx^n}n^\delta \quad &\geq \quad |x|^n\frac{d^n}{dx^n}(\frac{1}{(x-y)^{p\delta}}) \\
&= \quad \frac{|x|^n(-1)^n(p\delta+n-1)!}{(p\delta)!(x-y)^{p\delta+n}} \\
&\geq \quad |x|^n\frac{(-1)^n(p\delta+n-1)!}{(p\delta)!Cr_n(\log n)^{2/p+1}(\log n)^{2\delta/p+1}} \\
&\to \quad \infty \text{ as } n\to\infty.
\end{aligned}$$

In view of Sawyer's Principle [3] there exists a functions $f \in L^q([0,1)) \subseteq L^q(R)$ such that $\limsup_n |x|^n\frac{d^n}{dx^n}(\phi_n * f) \to \infty$ a.e. on a set of positive measure in R, It follows that $\phi_n * f$ not belongs to $S(R)$ or $\phi_n * f \nrightarrow f$ or$\widehat{\hat{\phi}_n(\xi)\hat{f}(\xi)} \nrightarrow f(x)$

for $1 \leq q < p$.

(ii) Let p_n be a decreasing sequence of real numbers such that $p_1 > p_2 > \ldots p_n > \ldots\ldots \searrow p$. for each p_i we can construct a perturbation $\{\phi_n^i\}_n$ of $\{\varphi_n\}$ that is L^q-good for $q \geq p_i$, and L^q-bad for $1 \geq q < p_i$. Consider a sequence of blocks $\{B_\kappa\}_{\kappa \in N}$, where $B_\kappa = \{\phi^\kappa_{n_{\kappa-1}+1}, \ldots, \phi^\kappa_{n_\kappa},\}$ and $\{n_\kappa\}$ is a sequence of positive integers increasing to infinity. Let $D_\kappa = \{n_{\kappa-1}+1, \ldots, n_\kappa\}$, and let $\{\phi_n\}_n = U_\kappa B_\kappa$. Now fix $q > p$. There exists $n_o \in N$ so that for all $n > n_o$ we have $p_n < q$,

$$\begin{aligned} \sum_{\kappa=n_o}^{\infty} \sum_{n \in D_\kappa} (1-\alpha_n^\kappa)^q &\leq \sum_{\kappa=n_o}^{\infty} \sum_{n \in D_\kappa} \frac{1}{(n \log^2 n)^{q/p_{n_o}}} \\ &\leq \sum_n \left(\frac{1}{n \log^2 n}\right)^{q/p_{n_o}} < +\infty. \end{aligned}$$

Using Proposition 1.4(c) we get $\phi_n * f \to f$ for $f \in L^q(R)$, $q > p$, or $\hat{\phi}_n(\xi)\hat{f}(\xi) \to f(x)$ for $q > p$.

Now consider a sequence $C_i^N \to \infty$ as $i \to \infty$. Since $\{\phi_n^i\}_n$ is L^q-bad for all $q < p_i$, it is also L^p-bad. These exists $f_i \in L^p([0,1))$ and $\lambda_i^N > 0$ such that

$$\begin{aligned} |\{\sup_{n > n_{i-1}} \phi_n^i * f_i(x)\}| &> \int_{-J_\kappa} |\phi_n^i(x) f_i(y)|^p dy \\ &> C^N \parallel f_i(x-\lambda_i^N) \parallel_p^p \\ &= 2C_i^N, [\parallel f_i(x-\lambda_i^N) \parallel_p = 2^{1-i}, C^N = 2^{(i-1)p+1} C_i^N]. \end{aligned}$$

It follows that there exists $n_i > n_{i-1}$, so that

$$|\{\sup_{n_{i-1} < n \leq n_i} (\phi_n^i * f_i)\}| > C_i^N.$$

Set

$$\tilde{f} = \sum_i f_i, \text{ then } \parallel \tilde{f} \parallel_p \leq \sum_i \parallel f_i \parallel_p \leq 2.$$

Suppose that $\{\phi_n\}$satisfies a weak (p, p) inequality in $L^p([0,1))$. We know that if μ be a finite positive Borel measure, then these exists a sequence μ_n of atomic measure that converges to μ weakly or if f has compact support then

$$\int_R d\mu_n f(x) \to \int_R f(x) d\mu$$

or

$$\mu_n \to \mu \text{ weakly .}$$

If $f \in L^1(R), d\mu = |f(x)|dx$ is a finite Borel measure, so we can find

$$\mu_n = \sum_{\imath=1}^{N} C_i^N \delta_{\lambda_i^N} \to \mu \text{ weakly.}$$

Consider

$$\begin{aligned}
|\{\sup_n(\phi_n^i * f)\}| &= \int_{-J_\kappa} |\phi_n^i(y) f(x-y)|^p dy \\
&\leq \int_{-J_\kappa} |\phi_n^i(y) d_{\mu_n}(x-y)|^p dy \\
&\leq \| \sum_{i=1}^{N} f(x-\lambda_i^N) C_i^N \|_p^p \\
&\leq \sum_{i=1}^{N} C_i^N \| f(x-\lambda_i^N) \|_p^p \\
&\leq C_o^N \| f \|_p^p \\
&= 2^p C_o^N. \qquad (1)
\end{aligned}$$

