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On the Spectra of the Operator of the First Difference on the Spaces W_τ and W_τ^0 and Application to Matrix Transformations

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Abstract

Given any sequence $\tau = (\tau_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_τ for the set of all sequences $x = (x_n)_{n\geq 1}$ such that $x/a = (x_n/a_n)_{n\geq 1} \in E$. We define the sets $W_\tau = (w_\infty)_\tau$ and $W_\tau^0 = (w_0)_\tau$, where w_∞ is the set of all sequences such that $\sup_n \left(n^{-1}\sum_{m=1}^n |x_m|\right) < \infty$, and w_0 is the set of all sequences such that $\lim_{n\to\infty} \left(n^{-1}\sum_{m=1}^n |x_m|\right) = 0$. Then we explicitly calculate the spectra $\sigma(\Delta, W_\tau)$ and $\sigma(\Delta, W_\tau^0)$ of the operator of the first difference on each of the sets W_τ and W_τ^0. We then determine the sets (E, F) of all matrix transformations mapping E to F, with $E = W_\tau\left((\Delta - \lambda I)^h\right)$, or $W_\tau^0\left((\Delta - \lambda I)^h\right)$ and $F = s_\xi$, or s_ξ^0 for complex numbers λ and h and obtain simplifications of these sets for some values of λ.

Keywords: *spectrum of an operator, operator of the first difference, matrix transformations, sets of strongly C_1 summable to zero and bounded sequences.*

1 Introduction

In this paper we consider spaces that generalize the well known sets w_0 and w_∞ introduced and studied by Maddox [15]. Recall that w_0 and w_∞ are the sets of *strongly C_1 summable to zero and bounded sequences*. In [19] Malkowsky and

Rakočević gave characterizations of matrix maps between w_0, w, or w_∞ and w_∞^p and between w_0, w, or w_∞ and l_1. In [11, 4] were defined the spaces $w_\alpha(\lambda)$, $w_\alpha^{(c)}(\lambda)$ and $w_\alpha^0(\lambda)$ of all sequences that are $\alpha-$*strongly bounded, summable and summable to zero* respectively. For instance recall that $w_\alpha(\lambda)$ is the set of all sequences $(x_n)_n$ such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}|x_k| = \alpha_n O(1) \quad (n \to \infty).$$

It was shown that these spaces can be written in the form s_ξ, $s_\xi^{(c)}$ and s_ξ^0 under some conditions on α and λ, where s_ξ, $s_\xi^{(c)}$ and s_ξ^0 were defined for positive sequences ξ by $(1/\xi)^{-1} * \chi$ and $\chi = \ell_\infty$, c, c_0, respectively, (cf. [4]). More recently in [18] it was shown that if λ is a *sequence exponentially bounded* then $(w_\infty(\lambda), w_\infty(\lambda))$ is a Banach algebra. This result led to consider bijective operators mapping $w_\infty(\lambda)$ into itself.

In [8] de Malafosse and Malkowsky gave among other things properties of the spectrum of the matrix of weighted means $\overline{N}_q$ considered as operator in the set s_a. In [12] were given simplifications of the set $s_\alpha^0\left((\Delta-\lambda I)^h\right) + s_\beta^{(c)}\left((\Delta-\mu I)^l\right)$ where h, l are complex numbers, α, β are given sequences, using spectral properties of the operator of the first difference in the sets s_α^0 and $s_\beta^{(c)}$, then characterizations of matrix transformations in this set were stated.

Here we deal with the spectrum of the operator of the first difference over the spaces $W_\tau = D_\tau w_\infty$ and $W_\tau^0 = D_\tau w_0$, and we characterize matrix transformations in the sets $W_\tau\left((\Delta-\lambda I)^h\right)$ and $W_\tau^0\left((\Delta-\lambda I)^h\right)$. We then obtain simplifications for these sets under some conditions on λ, h and on the sequence τ.

This paper is organized as follows. In Section 2 we recall some results on matrix transformations and define the sets w_0 and w_∞ of *strongly C_1 summable to zero and bounded sequences*. In Section 3 we give some properties of the sets W_τ and W_τ^0. In Section 4 we deal with the spectra of the operator of the first difference on W_τ and W_τ^0. In Section 5 we determine the sets (E, F) of matrix transformations mapping E to F, with $E = W_\tau\left((\Delta-\lambda I)^h\right)$, or $W_\tau^0\left((\Delta-\lambda I)^h\right)$ and $F = s_\xi$, or s_ξ^0, for complex numbers λ and h and obtain simplifications for these sets for some values of λ.

