Gen. Math. Notes, Vol. 5, No. 1, July 2011, pp. 7-14
ISSN 2219-7184; Copyright © ICSRS Publication, 2011
www.i-csrs.org
Available free online at http://www.geman.in

Free Actions on Semiprime Gamma Rings

Kalyan Kumar Dey[1] and Akhil Chandra Paul[2]

[1]Department of Mathematics, Rajshahi University,
Rajshahi-6205, Bangladesh
E-mail: kkdmath@yahoo.com
[2]Department of Mathematics, Rajshahi University,
Rajshahi-6205, Bangladesh
E-mail: acpaulrubd_math@yahoo.com

(Received: 25-5-11/Accepted: 13-6-11)

Abstract

Let M be a semiprime Γ-ring. We study on some mappings related to left centralizers, centralizers, derivations, (σ, τ)-derivations and generalized (σ, τ)-derivations which are free actions on semiprime Γ-rings. If $\varphi(x) = T(x)\alpha x - x\alpha T(x)$ for all $x \in M$, $\alpha \in \Gamma$ is a mapping from M into M. Then we show that it is a free action. If $F : M \rightarrow M$ is a generalized (σ, τ)-derivation with associate (σ, τ)-derivation d, and a in F is a dependent element, then we also show that it is a dependent element of $(\sigma + d)$. Furthermore, we prove that for centralizer f and a derivation d of a semiprime Γ-ring M, $\varphi = d \circ f$ is a free action.

Keywords: *prime Γ-ring, semiprime Γ-ring, dependent element, free action, centralizer, derivation.*

1 Introduction

Let M and Γ be additive abelian groups. M is called a Γ-ring if for all $a,b,c \in M$, $\alpha,\beta \in \Gamma$ the following conditions are satisfied :

(i) $a\beta b \in M$,
(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Throughout, Z(M) denote the center of M. As usual, the commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$. We know that $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y + x[\beta,\alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z + y[\beta,\alpha]_x z$, for all $x,y,z\in M$ and for all $\alpha,\beta\in\Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$. Using the assumption (*) the identities $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z$, for all $x,y,z\in M$ and for all $\alpha,\beta\in\Gamma$ are used extensively in our results. Recall that a Γ-ring M is prime if $a\Gamma M\Gamma b = 0$ implies that either $a = 0$ or $b = 0$, and is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. An additive mapping $D: M \to M$ is called a derivation provided $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all pairs $x,y\in M$, $\alpha\in\Gamma$. Let σ be an automorphism of a Γ-ring M. An additive mapping $D: M \to M$ is called an σ-derivation if $D(x\alpha y) = D(x)\alpha\sigma(y) + x\alpha D(y)$ holds for all $x,y\in M$, $\alpha\in\Gamma$. Note that the mapping, $D = \sigma - I$, where I denotes the identity mapping on M, is an σ-derivation. Of course, the concept of an σ-derivation generalizes the concept of a derivation, since any I-derivation is a derivation. σ-derivations are further generalized as (σ, τ)-derivations. Let σ, τ be automorphisms of M, then an additive mapping $D: M \to M$ is called an (σ, τ)-derivation if $D(x\alpha y) = D(x)\alpha\sigma(y) + \tau(x)\alpha D(y)$ holds for all pairs $x,y\in M$, $\alpha\in\Gamma$. σ-derivations and (σ, τ)-derivations have been applied in various situations; in particular, in the solution of some functional equations. An additive mapping F of a Γ-ring M into itself is called a generalized derivation, with the associated derivation d, if there exists a derivation d of M such that $F(x\alpha y) = F(x)\alpha y + x\alpha d(y)$ for all $x,y\in M$, $\alpha\in\Gamma$. The concept of a generalized derivation covers both the concepts of a derivation and of a left centralizer provided F = d and d = 0, respectively. An additive mapping $f : M \to M$ is called centralizing (commuting) if $[f(x), x]_\alpha \in Z(M)$ ($[f(x), x]_\alpha = 0$) for all $x\in M$, $\alpha\in\Gamma$. An additive mapping $T : M \to M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) for all $x, y\in M$, $\alpha\in\Gamma$. If $a\in M$, then $L_a(x) = a\alpha x$ and $R_a(x) = x\alpha a$ ($x\in M$, $\alpha\in\Gamma$) define a left centralizer and a right centralizer of M, respectively.