On the other hand,

$$|\{\sup_n(\phi_n * f)\}| \leq |\{ \sup_{n_{i-1}<n\leq n_i} (\phi_n^i * f(i))\}| > C_i^N \qquad (2)$$

Combining Equations (2.1) and (2.2) we get

$$C_o^N > C_i^N$$

But $C_i^N \to \infty$ as $i \to +\infty$. Hence $\phi_n * f \nrightarrow f$ in $L^p([0,1))$. Since the spaces $L^q([0,1))$ are nested, $\{\phi_n\}$ is $L^q([0,1))$-bad for all $1 \leq q \leq p$. Therefore, such a choice of $\{n_\kappa\}$ makes $\{\phi_n\} L^q(R)-$bad for all $1 \leq q \leq p$. This implies that $\hat{\phi}_n(\xi)\hat{\hat{f}}(\xi) \nrightarrow f(x)$ for $1 \leq q \leq p$.

(iii) Let $\{\varphi_n\}_{n\in N}$ be a good approximate identity, and let $\{\zeta_n\}_{n\in N}$ be any approximate identity. Let $\{p_n\}$ be a sequence of real numbers satisfying

$$1 \leq p_1 < p_2 < \dots < p_n \nearrow \infty$$

Consider the blocks $\{B_\kappa\}$, where each block B_κ is related to p_κ. for $i \in D_\kappa$, let

$$\phi_i = \alpha_i^\kappa \varphi_i^\kappa + (1-\alpha_i^\kappa)\sigma_i^\kappa.$$

Choose n_κ such that $\alpha_i^\kappa \to 1$. Then since $\{\varphi_n\}$ is L^∞ good,

$$\varphi_n * f \to f \text{ a.e. for all } f \in L^\infty(R),$$

and

$$\alpha_i^\kappa \varphi_i^\kappa * f \to f \text{ a.e. for all } f \in L^\infty(R).$$

Since

$$\sigma_i^\kappa * f(x) \leq \| f \|_\infty .$$

$$(1-\alpha_i^\kappa)\sigma_i^\kappa * f \to 0 \text{ a.e. for all } f \in L^\infty(R).$$

It follows that $\phi_n * f \to f$ a.e, for all $f \in L^\infty(R)$. This implies that $(\hat{\phi}_n(\xi)\hat{f}(\xi)\hat{)} \to f(x)$ for $q = \infty$.

The approximate identity $\{\phi_n^\kappa\}_n$ is $L^{pm}-$bad for every $m \in \{1, ..., \kappa\}$, since it is L^q-bad for every $1 \leq q \leq p_\kappa$. There exists $f_m^\kappa \in L^{p_m}([0,1))$ with $\| f_m^\kappa(x - \lambda_m^{\kappa(N)}) \| = 2^{-\kappa}$, $\lambda_m^{\kappa(N)} > 0$ and $n_m^\kappa > m_{\kappa-1}$ so that

$$\begin{aligned} |\{\sup_{n_{\kappa-1}<n\leq n_m^\kappa} (\phi_n^\kappa * f_m^\kappa)\}| &> C^N \| f_m^\kappa(x-\lambda_m^{\kappa(N)}) \|_{p_m}^{p_m} \\ &= \frac{C_\kappa^N}{2^{\kappa p_m}} \end{aligned}$$

Let $\tilde{f} = \sum_{\kappa \geq \kappa_o} f_{\kappa_o}^\kappa$, then $\| \tilde{f} \|_{p_{\kappa_o}} < 2$.

So

$$\begin{aligned} |\{\sup_n(\phi_n * \tilde{f})\}| &\leq C_0^{\bowtie} \| \tilde{f} \|_{p_{\kappa_o}}^{p_{\kappa_o}} \\ &\leq 2^{p_{\kappa_o}} C_o^N. \end{aligned} \tag{3}$$

Hence

$$\begin{aligned} |\{\sup_n(\phi_n * \tilde{f})\}| &\geq |\{\sup_{n_{\kappa-1}<n\leq n_\kappa} (\phi_n^\kappa * f_{\kappa_o}^\kappa)\}| \\ &> \frac{C_\kappa^N}{2^{\kappa p_{\kappa_o}}} \end{aligned} \tag{4}$$

using (2.3) and (2.4) we get

$$C_o^N > \frac{C_\kappa^N}{2^{\kappa p_{\kappa_o}(\kappa+1)}} \to +\infty$$

.Thus we conclude that

$$\hat{\phi}_n(\xi)\hat{f}(\xi)\hat{} \nrightarrow f(x) \text{ For } 1 \leq q < \infty.$$

Hence the proof is completed.

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