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# Hoehnke and Hereditary Radical Class

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**Abstract**

*We introduced the notion of Hoehnke Radical class for associative semirings in [7]. We give here some consequences of Hoehnke radical and hereditary Kurosh-Amitsur radical class.*

**Keywords:** *Semirings, Ideal, Radical class, Hoehnke radical class, Hereditary class.*

## 1 Introduction

For the general radical theory of rings, the reader is referred to the classical monograph of N. J. Divinsky [3]. For definitions and properties of semirings, ideals, homomorphism, the reader is referred to [4]. The concepts of radical class for hemirings were given by D. M. Olson and T. L. Jenkins in 1983, see [6]. Moreover we introduced the notion of Hoehnke Radical class for associative semirings in [7]. In the present paper we have given some consequences of Hoehnke radical and hereditary Kurosh-Amitsur radical class.

## 2 Preliminaries

There are many different definitions of a semiring appearing in the literature. Throughout this paper, a Semirings, additively cancellative semirings, commutative semirings, semimodules, additively cancellative semimodules, ideals, k-ideals (subtractive ideals), homomorphisms semiring will be defined as follows:

**Definition 2.1.** *[4] A semiring is a set $R$ together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(R,+)$ is a commutative monoid with identity element $0_R$; $(R,\cdot)$ is a monoid with identity element 1; multiplication distributes over addition from either side and 0 is multiplicative absorbing, that is, $a\cdot 0 = 0\cdot a = 0$ for each $a\in R$.*

**Definition 2.2.** *[4] A semiring $R$ is said to have a unity if there exists $1_R\in R$ such that $1_R\cdot a = a\cdot 1_R = a$ for each $a\in R$.*

**For e.g.** The set $\mathbb{N}$ of non-negative integers with the usual operations of addition and multiplication of integers is a semiring with $1_{\mathbb{N}}$.

**Definition 2.3.** *[4] A semiring $R$ is commutative if $(R,\cdot)$ is a commutative semigroup.*

**Definition 2.4.** *[4] A subset $I$ of a semiring $R$ will be called an ideal of $R$ if $I$ is an additive subsemigroup of $(R,+)$, $IR\subseteq I$ and $RI\subseteq I$.*

**Definition 2.5.** *[4] An ideal $I$ of a semiring $R$ will be called subtractive (k-ideal) if for $a\in I, a+b\in I, b\in R$ imply $b\in I$.*

**Definition 2.6.** *[4] A semiring $R$ is said to be semisubtractive if for any arbitrary $a\neq b$ in $R$ there is always some $x\in R$ satisfying $b+x=a$ or some $y\in R$ satisfying $a+y=b$.*

Each homomorphism $\phi\colon S\to T$ of semirings corresponds to a congruence $k$ of $S$ and the homomorphic image $\phi(S)$ is isomorphic to the semiring $S/k$ of congruence classes. In this paper we mainly use congruences that are determined by an ideal $I$ of $S$ according to $sk_Is'\Leftrightarrow$ there are

$$a_i\in I \text{ satisfying } s+a_1=s'+a_2.$$

In this case one usually denotes $S/k_I$ by $S/I$. Moreover, $k_I=k_{\overline{I}}$ and thus $S/I=S/\overline{I}$ hold for all ideals $I$ of $S$ with the same k-closure $\overline{I}, S/I$ has always an absorbing zero, namely the congruence class $\overline{I}=[a]_I=[a]_{\overline{I}}$ determined by each $a\in I$. We also mention that a semiring has in general much more congruences than those determined by its ideals. For a last concept of this kind, let $\phi\colon S\to T$ be a surjective homomorphism for semirings which have a zero. Then $\phi$ is called a semi-isomorphism and denoted by $\phi\colon S\tilde{\to} T$ if $\phi(0_S)=0_T$ and $\phi^{-1}(0_T)=0_S$ are satisfied. We emphasize here that such a semi-isomorphism, despite of misleading name, has in general very little in common with an isomorphism.

Convention: Throughout $R\longmapsto S$ is a surjective homomorphism.

**Theorem 2.7.** *[5] Let $S$ be a semiring, $T$ a semiring with an absorbing zero $0_T$, and $\phi\colon S \to T$ a surjective homomorphism. Then $K = \phi^{-1}(0_T)$ is a k-ideal of $S$ (also called the kernel of $\phi$ ) and $\phi([s]_K) = \phi(s)$ for all $s \in S$ defines a semi-isomorphism $\phi\colon S/K \tilde{\to} T$ which satisfies $\phi \circ {k_K}^{\#} = \phi$, where ${k_K}^{\#}$ denotes the natural homomorphism of $S$ onto $S/K = S/k_K$.*

