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Cesàro Summability of Double Sequences of Sets

Fatih Nuray[1], Uğur Ulusu[2] and Erdinç Dündar[3]

[1,2,3]Department of Mathematics, Faculty of Science and Literature
Afyon Kocatepe University, Afyonkarahisar, Turkey
[1]E-mail: fnuray@aku.edu.tr
[2]E-mail: ulusu@aku.edu.tr
[3]E-mail: edundar@aku.edu.tr

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Abstract

In this paper, we study the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationship between them.

Keywords: *Lacunary sequence, Cesàro summability, double sequence of sets, Wijsman convergence.*

1 Introduction

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [3, 4, 5, 11, 16, 17, 18]). Nuray and Rhoades [11] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [15] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Ulusu and Nuray [16] introduced the concept of Wijsman strongly lacunary summability for set sequences and discused its relation with Wijsman strongly Cesàro summability.

Hill [8] was the first who applied methods of functional analysis to double sequences. Also, Kull [9] applied methods of functional analysis of matrix maps of double sequences. A lot of usefull developments of double sequences in summability methods, the reader may refer to [1, 10, 14, 19].

In this paper, we study the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationship between them.

2 Definitions and Notations

Now, we recall the basic definitions and concepts (See [1, 2, 3, 4, 5, 11, 12, 14, 16, 17, 18]).

For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

We say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

The sequence $\{A_k\}$ is said to be Wijsman Cesàro summable to A if $\{d(x, A_k)\}$ Cesàro summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A).$$

The sequence $\{A_k\}$ is said to be Wijsman strongly Cesàro summable to A if $\{d(x, A_k)\}$ strongly Cesàro summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

The sequence $\{A_k\}$ is said to be Wijsman strongly p-Cesàro summable to A if $\{d(x, A_k)\}$ strongly p-Cesàro summable to $\{d(x, A)\}$; that is, for each p positive real number and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p = 0.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$

will be abbreviated by q_r.

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is Wijsman lacunary convergent to A for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} d(x, A_k) = d(x, A).$$

In this case we write $A_k \to A(WN_\theta)$.

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is Wijsman strongly lacunary convergent to A for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN_\theta])$.

A double sequence $x = (x_{kj})_{k,j \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$ whenever $k, j > N_\varepsilon$. In this case we write

$$P - \lim_{k,j \to \infty} x_{kj} = L \quad or \quad \lim_{k,j \to \infty} x_{kj} = L.$$

A double sequence $x = (x_{kj})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{kj}| < M$ for all $k, j \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{k,j} |x_{kj}| < \infty.$$

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad as \quad r \to \infty$$

and

$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \;\; and \;\; j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\; and \;\; q_u = \frac{j_u}{j_{u-1}}.$$

Lemma 2.1 *[7, Lemma 3.2] If $b_1, b_2, ..., b_n$ are positive real numbers, and if $a_1, a_2, ..., a_n$ are real numbers satisfying*

$$\frac{|a_1 + a_2 + ... + a_n|}{b_1 + b_2 + ... + b_n} > \varepsilon > 0,$$

then $|a_i|/b_i > \varepsilon$ for some i, where $1 \leq i \leq n$.

3 Main Results

Throughout the paper, A, A_{kj} denote any non-empty closed subsets of X.

Definition 3.1 *The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if*

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad \text{or} \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case we write $W_2 - \lim A_{kj} = A$.

Example 3.2 *Let $X = \mathbb{R}^2$ and $\{A_{kj}\}$ be the following double sequence:*

$$A_{kj} = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{kj} \right\}.$$

This double sequence of sets is Wijsman convergent to the set $A = \{(0, 1)\}$.

Definition 3.3 *The double sequence $\{A_{kj}\}$ is said to be Wijsman Cesàro summable to A if $\{d(x, A_{kj})\}$ Cesàro summable to $\{d(x, A)\}$; that is, for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) = d(x, A).$$

In this case we write $A_{kj} \overset{(W_2\sigma_1)}{\longrightarrow} A$.

Definition 3.4 *The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly Cesàro summable to A if $\{d(x, A_{kj})\}$ strongly Cesàro summable to $\{d(x, A)\}$; that is, for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case we write $A_{kj} \overset{[W_2\sigma_1]}{\longrightarrow} A$.