2 Preliminaries and Well Known Results

For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$, by $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$, where $x = (x_n)_{n\geq 1}$ and the series are assumed to be convergent. So we are led to the study of the infinite linear system $A_n(x) = b_n$ with $n = 1, 2, ...$ where $b = (b_n)_{n\geq 1}$ is a one-column matrix and x is the unknown, see for instance [4-8]. The equations $A_n(x) = b_n$ for $n = 1, 2, ...$ can be written in the form $Ax = b$, where $Ax = (A_n(x))_{n\geq 1}$. Let E and F be two sets of sequences, then (E, F) denotes the set of all operators mapping E to F, [15]. We write s for the set of all complex sequences, ℓ_∞ and c_0 for the sets of all *bounded and null sequences*. It is well known that $A \in (\ell_\infty, \ell_\infty)$ if and only if

$$\sup_n \sum_{m=1}^{\infty} |a_{nm}| < \infty; \tag{1}$$

and $A \in (c_0, c_0)$ if and only if (1) holds and $\lim_{n\to\infty} a_{nm} = 0$ for all $m \geq 1$.

A Banach space E of complex sequences with the norm $\|\|_E$ *is a BK space if each projection* $P_n : x \mapsto P_n x = x_n$ *is continuous.* We will write $e = (1, ..., 1, ...)$, and define by $e^{(m)}$ the sequence with 1 in the $m - th$ position and 0 otherwise. A *BK space* $E \subset s$ *is said to have AK if every sequence* $x = (x_m)_{m\geq 1} \in E$ *has a unique representation* $x = \sum_{m=1}^{\infty} x_m e^{(m)}$. The set $B(E)$ of all operators $L : E \longrightarrow E$ with the norm $\|L\|^*_{B(E)} = \sup_{x\neq 0} (\|L(x)\|_E / \|x\|_E)$ is a Banach algebra and it is well known that if E is a BK space with AK, then $B(E) = (E, E)$. In all what follows we will use the set U^+ of all sequences $(u_n)_{n\geq 1}$ with $u_n > 0$ for all n. For any given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we write D_τ for the infinite diagonal matrix defined by $[D_\tau]_{nn} = \tau_n$. For any subset E of s, $D_\tau E$ *is the set of all sequences* $x = (x_n)_n$ *such that* $(x_n/\tau_n)_{n\geq 1} \in E$. Note that have $D_\tau E = E_\tau$. Then we put $D_\tau c_0 = s^0_\tau$, $D_\tau \ell_\infty = s_\tau$ and $D_\tau c = s^{(c)}_\tau$. It is well known that each of the spaces s^0_τ, s_τ and $s^{(c)}_\tau$ is a BK space normed by $\|x\|_{s_\tau} = \sup_n (|x_n| / \tau_n)$, (cf. [6]). Recall the next elementary and useful result.

Lemma 1 *Let* τ, $\xi \in U^+$, *and* E, $F \subset \omega$. *Then* $A \in (D_\tau E, D_\xi F)$ *if and only if* $D_{1/b} A D_\xi \in (E, F)$.

For $\lambda = (\lambda_n)_{n\geq 1} \in U^+$ define the triangle $C(\lambda)$ by $[C(\lambda)]_{nm} = 1/\lambda_n$ for $m \leq n$. It can be easily shown that the matrix $\Delta(\lambda)$ defined by

$$[\Delta(\lambda)]_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise}, \end{cases}$$

is the inverse of $C(\lambda)$. Using the notation $|x| = (|x_n|)_n$, we have $[C(\lambda)|x|]_n = \lambda_n^{-1} \sum_{m=1}^{n} |x_m|$. In this way we consider the spaces of *strongly bounded and summable sequences* $w_\infty(\lambda)$ and $w_0(\lambda)$ defined by

$$\begin{aligned} w_\infty(\lambda) &= \left\{ x = (x_n)_{n\geq 1} \in s \ : C(\lambda)|x| \in \ell_\infty \right\}, \\ w_0(\lambda) &= \left\{ x \in s : C(\lambda)|x| \in c_0 \right\}. \end{aligned}$$

These spaces were studied by Malkowsky, with the concept of *exponentially bounded sequences*, see for instance [19]. Recall that Maddox [16] defined and studied the previous sets where $\lambda_n = n$ for all n and it is written $w_\infty(\lambda) = w_\infty$ and $w_0(\lambda) = w_0$.

3 The Sets W_τ and W_τ^0

In this section we state some results on the sets $W_\tau = D_\tau w_\infty$ and $W_\tau^0 = D_\tau w_0$ and deal with triangles Δ_ρ and Δ_ρ^T mapping from W_τ to itself.