An additive mapping $T: M \to M$ is called a centralizer if $T(x\alpha y) = T(x)\alpha y = x\alpha T(y)$ for all $x, y\in M$, $\alpha\in\Gamma$. An element $a\in M$ is called a dependent element of a mapping $F : M \to M$ if $F(x)\alpha a = a\alpha x$ holds for all $x\in M$, $\alpha\in\Gamma$. A mapping $F : M \to M$ is called a free action if zero is the only dependent element of F. For a mapping $F : M \to M$, D(F) denotes the collection of all dependent elements of F.

The notion of a free action has been introduced by Murray and Neumann [7] and von Neumann [8] to study abelian von Neumann algebras.

Laradji and Thaheem [6] introduced the dependent elements of the endomorphism of semiprime rings and obtained a number of results of [5] to semiprime rings.

Vukman and Kosi-Ulbl [12, 13] and Vukman [11] have made further study of dependent elements of various mappings related to automorphisms, derivations, (α, β)-derivations and generalized derivations of semiprime rings.

Chaudhry and Samman [3] studied on dependent elements of mappings and free actions of semiprime rings by the motivation of the work of Laradji and Thaheem [6], Vukman and Kosi-Ulbl [13] and Vukman [11].

In this paper, motivated the works in [6] we obtain the analogous results of [6] on Γ-rings.

2. Results

Lemma 2.1 *Let M be a semiprime Γ-ring satisfying the condition (*). Let* $a\beta[x, y]_\alpha = 0$, *for* $a,x,y\in M$, $\alpha,\beta\in\Gamma$, *then* $a\in Z(M)$.

Proof

Since $a\beta[x, y]_\alpha = 0$, for $a,x,y\in M$, $\alpha,\beta\in\Gamma$, then replace y by a, we get $a\beta[x, a]_\alpha = 0$, for $a,x\in M$, $\alpha,\beta\in\Gamma$. Thus we get $a\beta x\alpha a = a\beta a\alpha x$, for all $a,x\in M$, $\alpha,\beta\in\Gamma$.

Now $[a, x]_\alpha\beta[a, y]_\alpha = (a\alpha x - x\alpha a)\beta(a\alpha y - y\alpha a)$

$= a\alpha x\beta a\alpha y - a\alpha x\beta y\alpha a - x\alpha a\beta a\alpha y + x\alpha a\beta y\alpha a$

$= a\alpha(x\beta a)\alpha y - a\alpha(x\beta y)\alpha a - x\alpha a\beta a\alpha y + x\alpha a\beta(y\alpha a)$

$= a\alpha a\beta x\alpha y - a\alpha a\alpha x\beta y - x\alpha a\beta a\alpha y + x\alpha a\beta a\alpha y$

$= a\alpha a\beta x\alpha y - a\alpha a\alpha x\beta y = a\alpha a\beta x\alpha y - a\alpha a\beta x\alpha y = 0$, for all $a,x,y\in M$, $\alpha,\beta\in\Gamma$.

Hence $[a, x]_\alpha\beta[a, y]_\alpha = 0$, for all $a,x,y\in M$, $\alpha,\beta\in\Gamma$.

Replace y by $y\delta x$, we get, $[a, x]_\alpha\beta[a, y\delta x]_\alpha = 0$

$\Rightarrow[a, x]_\alpha\beta y\delta[a, x]_\alpha + [a, x]_\alpha\beta[a, y]_\alpha\delta x = 0$,

$\Rightarrow[a, x]_\alpha\beta y\delta[a, x]_\alpha = 0$, for all $a,x,y\in M$, $\alpha,\beta,\delta\in\Gamma$. By the semiprimeness of M we get, $[a, x]_\alpha = 0$, for all $a,x\in M$, $\alpha\in\Gamma$.

Hence $a\in Z(M)$, for all $a\in M$.

Theorem 2.2. *Let M be a semiprime Γ-ring satisfying the assumption (*) and T a left centralizer of M. Then $a\in D(T)$ if and only if $a\in Z(M)$ and $T(a) = a$.*

Proof. Let $a\in D(T)$. Then

(1) $T(x)\alpha a = a\alpha x$, $\alpha\in\Gamma$.