**Theorem 2.8.** *[5] For a semiring $S$ with an absorbing zero 0 let $S$ be a subsemiring which contains 0 and $B$ an ideal of $S$. Then $\phi([a]_{A\cap\overline{B}}) = [a]_B$ for all $a \in A \subseteq A + B$ defines a semi-isomorphism*

$$\phi\colon A/A \cap \overline{B} \tilde{\to} A + B/B.$$

**Theorem 2.9.** *[5] Let $A$, $B$ be ideals of a semiring $S$ with the additional condition $A \subseteq B$. Then $\overline{\phi}([s]_B) = [[s]_A]_{\overline{B}/A}$ for all $s \in S$ defines an isomorphism*

$$\overline{\phi}\colon S/B \to (S/A)/(\overline{B}/A).$$

# 3 Radical Class

There are some definitions of radical class appearing in the semiring literature. But we were looking for the definition given by HMJ-Althani [1], who has introduced the definition of radical class in a different way. In [8] we have discuss useful equivalent conditions for a subclass of a fixed universal class to be a semisimple radical class and given some consequences of Upper radical class. In this paper we give some useful interrelationship between Hereditary Kurosh-Amitsur radical and Hoehnke radical.

**Definition 3.1.** *[1] Let $\mathcal{R}$ be a class of semirings. A semiring (ideal) belonging to the class $\mathcal{R}$, will be called a $\mathcal{R}$-semiring ($\mathcal{R}$-ideal).*

**Definition 3.2.** *[1] A class $\mathcal{R}$ of semirings is called a radical class whenever the following three conditions are satisfied:*

*(a) $\mathcal{R}$ is homomorphically closed; i.e. if $S$ is a homomorphic image of a $\mathcal{R}$-semiring $R$ then $S$ is also a $\mathcal{R}$-semiring*

*(b) Every semiring $R$ contains a $\mathcal{R}$-ideal $\mathcal{R}(R)$ which in turn contains every other $\mathcal{R}$-ideal of $R$.*

*(c) The factor semiring $R/\mathcal{R}(R)$ does not contain any nonzero $\mathcal{R}$-ideal; i.e. $\mathcal{R}(R/\mathcal{R}(R)) = 0$.*

**Proposition 3.3.** *[7] Assuming conditions (a) and (b) on a class $\mathcal{R}$ of semirings, condition (c) is equivalent to*
*(c') If $I$ is an ideal of the semiring $R$ and if both $I$ and $R/I$ are in $\mathcal{R}$, then $R$ itself is in $\mathcal{R}$.*

**Proposition 3.4.** *[7] Assuming conditions (a) and (c') on a class $\mathcal{R}$ of semirings, condition (b) is equivalent to*
*(b') if $I_1 \subset I_2 \subset \cdots \subset I_\lambda \subset \ldots$ is an ascending chain of ideals of a semiring $R$ and if each $I_\lambda$ is in $\mathcal{R}$, then $\bigcup I_\lambda$ is in $\mathcal{R}$.*

**Theorem 3.5.** *[7] A non-empty sub class $\mathcal{R}$ of a universal class $\mathbb{U}$ is a radical class if and only if*
*a) $\mathcal{R}$ is homomorphically closed.*
*b') $\mathcal{R}$ has the inductive property.*
*c') $\mathcal{R}$ is closed under extensions.*

# 4 Hoehnke and Hereditary Radical Class

**Definition 4.1.** *[7] From an axiomatic point of view a radical $\mathcal{R}$ may be defined as an assignment $\mathcal{R} : R \longrightarrow \mathcal{R}(R)$ designating a certain ideal $\mathcal{R}(R)$ to each semiring $R$. Such an assignment $\mathcal{R}$ is called Hoehnke radical if*

*(i) $\phi(\mathcal{R}(R)) \subseteq \mathcal{R}(\phi(R))$ for any homomorphism $\phi : R \mapsto \phi(R)$.*

*(ii) $\mathcal{R}(R/\mathcal{R}(R)) = 0$.*

*A Hoehnke radical $\mathcal{R}$ may also satisfy the following conditions:*

*(iii) $\mathcal{R}$ is complete: If $I \triangleleft R$ and $\mathcal{R}(I) = I$, then $I \subseteq \mathcal{R}(R)$.*

*(iv) $\mathcal{R}$ is idempotent: $\mathcal{R}(\mathcal{R}(R)) = \mathcal{R}(R)$, for every semiring $R$.*

**Theorem 4.2.** *[7] If $\mathcal{R}$ is a Kurosh-Amitsur radical then the assignment $R \rightarrow \mathcal{R}(R)$ is a complete, idempotent, Hoehnke radical. Conversely, if $\mathcal{R}$ is a complete, idempotent, Hoehnke radical, then there is a Kurosh-Amitsur radical $\varrho$ such that $\mathcal{R}(R) = \varrho(R)$ for every semiring $R$. Moreover $\varrho = \{R \mid \mathcal{R}(R) = R\}$.*

**Definition 4.3.** *[5] A class $\mathcal{R}$ of semirings is a hereditary radical class if $R \in \mathcal{R}$ and $I$ is an ideal of $R$, then $I \in \mathcal{R}$.*

**Definition 4.4.** *[5] A class $\mathcal{R}$ is said to be regular if for every semiring $R \in \mathcal{R}$, every nonzero ideal of $R$ has a nonzero homomorphic image in $\mathcal{R}$.*

In particular, every hereditary class is regular.