Example 3.5 *Let $X = \mathbb{R}^2$ and define the double sequence $\{A_{kj}\}$ by*

$$A_{kj} = \begin{cases} \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = k\} & , \ j = 1, \ \text{for all } k \\ \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = j\} & , \ k = 1, \ \text{for all } j \\ \{(0, 0)\} & , \ \text{otherwise.} \end{cases}$$

Then $\{A_{kj}\}$ is Wijsman convergent to the set $A = \{(0, 0)\}$ but

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj})$$

does not tend to a finite limit. Hence, $\{A_{kj}\}$ is not Wijsman Cesàro summable. Also, $\{A_{kj}\}$ is not Wijsman strongly Cesàro summable.

Definition 3.6 *The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly p-Cesàro summable to A if $\{d(x, A_{kj})\}$ strongly p-Cesàro summable to $\{d(x, A)\}$; that is, for each p positive real number and for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p = 0.$$

In this case we write $A_{kj} \stackrel{[W_2\sigma_p]}{\longrightarrow} A$.

Definition 3.7 *Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman lacunary convergent to A if for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} d(x, A_{kj}) = d(x, A).$$

In this case we write $A_{kj} \stackrel{(W_2N_\theta)}{\longrightarrow} A$.

Definition 3.8 *Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case we write $A_{kj} \stackrel{[W_2N_\theta]}{\longrightarrow} A$.

Definition 3.9 *Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly p-lacunary convergent to A if for each p positive real number and for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)|^p = 0.$$

In this case we write $A_{kj} \stackrel{[W_2^pN_\theta]}{\longrightarrow} A$.

Theorem 3.10 *For any double lacunary sequence θ, if $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then $[W_2\sigma_1] \subseteq [W_2N_\theta]$.*

Proof: Assume that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then there exist $\lambda, \mu > 0$ such that $q_r \geq 1 + \lambda$ and $q_u \geq 1 + \mu$ for all $r, u \geq 1$, which implies that

$$\frac{k_r j_u}{h_r\bar{h}_u} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.$$

Let $A_{kj} \overset{[W_2\sigma_1]}{\longrightarrow} A$. We can write

$$\begin{aligned}
\frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}}|d(x,A_{kj})-d(x,A)| &= \frac{1}{h_r\bar{h}_u}\sum_{i,s=1,1}^{k_r,j_u}|d(x,A_{is})-d(x,A)| \\
&\quad -\frac{1}{h_r\bar{h}_u}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}}|d(x,A_{is})-d(x,A)| \\
&= \frac{k_rj_u}{h_r\bar{h}_u}\Big(\frac{1}{k_rj_u}\sum_{i,s=1,1}^{k_r,j_u}|d(x,A_{is})-d(x,A)|\Big) \\
&\quad -\frac{k_{r-1}j_{u-1}}{h_r\bar{h}_u}\Big(\frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}}|d(x,A_{is})-d(x,A)|\Big).
\end{aligned}$$

Since $A_{kj} \overset{[W_2\sigma_1]}{\longrightarrow} A$, the terms

$$\frac{1}{k_rj_u}\sum_{i,s=1,1}^{k_r,j_u}|d(x,A_{is})-d(x,A)| \quad \text{and} \quad \frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}}|d(x,A_{is})-d(x,A)|$$

both tend to 0, and it follows that

$$\frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}}|d(x,A_{kj})-d(x,A)| \to 0,$$

that is, $A_{kj} \overset{[W_2N_\theta]}{\longrightarrow} A$. Hence, $[W_2\sigma_1] \subseteq [W_2N_\theta]$.

Theorem 3.11 *For any double lacunary sequence θ, if $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$ then $[W_2N_\theta] \subseteq [W_2\sigma_1]$.*