3.1 Some Properties of the Sets W_τ and W_τ^0

Here we consider the sets $W_\tau = D_\tau w_\infty$ and $W_\tau^0 = D_\tau w_0$, (see [17, 9]), which can be written as

$$W_\tau = \left\{ x \in s : \|x\|_{W_\tau} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{\tau_m} \right) < \infty \right\}$$

and

$$W_\tau^0 = \left\{ x \in s : \lim_{n\to\infty} \left(\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{\tau_m} \right) = 0 \right\}.$$

For $\tau \in U^+$ it was shown in [9] that the sets W_τ and W_τ^0 are BK spaces normed by $\|\|_{W_\tau}$ and W_τ^0 has AK, [9, Proposition 3.1, p. 54]. So $W_e = w_\infty$ *and* $W_e^0 = w_0$. It was shown in [18, 7] that the class (w_∞, w_∞) is a *Banach algebra* normed by $\|A\|^*_{(w_\infty,w_\infty)} = \sup_{x\neq 0} \left(\|Ax\|_{w_\infty} / \|x\|_{w_\infty} \right)$. In the following we will write $D_r = D_{(r^n)_n}$ for any given $r > 0$, and define the sets

$$W_r = D_r w_\infty = \left\{ x : \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{r^m} \right) < \infty \right\}$$

and

$$W_r^0 = D_r w_0 = \left\{ x : \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{r^m} = 0 \right\}.$$

Note that we have $W_1 = w_\infty$ and $W_1^0 = w_0$.

3.2 On the Operators Δ_ρ and Δ_ρ^+ Considered as Maps in W_τ and W_τ^0

On the operators Δ_ρ and Δ_ρ^+ considered as operators in W_τ. In all what follows we use the convention $x_0 = 0$. For given $\rho = (\rho_n)_{n\geq 1}$ we will consider the operator Δ_ρ defined by $[\Delta_\rho x]_n = x_n - \rho_{n-1}x_{n-1}$ for all $n \geq 1$. Then putting $\Delta_\rho^+ = (\Delta_\rho)^T$ we obtain $\left[\Delta_\rho^+ x\right]_n = x_n - \rho_n x_{n+1}$ for all $n \geq 1$. To state the next Lemma we will put for $\tau \in U^+$ and any integer k

$$\rho_n^-(\tau) = \rho_n \frac{\tau_{n-1}}{\tau_n} \text{ and } \rho_n^+(\tau) = \rho_n \frac{\tau_{n+1}}{\tau_n} \text{ for all } n.$$

Now recall the next lemma which is a direct consequence of [9, Proposition 3.3, pp. 56-57]

Lemma 2 *Let $\rho, \tau \in U^+$.*

i) Let χ be any of the symbols W or W^0.

a) If $\rho^-(\tau) \in \ell_\infty$, then $\Delta_\rho \in (\chi_\tau, \chi_\tau)$ and

$$\|\Delta_\rho\|^*_{(W_\tau, W_\tau)} \leq 1 + \left\|\rho^-(\tau)\right\|_{l_\infty}.$$

b) If $\overline{\lim}_{n\to\infty} |\rho_n^-(\tau)| < 1$, then the operator Δ_ρ is a bijection from χ_τ to itself and

$$\chi_\tau(\Delta_\rho) = \chi_\tau.$$

ii) a) If $\rho^+(\tau) \in \ell_\infty$, then $\Delta_\rho^+ \in (W_\tau, W_\tau)$ and

$$\left\|\Delta_\rho^+\right\|^*_{(W_\tau, W_\tau)} \leq 1 + 2\left\|\rho^+(\tau)\right\|_{l_\infty}.$$

b) If $\overline{\lim}_{n\to\infty} |\rho_n^+(\tau)| < 1$, then the operator Δ_ρ^+ is a bijection from W_τ to itself.

Remark 3 *The proof of i) b) for $\chi = W^0$ comes from the fact that W_τ^0 is a BK space with AK which implies $B(W_\tau^0) = (W_\tau^0, W_\tau^0)$ is a Banach algebra.*

4 On the Spectra of the Operator of the First Difference on W_τ^0 and W_τ

In this section we deal with the spectra of the operator of the first difference Δ defined by $\Delta x_n = \Delta_e x_n = x_n - x_{n-1}$ for all n, considered as an operator from W_τ^0 to itself and from W_τ to itself.

Let E be a BK space and A be an operator mapping E to itself, (note that A is continuous since E is a BK space). We denote by $\sigma(A, E)$ the set of all complex numbers λ such that $A - \lambda I$ considered as an operator from E to itself is not invertible. Then we write $\rho(A, E) = [\sigma(A, E)]^c$ for the resolvent set, which is the set of all complex numbers λ such that $\lambda I - A$ considered as an operator from E to itself is bijective. Recall that the resolvent set of a linear operator on E is an open subset of the complex plane $\mathbb{C}$. We use the notation $\overline{D}(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\}$ for $\lambda_0 \in \mathbb{C}$ and $r > 0$. Recently the fine spectra of the operator of the first difference over the sequence spaces ℓ_p and bv_p, were studied in [1], where bv_p is the space of p-bounded variation sequences, with $1 \leq p < \infty$. In [2] there is a study on the fine spectrum of the generalized difference operator $B(r, s)$ on each of the sets ℓ_p and bv_p. In [14] there is a study of the spectrum of the operator of the first difference on the sets s_α, s_α^0, $s_\alpha^{(c)}$ and $\ell_p(\alpha)$ $(1 \leq p < \infty)$. In [13], among other things there is a study of the spectrum of the operator $B(r, s)$ on the sets s_α and s_α^0. In the following we deal with the spectra of Δ in the sets W_τ and W_τ^0. For this we need the next lemmas.