Replacing x by $x\beta y$ in (1), we get $T(x\beta y)\alpha a = a\alpha x\beta y$, $x,y\in M$, $\alpha,\beta\in\Gamma$. That is,

(2) $T(x)\beta y\alpha a = a\alpha x\beta y$, $x,y\in M$, $\alpha,\beta\in\Gamma$.

Multiplying (2) by δz on the right, we get

(3) $T(x)\beta y a\delta z = a\alpha x\beta y\delta z$, $x,y,z\in M$, $\alpha,\beta,\delta\in\Gamma$.

Replacing y by $y\delta z$ in (2), we get

(4) $T(x)\beta y\delta z\alpha a = a\alpha x\beta y\delta z$, $x,y,z\in M$, $\alpha,\beta,\delta\in\Gamma$.

Subtracting (4) from (3), we get $T(x)\beta y\delta(a\alpha z - z\alpha a) = T(x)\beta y\delta[a, z]_\alpha = 0$, $x,y,z\in M$, $\alpha,\beta,\delta\in\Gamma$. Replacing y by $a\delta y$ and then using semiprimeness of M, we get $T(x)\beta a\delta[a, z]_\alpha = 0$. That is, $a\beta x\delta[a, z]_\alpha = 0$, which, by semiprimeness of M, implies $a\beta[a, z]_\alpha = 0$ for all $a\in M$, $\alpha,\beta\in\Gamma$. By lemma 2.1 we get $a\in Z(M)$. Since $a\in Z(M)$, we have $a\alpha y = y\alpha a$, $\alpha\in\Gamma$. Thus $T(a\alpha y) = T(y\alpha a)$, $\alpha\in\Gamma$. That is, $T(a)\alpha y = T(y)\alpha a = a\alpha y$. So $(T(a) - a)\alpha y = 0$, Thus we get,

$(T(a) - a)\alpha y(T(a) - a) = 0$. By semiprimeness of M, implies $T(a) - a = 0$. Thus $T(a) = a$.

Conversely, let $T(a) = a$ and $a\in Z(M)$. Then $T(x)\alpha a = T(x\alpha a) = T(a\alpha x) = T(a)\alpha x = a\alpha x$. Thus $a\in D(T)$.

Theorem 2.3. *Let M be a prime Γ-ring and $T \neq I$ a left centralizer of M. Then T is a free action on M.*

Proof. Let $a\in D(T)$. Then $T(x)\alpha a = a\alpha x$, $\alpha\in\Gamma$.Moreover, $a\in Z(M)$ by Theorem 2.2. Thus $T(x)\alpha a = x\alpha a$, $\alpha\in\Gamma$. That is,

(5) $(T(x) - x)\alpha a = 0$, $\alpha\in\Gamma$.

Since $a\in Z(M)$, from (5) we get $(T(x) - x)\alpha z\beta a = 0$ for all $z\in M$, $\alpha,\beta\in\Gamma$. Since $T \neq I$ and M is prime, we have $a = 0$. So T is a free action.

Theorem 2.4. *Let M be a semiprime Γ-ring satisfying the condition (*) and T an injective left centralizer of M. Then $\varphi = T + I$ is a free action on M.*

Proof . Obviously T + I is a left centralizer of M. Let $a \in D(T + I)$. Then by Theorem 2.2, $a \in Z(M)$ and $(T + I)(a) = T(a) + I(a) = T(a) + a = a$. Thus $T(a) = 0$. So $a \in Ker(T)$. Since T is injective, we have $a = 0$. Hence T is a free action.

Theorem 2.5. *Let T be a left centralizer of a semiprime Γ-ring M satisfying the condition (*). Then* $\varphi : M \to M$, *defined by* $\varphi(x) = [T(x), x]_\alpha$ *for all* $x \in M, \alpha \in \Gamma$ *is a free action.*