Proof: Assume that $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there exists $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r, u. Let $\{A_{kj}\} \in [W_2N_\theta]$ and $\varepsilon > 0$. Then we can find $R, U > 0$ and $K > 0$ such that

$$\sup_{i\geq R, s\geq U} \tau_{is} < \varepsilon \qquad \text{and} \qquad \tau_{is} < K \qquad \text{for all } i, s = 1, 2, \cdots,$$

where

$$\tau_{ru} = \frac{1}{h_r\bar{h}_u}\sum_{I_{ru}}|d(x,A_{kj})-d(x,A)|.$$

If t, v are any integers with $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$, where $r > R$ and $u > U$, then we can write

$$\begin{aligned}
\frac{1}{tv}\sum_{i,s=1,1}^{t,v}|d(x,A_{is})-d(x,A)| &\leq \frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_r,j_u}|d(x,A_{is})-d(x,A)| \\
&= \frac{1}{k_{r-1}j_{u-1}}\Bigg(\sum_{I_{11}}|d(x,A_{is})-d(x,A)| \\
&\quad +\sum_{I_{12}}|d(x,A_{is})-d(x,A)| \\
&\quad +\sum_{I_{21}}|d(x,A_{is})-d(x,A)| \\
&\quad +\sum_{I_{22}}|d(x,A_{is})-d(x,A)| \\
&\quad +\cdots+\sum_{I_{ru}}|d(x,A_{is})-d(x,A)|\Bigg) \\
&\leq \frac{k_1 j_1}{k_{r-1}j_{u-1}}.\tau_{11}+\frac{k_1(j_2-j_1)}{k_{r-1}j_{u-1}}.\tau_{12} \\
&\quad +\frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}.\tau_{21} \\
&\quad +\frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}j_{u-1}}.\tau_{22} \\
&\quad +\cdots+\frac{(k_R-k_{R-1})(j_U-j_{U-1})}{k_{r-1}j_{u-1}}\tau_{RU} \\
&\quad +\cdots+\frac{(k_r-k_{r-1})(j_u-j_{u-1})}{k_{r-1}j_{u-1}}\tau_{ru} \\
&\leq \left(\sup_{i,s\geq 1,1}\tau_{is}\right)\frac{k_R j_U}{k_{r-1}j_{u-1}} \\
&\quad +\left(\sup_{i\geq R,s\geq U}\tau_{is}\right)\frac{(k_r-k_R)(j_u-j_U)}{k_{r-1}j_{u-1}} \\
&\leq K\frac{k_R j_U}{k_{r-1}j_{u-1}}+\varepsilon MN.
\end{aligned}$$

Since $k_{r-1}, j_{u-1} \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv} \sum_{i,s=1,1}^{t,v} |d(x, A_{is}) - d(x, A)| \to 0$$

and consequently $\{A_{kj}\} \in [W_2\sigma_1]$. Hence, $[W_2N_\theta] \subseteq [W_2\sigma_1]$.

Theorem 3.12 *For any double lacunary sequence θ, if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ and $1 < \liminf_u q_u \leq \limsup_u q_u < \infty$, then $[W_2N_\theta] = [W_2\sigma_1]$.*

Proof: This follows from Theorem 3.10 and Theorem 3.11.

Theorem 3.13 *For any double lacunary sequence θ, let $\{A_{kj}\} \in [W_2N_\theta] \cap [W_2\sigma_1]$. If $A_{kj} \xrightarrow{[W_2N_\theta]} A$ and $A_{kj} \xrightarrow{[W_2\sigma_1]} B$ then $A = B$.*

Proof: Let $A_{kj} \xrightarrow{[W_2\sigma_1]} A$, $A_{kj} \xrightarrow{[W_2N_\theta]} B$ and suppose that $A \neq B$. We can write

$$\begin{aligned}
\upsilon_{ru} + \tau_{ru} &= \frac{1}{h_r\overline{h}_u} \sum_{k,j \in I_{ru}} |d(x, A_{kj}) - d(x, A)| + \frac{1}{h_r\overline{h}_u} \sum_{k,j \in I_{ru}} |d(x, A_{kj}) - d(x, B)| \\
&\geq \frac{1}{h_r\overline{h}_u} \sum_{k,j \in I_{ru}} |d(x, A) - d(x, B)| \\
&= |d(x, A) - d(x, B)|,
\end{aligned}$$

where

$$\upsilon_{ru} = \frac{1}{h_r\overline{h}_u} \sum_{k,j \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \text{ and } \tau_{ru} = \frac{1}{h_r\overline{h}_u} \sum_{k,j \in I_{ru}} |d(x, A_{kj}) - d(x, B)|.$$