Lemma 4 *Let u be a sequence with $u_n \neq 0$ for all n, and assume $(u_n/u_{n-1})_n \in c$. We have*

$$u \in \ell_\infty \text{ implies } \lim_{n\to\infty} \left|\frac{u_n}{u_{n-1}}\right| \leq 1.$$

Proof. Assume $\lim_{n\to\infty} |u_n/u_{n-1}| = L > 1$. Then for $0 < \varepsilon < L - 1$, there is an integer N such that

$$\left|\frac{u_n}{u_{n-1}}\right| \geq L - \varepsilon > 1 \text{ for all } n \geq N.$$

So we obtain

$$|u_n| = \left|\frac{u_n}{u_{n-1}}\right| \left|\frac{u_{n-1}}{u_{n-2}}\right| ... \left|\frac{u_N}{u_{N-1}}\right| |u_{N-1}| \geq (L-\varepsilon)^{n-N+1} |u_{N-1}| \text{ for all } n \geq N.$$

Since $(L-\varepsilon)^{n-N+1} \to \infty$ $(n \to \infty)$ we conclude $u \notin \ell_\infty$. ■

Lemma 5 *We have*

$$(w_0, w_0) \subset \left(c_0, s^0_{(n)_n}\right) \text{ and } (w_\infty, w_\infty) \subset \left(\ell_\infty, s_{(n)_n}\right). \tag{2}$$

Proof. Trivially we have $c_0 \subset w_0$, and since $|x_n|/n \leq n^{-1}\sum_{k=1}^n |x_k|$ we deduce $w_0 \subset s^0_{(n)_n}$. Thus we have $(w_0, w_0) \subset \left(c_0, s^0_{(n)_n}\right)$. By similar arguments we obtain $(w_\infty, w_\infty) \subset \left(\ell_\infty, s_{(n)_n}\right)$. ■

In the next result we put $\tau^\bullet = (\tau_{n-1}/\tau_n)_{n\geq 2}$.

Theorem 6 *Let χ be any of the symbols W, or W^0. Then*

(i) If $\tau^\bullet \in \ell_\infty$, then we have

$$\sigma(\Delta, \chi_\tau) \subset \overline{D}\left(1, \overline{\lim_{n\to\infty}} \tau_n^\bullet\right). \tag{3}$$

(ii) If $\tau^{\bullet} \in c$*, then we have*

$$\sigma\left(\Delta, \chi_{\tau}\right) = \overline{D}\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right).$$

.

(iii) For any given $r > 0$*, we have*

$$\sigma\left(\Delta, \chi_{r}\right) = \overline{D}\left(1, 1/r\right).$$

Proof. (i) We only consider the case $\chi = W$, the case $\chi = W^0$ can be obtained in a similar way. Let $\lambda \in \left[\overline{D}\left(1, \overline{\lim}_{n\to\infty}\tau_n^{\bullet}\right)\right]^c$, that is, $\lambda \neq 1$ and

$$\overline{\lim_{n\to\infty}} \tau_n^{\bullet} < |\lambda - 1|. \tag{4}$$

Putting $\rho_n = 1/|\lambda - 1|$ for all n we have

$$\rho_n^-\left(\tau\right) = \left(\frac{1}{|\lambda - 1|}\tau_n^{\bullet}\right)_{n\geq 1} \in \ell_{\infty}$$

and inequality (4) means that $\overline{\lim}_{n\to\infty} |\rho_n^-\left(\tau\right)| < 1$. By Lemma 2 where

$$\Delta_{\rho} = \frac{1}{1-\lambda}\left(\Delta - \lambda I\right)$$

we deduce $\Delta - \lambda I$ is bijective from W_{τ} to itself. This shows that

$$\left[\overline{D}\left(1, \overline{\lim_{n\to\infty}} \tau_n^{\bullet}\right)\right]^c \subset \rho\left(\Delta, W_{\tau}\right)$$

and (3) in (i) is satisfied for $\chi = W$. This concludes the proof of (i).