Proof . Let $a \in D(\varphi)$. Then
(6) $[T(x), x]_\alpha\beta a = a\beta x$ for all $x \in M, \alpha,\beta \in \Gamma$.
Linearizing (6) and using (6) after linearization, we get
(7) $[T(x), y]_\alpha\beta a + [T(y), x]_\alpha\beta a = 0$.
Replacing y by $a\delta y$ in (7), we get
$0 = [T(x), a\delta y]_\alpha\beta a + [T(a\delta y), x]_\alpha\beta a = a\delta[T(x), y]_\alpha\beta a + [T(x), a]_\alpha\delta y\beta a + [T(a)\delta y, x]_\alpha\beta a$
$= a\delta[T(x), y]_\alpha\beta a + [T(x), a]_\alpha\delta y\beta a + T(a)\delta[y, x]_\alpha\beta a + [T(a), x]_\alpha\delta y\beta a$.
That is,
(8) $a\delta[T(x), y]_\alpha\beta a + [T(x), a]_\alpha\delta y\beta a + T(a)\delta[y, x]_\alpha\beta a + [T(a), x]_\alpha\delta y\beta a = 0$.
Using (7), from (8) we get
$- a\delta[T(y), x]_\alpha\beta a + [T(x), a]_\alpha\delta y\beta a + T(a)\delta[y, x]_\alpha\beta a + [T(a), x]_\alpha\delta y\beta a = 0$, which implies
(9) $- a\delta[T(a), a]_\alpha\beta a + [T(a), a]_\alpha\delta a\beta a + [T(a), a]_\alpha\delta a\beta a = 0$.
using (6), from (9) we get $- a\delta a\beta a + a\delta a\beta a + a\delta a\beta a = 0$. That is, $a\delta a\beta a = 0$. Putting $\delta = \beta$, we have $a\delta a\delta a = 0 \Rightarrow (a\delta)^2 a = 0$. Hence a is nilpotent. But we know that a semiprime Γ-ring does not contain nonzero nilpotent element. Hence $a = 0$. Hence φ is a free action.

Theorem 2.6. *Let M be a semiprime Γ-ring satisfying the condition (*) and* $d: M \to M$ *a derivation. Then the mapping* $\varphi: M \to M$, *defined by* $\varphi(x) = [d(x), x]_\alpha$ *for all* $x \in M, \alpha \in \Gamma$ *is a free action.*

Proof. Let $a \in D(\varphi)$. Then
(10) $\varphi(x)\beta a = [d(x), x]_\alpha\beta a = a\beta x, \alpha,\beta \in \Gamma$.
Linearizing (10) and using (10) after linearization, we get
(11) $[d(x), y]_\alpha\beta a + [d(y), x]_\alpha\beta a = 0$ for all $x, y \in M, \alpha,\beta \in \Gamma$.
Replacing y by x in (11), we get
(12) $2[d(x), x]_\alpha\beta a = 0$ for all $x \in M, \alpha,\beta \in \Gamma$.
Replacing y by $x\delta y$ in (11), we get
$0 = [d(x), x\delta y]_\alpha\beta a + [d(x\delta y), x]_\alpha\beta a$
$= x\delta[d(x), y]_\alpha\beta a + [d(x), x]_\alpha\delta y\beta a + [d(x)\delta y + x\delta d(y), x]_\alpha\beta a$
$= x\delta[d(x), y]_\alpha\beta a + [d(x), x]_\alpha\delta y\beta a + d(x)\delta[y, x]_\alpha\beta a + [d(x), x]_\alpha\delta y\beta a + x\delta[d(y), x]_\alpha\beta a$
$+ [x, x]_\alpha\delta d(y)\beta a$.
That is,
(13) $0 = x\delta\{[d(x), y]_\alpha\beta a + [d(y), x]_\alpha\beta a\} + 2[d(x), x]_\alpha\delta y\alpha a + d(x)\delta[y, x]_\alpha\beta a$.
Using (11), from (13) we get
(14) $2[d(x), x]\alpha\delta y\beta a + d(x)\delta[y, x]\alpha\beta a = 0$ for all $x, y \in M$.
Replacing y by $y\gamma a$ in (14), we get
$0 = 2[d(x), x]_\alpha\delta y\gamma a\beta a + d(x)\delta[y\gamma a, x]_\alpha\beta a$
$= 2[d(x), x]_\alpha\delta y\gamma a\beta a + d(x)\delta[y, x]_\alpha\gamma a\beta a + d(x)\delta y\gamma[a, x]_\alpha\beta a$.
That is,
(15) $(2[d(x), x]_\alpha\delta y\gamma a + d(x)\delta[y, x]_\alpha\gamma a)\beta a + d(x)\delta y\gamma[a, x]_\alpha\beta a = 0$.
Using (14), from (15) we get