Since $\{A_{kj}\} \in [W_2N_\theta]$, $\tau_{ru} \to 0$. Thus for sufficiently large r, u we must have

$$\upsilon_{ru} > \frac{1}{2}|d(x, A) - d(x, B)|.$$

Observe that

$$\begin{aligned}
\frac{1}{k_rj_u} \sum_{i,s=1,1}^{k_r,j_u} |d(x, A_{is}) - d(x, A)| &\geq \frac{1}{k_rj_u} \sum_{I_{ru}} |d(x, A_{is}) - d(x, A)| \\
&= \frac{(k_r - k_{r-1})(j_u - j_{u-1})}{k_rj_u} . \upsilon_{ru} \\
&= \left(1 - \frac{1}{q_r}\right)\left(1 - \frac{1}{q_u}\right) . \upsilon_{ru} \\
&> \frac{1}{2}\left(1 - \frac{1}{q_r}\right)\left(1 - \frac{1}{q_u}\right) . |d(x, A) - d(x, B)|
\end{aligned}$$

for sufficiently large r, u. Since $\{A_{kj}\} \in [W_2\sigma_1]$, the left hand side of the inequality above convergent to 0, so we must have $q_r \to 1$ and $q_u \to 1$. But this implies, by proof of Theorem 3.11, that

$$[W_2 N_\theta] \subset [W_2\sigma_1].$$

That is, we have

$$A_{kj} \stackrel{[W_2N_\theta]}{\longrightarrow} B \Rightarrow A_{kj} \stackrel{[W_2\sigma_1]}{\longrightarrow} B,$$

and therefore

$$\frac{1}{tv}\sum_{i,s=1,1}^{t,v} |d(x, A_{is}) - d(x, B)| \to 0.$$

Then, we have

$$\begin{aligned} \frac{1}{tv}\sum_{i,s=1,1}^{t,v} |d(x, A_{is}) - d(x, B)| \quad + \quad & \frac{1}{tv}\sum_{i,s=1,1}^{t,v} |d(x, A_{is}) - d(x, A)| \\ & \geq |d(x, A) - d(x, B)| > 0, \end{aligned}$$

which yields a contradiction to our assumption, since both terms on the left hand side tend to 0. That is, for each $x \in X$,

$$|d(x, A) - d(x, B)| = 0,$$

and therefore $A = B$.

Definition 3.14 *The double sequence $\theta' = \{(k_r', j_u')\}$ is called double lacunary refinement of the double lacunary sequence $\theta = \{(k_r, j_u)\}$ if $\{k_r\} \subseteq \{k_r'\}$ and $\{j_u\} \subseteq \{j_u'\}$.*

Theorem 3.15 *If θ' is a double lacunary refinement of double lacunary sequence θ and if $\{A_{kj}\} \notin [W_2N_\theta]$, then $\{A_{kj}\} \notin [W_2N_{\theta'}]$.*

Proof: Let $\{A_{kj}\} \notin [W_2N_\theta]$. Then, for any non-empty closed subset $A \subseteq X$ there exists $\varepsilon > 0$ and a subsequence (k_{r_n}) of (k_r) and (j_{u_n}) of (j_u) such that

$$\tau_{r_n u_n} = \frac{1}{h_{r_n}\bar{h}_{u_n}} \sum_{k,j=1,1}^{k_{r_n}, j_{u_n}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon.$$

Writing

$$I_{r_n u_n} = I'_{s+1,t+1} \cup I'_{s+1,t+2} \cup I'_{s+2,t+1} \cup I'_{s+2,t+2} \cup ... \cup I'_{s+p,t+p}$$

where

$k_{r_n-1} = k'_s < k'_{s+1} < ... < k'_{s+p} = k_{r_n}$ and $j_{u_{n-1}} = j'_t < j'_{t+1} < ... < j'_{t+p} = j_{u_n}$.

Then we have

$$\tau_{r_n u_n} = \frac{\sum\limits_{I'_{s+1,t+1}} |d(x, A_{kj}) - d(x, A)| + ... + \sum\limits_{I'_{s+p,t+p}} |d(x, A_{kj}) - d(x, A)|}{h'_{s+1}\overline{h}'_{t+1} + ... + h'_{s+p}\overline{h}'_{t+p}}.$$

It follows from Lemma 2.1 that

$$\frac{1}{h'_{s+p}\overline{h}'_{t+p}} \sum_{I'_{s+p,t+p}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon$$

for some j and consequently, $\{A_{kj}\} \notin [W_2 N_{\theta'}]$.

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