A Study on Series Solutions of Two Species Lotka Volterra Equations by Adomian Decomposition and Homotopy Perturbation Methods

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Abstract

In this work, series solutions are obtained for the two species Lotka –Volterra, prey-predator interaction species which are governed by a system of non linear differential equations. For this purpose, the Adomian decomposition method (ADM) and Homotopy Perturbation method (HPM) are employed and it is shown that the Homotopy Perturbation method is much easier and a more efficient method.

Keywords: *Adomian Decomposition, Homotopy Perturbation, Lotka - Volterra, prey-predator.*

1 Introduction

In the study of non linear system of differential equations such as the Lotka – Volterra equations, analytical solutions are usually unknown. In order to analyze

the behaviour of the system, one usually resorts to numerical integration techniques such as perturbation techniques. Perturbation technique depends on the existence of small or large parameters in the nonlinear problems. In this paper, the Adomian decomposition method (ADM) and Homotopy Perturbation method (HPM) will be employed to obtain series solutions to two species Lotka – Volterra, prey – predator interaction species which are governed by a system of non linear differential equations. The first method is Adomian decomposition method [3] which was introduced by Adomian in the 1980s, an iterative method which provides approximate analytical solutions in the form of an infinite series for non linear equations. This method avoids linearization, discretization and scientifically unrealistic assumptions. The second method is the Homotopy perturbation method [4] which was proposed by 'He' in 1999. In this method, the solutions are obtained as an infinite series, the summation of which is an analytical solution. In this method, a homotopy is constructed with an embedding parameter $p \in (0, 1)$, which is valid not only for small parameters but also for very large parameters. We also show with the help of an example that although the solutions obtained by both ADM and HPM are the same, HPM is convenient and efficient when compared to ADM.

2   Adomian Decomposition Method

2.1   Analysis of the Method for Interacting Species

In this section we consider the system of equations of the form

$$\frac{dx_1}{dt} = x_1(a_1 - b_1 x_1 - c_1 x_2)$$

$$\frac{dx_2}{dt} = x_2(-a_2 + c_2 x_1) \qquad\qquad (2.1)$$

where a_1, b_1, c_1, a_2, c_2 are positive constants.

x_1 and x_2 represent the prey and predator population at time, t. We non dimensionalize the system of equations (2.1) by setting

$$u(\tau) = \frac{c_2}{a_2} x_1(t), \qquad v(\tau) = \frac{c_1 c_2}{a_1 c_2 - a_2 b_1} x_2(t), \qquad \tau = a_1 t$$

$$\alpha = \frac{a_2}{a_1}, \qquad \beta = \frac{b_1 a_2}{c_2 a_1} \qquad\qquad (2.2)$$

The equations (2.1) are then transformed to

$$\frac{du}{d\tau} = u(\tau)\left(1 - \beta u(\tau) - (1 - \beta) v(\tau)\right)$$

$$\frac{dv}{d\tau} = \alpha v(\tau)\left(-1 + u(\tau)\right) \qquad\qquad (2.3)$$

Considering f and g to be non linear functions of u and (u, v) respectively, the equations (2.3) can now be written in the form

$$\frac{du}{d\tau} = u(\tau) - \beta f(u) - (1-\beta)\, g(u,v)$$

$$\frac{dv}{d\tau} = -\alpha v(\tau) + \alpha g(u, v) \qquad (2.4)$$

with initial conditions $u(0) = u_0$, $u(0) = u_0$.

Now, to find the solutions of u and v that satisfy equation (2.4), the decomposition method is used. It consists of approximating the solutions of (2.4) as an infinite series in the following form.

$$u = \sum_{n=0}^{\infty} U_n$$

$$v = \sum_{n=0}^{\infty} V_n \qquad (2.5)$$

Now, decomposing f and g as

$$f(u) = \sum_{n=0}^{\infty} A_n \qquad (2.6)$$

$$g(u, v) = \sum_{n=0}^{\infty} B_n \qquad (2.7)$$

where A_n, B_n are Adomian polynomials that can be generated for any form of non linearity. Writing equation (2.4) as

$Lu = u(\tau) - \beta f\,(u(\tau)) - (1-\beta)\, g\,(u(\tau), v(\tau))$
$Lv = -\alpha v(\tau) + \alpha g(u(\tau), v(\tau)) \qquad (2.8)$

where $L = \dfrac{\partial}{\partial \tau}$, the linear differential operator. Assume the inverse operator L^{-1}

exists, which stands for integrations as defined by $L^{-1} = \displaystyle\int_{0}^{\tau} (.)d\tau$. Applying L^{-1} on

both sides of equation (2.8), we obtain

$u(\tau) = u(0) + L^{-1}\, u(\tau) - \beta L^{-1} f\,(u(\tau)) - (1-\beta)\, L^{-1}\, g\,(u(\tau), v(\tau))$
$v(\tau) = -\alpha L^{-1}\, v(\tau) + \alpha L^{-1} g\,(u(\tau), v(\tau)) \qquad (2.9)$