(16) $d(x)\delta y\gamma[a, x]_\alpha\beta a = 0$.
Replacing y by $x\lambda y$ in (16), we get
(17) $d(x)\delta x\lambda y\gamma[a, x]_\alpha\beta a = 0$.
Multiplying (16) by $x\lambda$ on the left, we get
(18) $x\lambda d(x)\delta y\gamma[a, x]_\alpha\beta a = 0$.
Subtracting (18) from (17), we get $[d(x), x]_\lambda\delta y\gamma[a, x]_\alpha\beta a = 0$. Replacing y by $a\alpha y$ in the last identity and then using (10), we get
(19) $a\delta x\alpha y\gamma[a, x]_\alpha\beta a = 0$.
Replacing y by $a\beta a\delta y$ in (19), we get
(20) $a\delta x\alpha a\beta a\delta y\gamma[a, x]_\alpha\beta a = 0$.
Multiplying (19) on the left by a and replacing y by $a\delta y$ in (19), we get
(21) $a\alpha a\delta x\alpha a\delta y\gamma[a, x]_\alpha\beta a = 0$.
Subtracting (20) from (21), we get
(22) $a\delta(a\alpha x - x\alpha a)\beta a\delta y\lambda[a, x]_\alpha\beta a = 0$.
Replacing y by $y\delta a$ in (22), we get $a\delta[a, x]_\alpha\beta a\delta y\delta a\lambda[a, x]_\alpha\beta a = 0$, which, by semiprimeness of M, implies that $a\delta[a, x]_\alpha\beta a = 0$. In particular,
$a\delta[d(a), a]_\alpha\beta a = 0$. This, by (10), implies that $a\delta a\beta a = 0$. Putting $\delta = \beta$, we have $a\delta a\delta a = 0 \Rightarrow (a\delta)^2a = 0$. Hence a is nilpotent. But we know that a semiprime Γ-ring does not contain a nonzero nilpotent element. Hence a = 0. Hence we get that $\varphi(x) = [d(x), x]_\alpha$ is a free action on M.
We now define a generalized (σ, τ)-derivation of a Γ-ring M.

Definition 2.7. *Let σ and τ be automorphisms of a Γ-ring M. An additive mapping F : M → M is called a generalized (σ, τ)-derivation, with the associated (σ, τ)-derivation d, if there exists an (σ, τ)-derivation d of M such that* $F(x\alpha y) = \sigma(x)\alpha F(y) + d(x)\alpha\tau(y)$.

Remark 2.8. *We note that for F = d, F is an (σ, τ)-derivation and for d = 0 and σ = I, F is a right centralizer. So a generalized (σ, τ)-derivation covers both the concepts of an (σ, τ)-derivation and a right centralizer.*

Theorem 2.9. *Let M be a semiprime Γ-ring satisfying the condition (*). Let σ, τ be centralizing automorphisms of M and let F : M → M be a generalized (σ, τ)-derivation with the associated (σ, τ)-derivation d. If a is a dependent element of F, then* $a\in D(\sigma + d)$.

Proof . Let $a\in D(F)$. Then
(23) $F(x)\alpha a = a\alpha x$ for all $x\in M$, $\alpha\in\Gamma$.
Replacing x by $x\beta y$ in (23), we get $F(x\beta y)\alpha a = a\alpha x\beta y$, which implies $\sigma(x)\beta F(y)\alpha a + d(x)\beta\tau(y)\alpha a = a\alpha x\beta y$, $\alpha,\beta\in\Gamma$. That is, $\sigma(x)\beta a\alpha y + d(x)\beta\tau(y)\alpha a = a\alpha x\beta y = F(x)\alpha a\beta y$. Thus
(24) $(F(x)\alpha a - \sigma(x)\alpha a)\beta y = d(x)\beta\tau(y)\alpha a$.
Multiplying (24) by δz on the right, we get
$(F(x)\alpha a - \sigma(x)\alpha a)\beta y\delta z = d(x)\beta\tau(y)\alpha a\delta z$.
(25) $(F(x)\alpha a - \sigma(x)\alpha a)\beta y\delta z = d(x)\beta\tau(y)\delta a\alpha z$, $\alpha,\beta,\delta\in\Gamma$.
Replacing y by $y\delta z$ in (24), we get
(26) $(F(x)\alpha a - \sigma(x)\alpha a)\beta y\delta z = d(x)\beta\tau(y)\delta\tau(z)\alpha a$.
Subtracting (25) from (26), we get $d(x)\beta\tau(y)\delta(\tau(z)\alpha a - a\alpha z) = 0$, which, due to surjectivity of τ, implies
(27) $d(x)\beta y\delta(\tau(z)\alpha a - a\alpha z) = 0$.