**Proposition 4.5.** *A radical class $\mathcal{R}$ is hereditary if and only if $I \cap \mathcal{R}(R) \subseteq \mathcal{R}(I)$ for every ideal $I$ of a semiring $R$.*

*Proof.* If $I \triangleleft R$ and $\mathcal{R}$ is hereditary, then $I \cap \mathcal{R}(R)$ is an ideal in $\mathcal{R}(R) \in \mathcal{R}$, implies that $I \cap \mathcal{R}(R) \in \mathcal{R}$. Therefore, by $I \cap \mathcal{R}(R)$ is an ideal in $I$ and $I \cap \mathcal{R}(R) \subseteq \mathcal{R}(I)$.

Conversely, assume that $I \triangleleft R \in \mathcal{R}$ and $I \cap \mathcal{R}(R) \subseteq \mathcal{R}(I)$. Then $I = I \cap R = I \cap \mathcal{R}(R) \subseteq \mathcal{R}(I) \in \mathcal{R}$, showing that $I \in \mathcal{R}$. Thus every ideal $I$ of a semiring $R \in \mathcal{R}$ is also in $\mathcal{R}$. Hence $\mathcal{R}$ is hereditary. □

In ring theoretic sense, for a ring $R$, $I \triangleleft J \triangleleft R$ does in general not imply $I \triangleleft R$. Therefore it was an important result for the radical theory of (associative) rings by T. Anderson N. Divinsky and A. Sulinski in [2] that at least each radical $\mathcal{R}(I)$ of an ideal $I$ of a ring $R$ is an ideal of $R$.

In this context one speaks about the A-D-S-property of a radical class. In [5] it has been proved that this property also holds true for each radical class of semirings, and we deal with some consequences of the A-D-S-property.

**Lemma 4.6.** *[5] Assume $I \triangleleft J \triangleleft R$ and $r \in R$ for a semiring $R$. Then $rI + I$ is an ideal of $R$ and $\varphi(b) = [rb]_I$ defines a surjective homomorphism $\varphi : I \to (rI + I)/I$.*

**Theorem 4.7.** *[5] Let $\mathcal{R}$ be a radical class of a universal class $\mathbb{U}$ of semirings and $\rho = \rho_{\mathcal{R}}$ the corresponding radical operator. Then, for each ideal $I$ of a semiring $R \in \mathbb{U}$ the radical $\rho(I)$ of $I$ is an ideal of $R$, which in particular yields $\rho(I) \subseteq \rho(R) \cap I$.*

**Theorem 4.8.** *[5] Let $\mathcal{R}$ be a radical class of $\mathbb{U}$ and $\varrho = \varrho_{\mathcal{R}}$ the corresponding radical operator. Then $\mathcal{R}$ is hereditary if and only if $\varrho(I) \supseteq I \cap \varrho(R)$ holds for each ideal $I$ of any semiring $R \in \mathbb{U}$. By Theorem 4.7 this inclusion is equivalent to $\varrho(I) = I \cap \varrho(R)$.*

Together with these results we can prove the following.

**Corollary 4.9.** *A radical class $\mathcal{R}$ is hereditary if and only if $\mathcal{R}(I) = I \cap \mathcal{R}(R)$, for any ideal $I$ of a semiring $R$.*

**Theorem 4.10.** *A Hoehnke radical $\mathcal{R}$ satisfies the condition*

$$\mathcal{R}(I) = I \cap \mathcal{R}(R) \quad \textit{for all} \quad I \triangleleft R \tag{1}$$

*if and only if $\mathcal{R}$ is a hereditary Kurosh-Amitsur radical.*

*Proof.* Let $\mathcal{R}$ be a Hoehnke radical with (1). In a view of Theorem 4.2 and above corollary it suffices to show that $\mathcal{R}$ is complete and idempotent. If $I \triangleleft R$ and $\mathcal{R}(I) = I$, then $I = \mathcal{R}(I) = I \cap \mathcal{R}(R)$ holds implying that $I \subseteq \mathcal{R}(R)$. Shows that $\mathcal{R}$ is complete.

Further, for $I = \mathcal{R}(R)$ we have $\mathcal{R}(\mathcal{R}(R)) = \mathcal{R}(R) \cap \mathcal{R}(R) = \mathcal{R}(R)$, and hence $\mathcal{R}$ is idempotent.

Conversely, a hereditary Kurosh -Amitsur radical $\mathcal{R}$ is a Hoehnke radical by Theorem 4.2 and satisfies (1) by above corollary. $\square$

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