(ii) Case $\chi = W^0$. First we show

$$D\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right) \subset \sigma\left(\Delta, W_{\tau}^0\right) \subset \overline{D}\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right).$$

The inclusion $\sigma\left(\Delta, W_{\tau}^0\right) \subset \overline{D}\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right)$ is a direct consequence of (i), since we have $\tau^{\bullet} \in c$. Now we show

$$D\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right) \subset \sigma\left(\Delta, W_{\tau}^0\right). \tag{5}$$

Since the inclusion $\rho\left(\Delta, W_{\tau}^0\right) \subset \left[D\left(1, \lim_{n\to\infty} \tau_n^{\bullet}\right)\right]^c$ is equivalent to (5), we will show if $\lambda I - \Delta$ considered as an operator from W_{τ}^0 to itself is invertible, then $\lambda \neq 1$ and

$$|\lambda - 1| \geq \lim_{n\to\infty} \tau_n^{\bullet}.$$

We have $\left(\lambda I - \Delta\right)^{-1} \in \left(W_{\tau}^0, W_{\tau}^0\right)$ if and only if

$$D_{1/\tau}\left(\lambda I - \Delta\right)^{-1} D_{\tau} \in \left(w_0, w_0\right).$$

Then by Lemma 5 we have $(w_0, w_0) \subset \left(c_0, s^0_{(n)_n}\right)$, and

$$D_{(1/n\tau_n)_n} (\lambda I - \Delta)^{-1} D_\tau \in (c_0, c_0) . \tag{6}$$

Now it is well known that $(\lambda I - \Delta)^{-1}$ is the triangle defined for $\lambda \neq 1$, by

$$\left[(\lambda I - \Delta)^{-1}\right]_{nm} = \frac{(-1)^{n-m}}{(\lambda - 1)^{n-m+1}} \text{ for } m \leq n.$$

We have

$$u_n = \left|\left[D_{(1/n\tau_n)_n} (\lambda I - \Delta)^{-1} D_\tau\right]_{n1}\right| = \frac{\tau_1}{n\tau_n |\lambda - 1|^n} \text{ for } n \geq 2,$$

and by Lemma 4 we obtain

$$\lim_{n\to\infty} \frac{u_n}{u_{n-1}} = \lim_{n\to\infty} \frac{n-1}{n} \frac{1}{|\lambda - 1|} \tau_n^\bullet \leq 1,$$

and

$$\frac{1}{|\lambda - 1|} \lim_{n\to\infty} \tau_n^\bullet \leq 1.$$

We conclude

$$\rho\left(\Delta, W^0_\tau\right) \subset \left\{\lambda \in \mathbb{C} : |\lambda - 1| \geq \lim_{n\to\infty} \tau_n^\bullet \text{ and } \lambda \neq 1\right\}$$

and

$$D\left(1, \lim_{n\to\infty} \tau_n^\bullet\right) \subset \sigma\left(\Delta, W^0_\tau\right).$$

We then have

$$D\left(1, \lim_{n\to\infty} \tau_n^\bullet\right) \subset \sigma\left(\Delta, W^0_\tau\right) \subset \overline{D}\left(1, \lim_{n\to\infty} \tau_n^\bullet\right)$$

and since $\sigma\left(\Delta, W^0_\tau\right)$ is a closed subset of $\mathbb{C}$, and $\overline{D}\left(1, \lim_{n\to\infty} \tau_n^\bullet\right)$ is the smallest closed set containing $D\left(1, \lim_{n\to\infty} \tau_n^\bullet\right)$, we conclude $\sigma\left(\Delta, W^0_\tau\right) = \overline{D}\left(1, \lim_{n\to\infty} \tau_n^\bullet\right)$.

Case $\chi = W$. The proof follows exactly the same lines that above. It is enough to notice that by Lemma 5, the condition $D_{1/\tau} (\lambda I - \Delta)^{-1} D_\tau \in (w_\infty, w_\infty)$ implies

$$D_{1/\tau} (\lambda I - \Delta)^{-1} D_\tau \in \left(\ell_\infty, s_{(n)_n}\right),$$

and

$$D_{(1/n\tau_n)_n} (\lambda I - \Delta)^{-1} D_\tau \in (\ell_\infty, \ell_\infty) = S_1.$$

This completes the proof of (ii).

(iii) is an immediate consequence of (ii) with $\tau_n = r^n$. ∎

5 Matrix Transformations in $W_\tau^0\left((\Delta-\lambda I)^h\right)$

In this section we recall results on the sets (E, F) where E is either w_0 and w_∞ and $F = \ell_\infty$, or c_0. Then we apply the results of Section 4 to determine the sets (E', F') where E' is either $W_\tau^0\left((\Delta-\lambda I)^h\right)$, or $W_\tau\left((\Delta-\lambda I)^h\right)$ and $F' = s_\xi$, or s_ξ^0.

5.1 Matrix Transformations in the Sets w_0 and w_∞

Here we recall some results that are direct consequence of [3, Theorem 2.4], where it is written

$$\left\|(a_n)_{n\geq 1}\right\|_{\mathcal{M}} = \sum_{\nu=1}^{\infty} 2^{\nu} \max_{2^{\nu}\leq m\leq 2^{\nu+1}-1} |a_m| \,. \tag{7}$$

sing the notation $A_n = (a_{nm})_{m\geq 1}$ we obtain the following.