Since τ is centralizing and M is semiprime, from (27) we get $d(x)\beta(\tau(z)\alpha a - a) = 0$. That is,
(28) $d(x)\beta\tau(z)\alpha a = d(x)\beta a\alpha z$ for all $x, z \in M$, $\alpha,\beta \in \Gamma$.
Using (28), from (24) we get $(F(x)\alpha a - \sigma(x)\alpha a)\beta y = d(x)\alpha a\beta y$. That is, $(F(x)\alpha a - \sigma(x)\alpha a - d(x)\alpha a)\beta y = 0$. Hence $(F(x)\alpha a - \sigma(x)\alpha a - d(x)\alpha a)\beta y\delta(F(x)\alpha a - \sigma(x)\alpha a - d(x)\alpha a) = 0$, $\alpha,\beta,\delta \in \Gamma$, which implies due to semiprimeness of M, $(F(x)\alpha a - \sigma(x)\alpha a - d(x)\alpha a) = 0$. Thus
(29) $F(x)\alpha a - (\sigma + d)(x)\alpha a = 0$.
Using (23), from (29) we get
(30) $(\sigma + d)(x)\alpha a = a\alpha x$.
This shows that $a \in D(\sigma + d)$.
We now have the following result as a corollary of Theorem 2.9.

Corollary 2.10. *If F is an (σ, τ)-derivation of a semiprime Γ-ring M satisfying the condition (*), then F is a free action.*

Proof . Let F = d. Then d is an (σ, τ)-derivation and so equation (30) gives
$(\sigma + F)(x)\alpha a = a\alpha x$. That is, $\sigma(x)\alpha a + F(x)\alpha a = a\alpha x$, which implies that $\sigma(x)\alpha a + a\alpha x = a\alpha x$. Thus $\sigma(x)\alpha a = 0$ for all $x \in M$. Since σ is onto, we have $x\alpha a = 0$ for all $x \in M$. Thus $a\alpha x\beta a = 0$. Putting $\alpha = \beta$, we have $a\alpha a\alpha a = 0 \Rightarrow (a\alpha)^2 a = 0$. Hence a is nilpotent. But we know that a semiprime Γ-ring does not contain a nonzero nilpotent element. Hence a = 0. Hence F is a free action.

Corollary 2.11. *Let M be a semiprime Γ-ring satisfying the condition (*) and σ a centralizing automorphism of M. Let $F : M \to M$ be an additive mapping satisfying $F(x\alpha y) = \sigma(x)\alpha F(y)$ for all $x, y \in M$, $\alpha \in \Gamma$. If $a \in D(F)$, then $a \in Z(M)$.*

Proof . We take d = 0 in Theorem 2.9. Then $F(x\alpha y) = \sigma(x)\alpha F(y)$ and $a \in D(F)$ implies that $a \in D(\sigma)$. Since σ is a centralizing automorphism, we obtained that $a \in Z(M)$.

Remark 2.12. *If in the above corollary we take $\sigma = I$, the identity automorphism, then F is a right centralizer. Thus all dependent elements of a right centralizer F of a semiprime Γ-ring M lie in Z(M).*

Theorem 2.13. *Let M be a semiprime Γ-ring. Let f be a centralizer and d a derivation of M. Then $\varphi = (d \circ f)$ is a free action.*