Lemma 7 *[3] (i) We have $(w_0, \ell_\infty) = (w_\infty, \ell_\infty)$ and $A \in (w_\infty, \ell_\infty)$ if and only if*

$$\sup_n \left(\|A_n\|_{\mathcal{M}}\right) = \sup_n \left(\sum_{\nu=1}^{\infty} 2^{\nu} \max_{2^{\nu}\leq m\leq 2^{\nu+1}-1} |a_{nm}|\right) < \infty, \tag{8}$$

(ii) $A \in (w_\infty, c_0)$ if and only if

$$\lim_{n\to\infty} \|A_n\|_{\mathcal{M}} = \lim_{n\to\infty} \left(\sum_{\nu=1}^{\infty} 2^{\nu} \max_{2^{\nu}\leq m\leq 2^{\nu+1}-1} |a_{nm}|\right) = 0.$$

(iii) $A \in (w_0, c_0)$ if and only if (8) holds and

$$\lim_{n\to\infty} a_{nm} = 0 \text{ for all } m.$$

5.2 Matrix Transformations in the Sets $w_0(T)$ and $w_\infty(T)$

To state the next result we consider the matrix Σ^+, by $[\Sigma^+]_{nm} = 1$ for $m \geq n$ and $[\Sigma^+]_{nm} = 0$ otherwise, and from any matrix $A = (a_{nm})_{n,m\geq 1}$ we define for any integer i, the triangle $W^{(i)}$ by

$$\left[W^{(i)}\right]_{nm} = \left[\Sigma^+ D_{(a_{in})_n} T^{-1}\right]_{nm} \text{ for } m \leq n.$$

So an elementary calculations yield

$$\left[W^{(i)}\right]_{nm} = \sum_{k=n}^{\infty} a_{ik} s_{km} \text{ for } m \leq n, \tag{9}$$

where T^{-1} is the triangle whose nonzero entries are defined by $[T^{-1}]_{nm} = s_{nm}$. From [3, Lemma 4.1 and Theorem 4.2], we obtain the following.

Lemma 8 *Let χ be any of the sets w_∞ or w_0 and Y be an arbitrary subset of s. Then $A \in (\chi(T), Y)$ if and only if*
(i) $AT^{-1} \in (\chi, Y)$,
(ii) $W^{(i)} \in (\chi, c_0)$ for all $i \geq 1$.

From (9) and (7) we easily see that for every $n \in \mathbb{N}$, there is $\nu(n)$ uniquely defined with $2^{\nu(n)} \leq n \leq 2^{\nu(n)+1} - 1$, and for any $i \geq 1$ we have

$$\left\|W_n^{(i)}\right\|_{\mathcal{M}} = \sum_{\nu=0}^{\nu(n)-1} 2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1} \left|\sum_{k=n}^{\infty} a_{ik} s_{km}\right| + 2^{\nu(n)} \max_{2^{\nu(n)} \leq m \leq n} \left|\sum_{k=n}^{\infty} a_{ik} s_{km}\right|.$$

Now we state the next lemma which is a direct consequence of Lemma 7 and Lemma 8, where we have $[AT^{-1}]_{nm} = \sum_{k=m}^{\infty} a_{nk} s_{km}$ for all n, m.

Lemma 9 *(i) $A \in (w_\infty(T), \ell_\infty)$ if and only if*
a)

$$\sup_n \left(\sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1} \left|\sum_{k=m}^{\infty} a_{nk} s_{km}\right|\right) < \infty. \tag{10}$$

b) For every $i \geq 1$ we have

$$\lim_{n\to\infty} \left\|W_n^{(i)}\right\|_{\mathcal{M}} = 0. \tag{11}$$

(ii) $A \in (w_\infty(T), c_0)$ if and only if (11) holds for all i, and

$$\lim_{n\to\infty} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1} \left|\sum_{k=m}^{\infty} a_{nk} s_{km}\right|\right) = 0. \tag{12}$$

(iii) $A \in (w_0(T), \ell_\infty)$ if and only if (10) holds, and for each i we have

$$\sup_n \left(\sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1} \left|\sum_{k=n}^{\infty} a_{ik} s_{km}\right|\right) < \infty, \tag{13}$$

and

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} a_{ik} s_{km} = 0 \text{ for all } m \tag{14}$$

(iv) $A \in (w_0(T), c_0)$ if and only if (10) holds, (14) and (13) hold for all i, and

$$\lim_{n\to\infty} \sum_{k=m}^{\infty} a_{nk} s_{km} = 0 \text{ for all } m.$$

5.3 The Operator $(\Delta - \lambda I)^h$, where $h \in \mathbb{C}$

For any given $h \in \mathbb{C}$, we put

$$\begin{pmatrix} -h+k-1 \\ k \end{pmatrix} = \begin{cases} \dfrac{-h(-h+1)\dots(-h+k-1)}{k!} & \text{if } k>0, \\ 1 & \text{if } k=0\ , \end{cases}$$

(cf. [10]). To simplify we will write

$$[-h,k] = \begin{pmatrix} -h+k-1 \\ k \end{pmatrix}.$$

It is known that $(\Delta - \lambda I)^h$ with $\lambda \neq 1$ is the triangle defined by

$$\left[(\Delta - \lambda I)^h\right]_{nm} = \frac{[-h,n-m]}{(1-\lambda)^{-h+n-m}} \text{ for } m \leq n,$$

see [10, Theorem 8, pp. 295-296].

5.4 The Sets (E,F) where E is either $W_\tau^0\left((\Delta - \lambda I)^h\right)$, or $W_\tau\left((\Delta - \lambda I)^h\right)$ and $F = s_\xi$, or s_ξ^0.

In the following we consider for $\lambda \neq 1$, matrix transformations mapping in the sets

$$W_\tau\left((\Delta - \lambda I)^h\right) = \left\{x \in s : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\frac{[-h,n-m]}{(1-\lambda)^{-h+n-m}}x_m\right|\right) < \infty\right\}$$

and

$$W_\tau^0\left((\Delta - \lambda I)^h\right) = \left\{x \in s : \lim_{n\to\infty} \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\frac{[-h,n-m]}{(1-\lambda)^{-h+n-m}}x_m\right|\right) = 0\right\}.$$

To state the next result we put

$$\chi_{nm}(i) = \sum_{k=n}^{\infty}[h,k-m]\frac{a_{ik}}{(1-\lambda)^{h+k-m}} \text{ for } n,\ m,\ i \geq 1 \text{ integers.}$$

Theorem 10 *(i) Let $\lambda \neq 1$. Then*

a) $A \in \left(W_\tau\left((\Delta - \lambda I)^h\right), s_\xi\right)$ if and only if

$$\sup_n\left(\frac{1}{\xi_n}\sum_{\nu=0}^{\infty}2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1}\left|\sum_{k=m}^{\infty}[h,k-m]\frac{a_{nk}}{(1-\lambda)^{h+k-m}}\tau_m\right|\right) < \infty \qquad (15)$$

and for every $n \in \mathbb{N}$*, there is* $\nu(n)$ *uniquely defined with* $2^{\nu(n)} \leq n \leq 2^{\nu(n)+1}-1$*, and for any* $i \geq 1$ *we have*

$$\lim_{n\to\infty}\left\{\frac{1}{\xi_n}\left(\sum_{\nu=0}^{\nu(n)-1}2^{\nu}\max_{2^{\nu}\leq m\leq 2^{\nu+1}-1}\left|\chi_{nm}(i)\right|\tau_m+2^{\nu(n)}\max_{2^{\nu(n)}\leq m\leq n}\left|\chi_{nm}(i)\right|\tau_m\right)\right\}=0. \tag{16}$$

b) $A \in \left(W_{\tau}\left((\Delta-\lambda I)^h\right), s_{\xi}^0\right)$ *if and only if (16) holds for every* $i \geq 1$ *and*

$$\lim_{n\to\infty}\left(\frac{1}{\xi_n}\sum_{\nu=0}^{\infty}2^{\nu}\max_{2^{\nu}\leq m\leq 2^{\nu+1}-1}\left|\sum_{k=m}^{\infty}[h,k-m]\frac{a_{nk}}{(1-\lambda)^{h+k-m}}\tau_m\right|\right)=0. \tag{17}$$

c) $A \in \left(W_{\tau}^0\left((\Delta-\lambda I)^h\right), s_{\xi}\right)$ *if and only if (15) holds and for every* $n \in \mathbb{N}$*, there is* $\nu(n)$ *uniquely defined with* $2^{\nu(n)} \leq n \leq 2^{\nu(n)+1}-1$*, and for any* $i \geq 1$ *condition (17) holds and*

$$\sup_n\left\{\frac{1}{\xi_n}\left(\sum_{\nu=0}^{\nu(n)-1}2^{\nu}\max_{2^{\nu}\leq m\leq 2^{\nu+1}-1}\left|\chi_{nm}(i)\right|\tau_m+2^{\nu(n)}\max_{2^{\nu(n)}\leq m\leq n}\left|\chi_{nm}(i)\right|\tau_m\right)\right\}<\infty. \tag{18}$$

d) $A \in \left(W_{\tau}^0\left((\Delta-\lambda I)^h\right), s_{\xi}^0\right)$ *if and only if (15) holds, (17) and (18) hold for all* i*, and*

$$\lim_{n\to\infty}\frac{1}{\xi_n}\sum_{k=m}^{\infty}[h,k-m]\frac{a_{nk}}{(1-\lambda)^{h+k-m}}\tau_m=0 \text{ for all } m.$$