Proof . Let $a \in D(\varphi)$. Then $\varphi(x)\alpha a = a\alpha x$. That is,
(31) $(d \circ f)(x)\alpha a = a\alpha x$ for all $x \in M$, $\alpha \in \Gamma$.
Replacing x by $x\beta y$ in (31), we get
$a\alpha x\beta y = (d \circ f)(x\beta y)\alpha a = d(f(x)\beta y)\alpha a = (d \circ f)(x)\beta y\alpha a + f(x)\beta d(y)\alpha a$.
That is, $(d \circ f)(x)\beta y\alpha a + f(x)\beta d(y)\alpha a = a\alpha x\beta y = (d \circ f)(x)\alpha a\beta y$. Thus,
(32) $(d \circ f)(x)\beta[a, y]_\alpha = f(x)\beta d(y)\alpha a$ for all $x, y \in M$, $\alpha,\beta \in \Gamma$.
Replacing y by $a\delta y$ in (32), we get $(d \circ f)(x)\beta[a, a\delta y] = f(x)\beta d(a\delta y)\alpha a$. That is,
(33) $(d \circ f)(x)\beta a\delta[a, y]_\alpha = f(x)\beta d(a)\delta y\alpha a + f(x)\beta a\delta d(y)\alpha a$, $\alpha,\beta,\delta \in \Gamma$.
Using (31), from (33) we get
(34) $a\beta x\delta[a, y]_\alpha = f(x)\beta d(a)\delta y\alpha a + f(x)\beta a\delta d(y)\alpha a$.
Multiplying (34) on the left by $z\alpha$, we get
(35) $z\alpha a\beta x\delta[a, y]_\alpha = z\alpha f(x)\beta d(a)\delta y\alpha a + z\alpha f(x)\beta a\delta d(y)\alpha a$.
Replacing x by $z\alpha x$ in (34), we get $a\beta z\alpha x\delta[a, y]_\alpha = f(z\alpha x)\beta d(a)\delta y\alpha a + f(z\alpha x)\beta a\delta d(y)\alpha a$

$= z\alpha f(x)\beta d(a)\delta y\alpha a + z\alpha f(x)\beta a\delta d(y)\alpha a$. That is,

(36) $a\alpha z\beta x\delta[a, y]_\alpha = z\alpha f(x)\beta d(a)\delta y\alpha a + z\alpha f(x)\beta a\delta d(y)$ for all $x, y, z\in M$.

Subtracting (35) from (36), we get $[a, z]_\alpha\beta x\delta[a, y]_\alpha = 0$. In particular, $[a, y]_\alpha\beta x\delta[a, y]_\alpha = 0$, which, by semiprimeness of M, implies $[a, y]_\alpha = 0$ for all $y\in M$, $\alpha\in\Gamma$. Thus $a\in Z(M)$, so from (32) we get

(37) $f(x)\beta d(y)\alpha a = 0$ for all $x, y\in M$, $\alpha,\beta\in\Gamma$.

Since $f(y) = y$ (by lemma 2.1) in (37) and then using (31) we get $f(x)\beta a\alpha y = 0$, that is $f(x)\beta a\alpha y\delta f(x)\beta a = 0$. By semiprimeness of M, this implies that

(38) $f(x)\beta a = 0$, $\beta\in\Gamma$.

Thus $d(f(x)\beta a) = d(0) = 0$. That is $(d\circ f)(x)\beta a + f(x)\beta d(a) = 0$, which implies that

(39) $(d\circ f)(x)\beta a\alpha a + f(x)\beta d(a)\alpha a = 0$.

Using (37) and (31), from (39) we get $a\alpha x\beta a = 0$. Putting $\alpha = \beta$, we have $a\alpha a\alpha a = 0 \Rightarrow (a\alpha)^2a = 0$. Hence a is nilpotent. But we know that a semiprime Γ-ring does not a contain nonzero nilpotent element. Hence $a = 0$, which implies that $(d\circ f)$ is a free action.

Theorem 2.14. *Let f be a left centralizer of a semiprime Γ-ring M satisfying the assumption (*). Let $\varphi(x) = f(x)\alpha x + x\alpha f(x)$. Then φ is a free action on M.*

Proof . Let $a\in D(\varphi)$. Then $\varphi(x)\alpha a = a\alpha x$, $\alpha\in\Gamma$. That is,

(40) $(f(x)\alpha x + x\alpha f(x))\beta a = a\beta x$.

Linearizing (40), we get

(41) $(f(x)\alpha y + f(y)\alpha x + y\alpha f(x) + x\alpha f(y))\beta a = 0$.