(ii) Let $h \in \mathbb{N}$*. Assume that* $\tau^{\bullet}=(\tau_{n-1}/\tau_n)_{n\geq 2}\in\ell_{\infty}$ *and let* λ *such that*

$$|\lambda-1|>\overline{\lim_{n\to\infty}}\,\tau_n^{\bullet}. \tag{19}$$

a) We have $\left(W_{\tau}\left((\Delta-\lambda I)^h\right), s_{\xi}\right)=\left(W_{\tau}^0\left((\Delta-\lambda I)^h\right), s_{\xi}\right)$ *and* $A \in \left(W_{\tau}\left((\Delta-\lambda I)^h\right), s_{\xi}\right)$ *if and only if*

$$\sup_n\left\{\frac{1}{\xi_n}\sum_{\nu=0}^{\infty}2^{\nu}\max_{2^{\nu}\leq m\leq 2^{\nu+1}-1}\left|a_{nm}\right|\tau_m\right\}<\infty. \tag{20}$$

b) We have $A \in \left(W_{\tau}^0\left((\Delta-\lambda I)^h\right), s_{\xi}^0\right)$ *if and only if (20) holds and* $\lim_{n\to\infty}a_{nm}/\xi_n=0$ *for all* $m \geq 1$*.*

c) We have $A \in \left(W_{\tau}\left((\Delta-\lambda I)^h\right), s_{\xi}^0\right)$ *if and only if*

$$\lim_{n\to\infty}\left\{\frac{1}{\xi_n}\sum_{\nu=0}^{\infty}2^{\nu}\max_{2^{\nu}\leq m\leq 2^{\nu+1}-1}\left|a_{nm}\right|\tau_m\right\}=0.$$

Proof. (i) By Lemma 9 with $T = (\Delta - \lambda I)^h$, we have $T^{-1} = (\Delta - \lambda I)^{-h}$ which is defined by

$$\left[(\Delta - \lambda I)^{-h}\right]_{nm} = \frac{[h, n-m]}{(1-\lambda)^{h+n-m}} \text{ for } m \leq n.$$

Then we have

$$\begin{aligned}\left[A(\Delta - \lambda I)^{-h}\right]_{nm} &= \sum_{k=m}^{\infty} a_{nk}\left[(\Delta - \lambda I)^{-h}\right]_{km} \\ &= \sum_{k=m}^{\infty} [h, k-m] \frac{a_{nk}}{(1-\lambda)^{h+k-m}} \text{ for all } n,\ m \geq 1.\end{aligned}$$

We also have

$$\left[W^{(i)}\right]_{nm} = \sum_{k=n}^{\infty} a_{ik}\left[(\Delta - \lambda I)^{-h}\right]_{km} = \chi_{nm}(i) \text{ for } m \leq n \text{ and for all } i \geq 1.$$

Then by Lemma 8 we have $A \in \left(W_\tau\left((\Delta - \lambda I)^h\right), s_\xi\right)$ if and only if

$$D_{1/\xi} A (\Delta - \lambda I)^{-h} D_\tau \in (w_\infty, \ell_\infty)$$

and

$$D_{1/\xi} W^{(i)} D_\tau \in (w_\infty, c_0) \text{ for all } i \geq 1.$$

So there is $\nu(n)$ uniquely defined with $2^{\nu(n)} \leq n \leq 2^{\nu(n)+1} - 1$, and for any $i \geq 1$ we have

$$\left\| \left[D_{1/\xi} W^{(i)} D_\tau\right]_n \right\|_{\mathcal{M}} = \frac{1}{\xi_n} \sum_{\nu=0}^{\nu(n)-1} 2^\nu \max_{2^\nu \leq m \leq 2^{\nu+1}-1} |\chi_{nm}(i)| \tau_m + \frac{1}{\xi_n} 2^{\nu(n)} \max_{2^{\nu(n)} \leq m \leq n} |\chi_{nm}(i)| \tau_m.$$

Applying Lemma 7 with a_{nm} replaced by

$$\frac{1}{\xi_n} \sum_{k=m}^{\infty} [h, k-m] \frac{a_{nk}}{(1-\lambda)^{h+k-m}} \tau_m$$

we obtain (i) a). Then replacing a_{nm} by

$$\frac{1}{\xi_n} \chi_{nm}(i) \tau_m \text{ for } m \leq n \text{ and for } i \geq 1,$$

we obtain (i) b). The statements (i) c) and (i) d) can be shown in the same way.

(ii) Since (19) holds, by Theorem 6, we have $\lambda \notin \sigma(\Delta, W_\tau^0)$, and $\Delta - \lambda I$ is bijective from W_τ^0 to itself and $W_\tau^0\left((\Delta - \lambda I)^h\right) = W_\tau^0$. Then

$$A \in \left(W_\tau^0\left((\Delta - \lambda I)^h\right), s_\xi\right) = \left(W_\tau^0, s_\xi\right)$$

if and only if $D_{1/\xi} A D_\tau \in (w_0, \ell_\infty)$, and we conclude applying Lemma 7. The cases b) and c) are direct consequences of Theorem 6 and Lemma 7. ■

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