Replacing both x and y by a in (41) and using (40), we get

$0 = (f(a)\alpha a + f(a)\alpha a + a\alpha f(a) + a\alpha f(a))\beta a = 2(f(a)\alpha a + a\alpha f(a))\beta a = 2a\beta a$. That is,

(42) $2a\alpha a = 0$.

Now replacing y by $x\delta a$ in (41) and using (40), we get

$0 = (f(x)\alpha x\delta a + f(x\delta a)\alpha x + x\delta a\alpha f(x) + x\alpha f(x\delta a))\beta a$

$= (f(x)\alpha x\delta a + f(x)\delta a\alpha x + x\delta a\alpha f(x) + x\alpha f(x)\delta a)\beta a$

$= (f(x)\alpha x + x\alpha f(x))\delta a\beta a + f(x)\delta a\alpha x\beta a + x\alpha f(x)\beta a$

$= a\alpha x\beta a + f(x)\delta a\beta x\alpha a + x\beta a\delta f(x)\alpha a$.

That is,

(43) $a\beta x\alpha a + f(x)\beta a\delta x\alpha a + x\beta a\delta f(x)\alpha a = 0$ for all $x\in M$.

Replacing x by a in (43) and using (40) and (42), we get $0 = a\beta a\alpha a+f(a)\delta a\beta a\alpha a+a\beta a\delta f(a)\alpha a = a\beta a\alpha a + f(a)\delta a\beta a\alpha a - a\beta a\delta f(a)\alpha a$. That is,

(44) $a\beta a\alpha + f(a)\delta a\beta a\alpha a - a\beta a\delta f(a)\alpha a = 0$.

Replacing x by a in (40), we get

(45) $f(a)\alpha a\beta a + a\beta f(a)\alpha a = a\beta a$.

Multiplying (45) by a on the left as well as on the right, we get

(46) $a\alpha f(a)\delta a\beta a + a\beta a\delta f(a)\alpha a = a\beta a\alpha$

and

(47) $f(a)\delta a\beta a\alpha a + a\beta f(a)\delta a\alpha a = a\beta a\alpha a$,

respectively. Subtracting (46) from (47), we get

(48) $f(a)\delta a\beta a\alpha a - a\beta a\delta f(a)\alpha a = 0$.

Using (48), from (44) we get $a\beta a\alpha a = 0$. Thus $a = 0$, which implies that φ is a free action.

References

[1] W.E. Barnes, On the Γ-rings of Nabusawa, *Pacific J. Math.*, 18 (1966), 411-422.

[2] M. Brešar, On the composition of (α, β)-derivations of rings and application to von Neumann algebras, *Acta Sci. Math*., 56 (1992), 369–375.

[3] M.A. Chaudhry and M.S. Samman, Free actions on semiprime rings, *Mathematica Bohemica*, 133(2) 2008, 197-208.

[4] M.A. Chaudhry and A.B. Thaheem, On (α, β)-derivations of semiprime rings, *Demonstratio Math.*, 36 (2003), 283–287.

[5] H. Choda, I. Kasahara and R. Nakamoto, Dependent elements of automorphisms of a C*-algebra, *Proc. Japan Acad.*, 48 (1972), 561–565.

[6] A. Laradji and A.B. Thaheem, On dependent elements in semiprime rings, *Math. Japonica*, 47 (1998), 29–31.

[7] F.J. Murray and J. V. Neumann, On rings of operators, *Ann. Math.*, 37 (1986), 116–229.

[8] J.V. Neumann, On rings of operators III, *Ann. Math.*, 41 (1980), 94–161.

[9] M.S. Samman, M.A. Chaudhry, *Dependent Elements of Left Centralizers of Semiprime Rings*, Preprint.

[10] J. Vukman, Centralizers of semiprime rings, *Comment. Math. Univ. Carolin.*, 42 (2001), 237–245.

[11] J. Vukman, On dependent elements and related problems in rings, *Int. Math. J.*, 6 (2005), 93–112.

[12] J. Vukman, I. Kosi-Ulbl, On dependent elements in rings, *Int. J. Math. Math. Sci.*, 53(56) (2004), 2895–2906.

[13] J. Vukman, I. Kosi-Ulbl, Centralizers on rings and algebras, *Bull. Austral. Math. Soc.*, 71 (2005), 225–234.