Using equations (2.5), (2.6) & (2.7), we get

$$\sum_{n=0}^{\infty} u_n = u(0) + L^{-1} \sum_{n=0}^{\infty} u_n - \beta\, L^{-1} \sum_{n=0}^{\infty} A_n - (1-\beta)\, L^{-1} \sum_{n=0}^{\infty} B_n$$

$$\sum_{n=0}^{\infty} v_n = v(0) - \alpha L^{-1} \sum_{n=0}^{\infty} v_n + \alpha L^{-1} \sum_{n=0}^{\infty} B_n \qquad (2.10)$$

Now, the iterates are determined in the following recursive way.

$u(0) = u_0$
$u_{n+1} = L^{-1}\, u_n - \beta\, L^{-1} A_n - (1-\beta)\, L^{-1} B_n \qquad (2.11)$
$v(0) = v_0$

$$v_{n+1} = -\alpha L^{-1} v_n + \alpha L^{-1} B_n \qquad (2.12)$$

We then define the solution (u, v) as

$$(u, v) = \left(\lim_{n \to \infty} \sum_{k=0}^{n} uk, \ \lim_{n \to \infty} \sum_{k=0}^{n} vk \right) \qquad (2.13)$$

2.2   Analysis of the Method for Interacting Species with Negative Feedback

In this section, we consider the nonlinear system of the form

$$\frac{dx_1}{dt} = x_1 \left(a_1 - b_1 x_1 - c_1 x_2 \right)$$

$$\frac{dx_2}{dt} = x_2 \left(-a_2 + b_2 x_1 - c_2 x_2 \right) \qquad (2.14)$$

Here x_1 and x_2 are prey, predator populations at time t respectively where $a_1, b_1, c_1,$ a_2, b_2, c_2 are positive constants. Each of the terms $-b_1 x_1^2$ and $-c_2 x_2^2$ is called negative feed back. The equilibrium of the system is positive provided the inequality $a_1 b_2 > a_2 b_1$ holds.
We non dimensionalize the system by setting

$$u(\tau) = \frac{b_1 c_2 + b_2 c_1}{a_1 c_2 + a_2 c_1} x_1(t), \qquad v(\tau) = \frac{b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1} x_2(t), \ \text{and} \ \tau = a_1 t \qquad (2.15)$$

Taking $\alpha = \dfrac{a_1 c_2 + a_2 c_1}{b_1 c_2 + b_2 c_1}, \qquad \beta = \dfrac{a_1 b_2 - a_2 b_1}{b_1 c_2 + b_2 c_1}$, the equations (2.14) are then

$$\frac{du}{d\tau} = u(\tau) - \frac{b_1}{a_1} \alpha u^2(\tau) - \frac{c_1}{a_1} \beta u(\tau) v(\tau)$$

$$\frac{dv}{d\tau} = -\frac{a_2}{a_1} v(\tau) + \frac{b_2}{a_1} \alpha u(\tau) v(\tau) - \frac{c_2}{a_1} \beta v^2(\tau) \qquad (2.16)$$

With $k_0 = \dfrac{a_2}{a_1}, \ k_1 = \dfrac{b_1}{a_1} \alpha, \ k_2 = \dfrac{c_1}{a_1} \beta, \ k_3 = \dfrac{b_2}{a_1} \alpha, \ k_4 = \dfrac{c_2}{a_1} \beta$ the equation (2.16) is now

$$\frac{du}{dT} = u(\tau) - k_1 u^2(\tau) - k_2 \ u(\tau) \ v(\tau)$$

$$\frac{dv}{dT} = -k_0 v(\tau) + k_3 u(\tau) v(\tau) - k_4 v^2(\tau) \tag{2.17}$$

Considering f, g and h to be non linear functions of u, v and (u, v) respectively, the equations (2.17) can now be written in the form

$$\frac{du}{d\tau} = u(\tau) - k_1 f(u) - k_2 g(u, v)$$

$$\frac{du}{d\tau} = -k_0 v(\tau) + k_3 g(u, v) - k_4 h(v) \tag{2.18}$$

with initial values $u(0) = u_0$, $v(0) = v_0$. To find the solutions of u and v that satisfy equation (2.18), the decomposition method is used. It consists of approximating the solutions of (2.18) as an infinite series in the following form.

$$u = \sum_{n=0}^{\infty} u_n \qquad v = \sum_{n=0}^{\infty} v_n \tag{2.19}$$

Decomposing f, h and g as

$$f(u) = \sum_{n=0}^{\infty} A_n \qquad h(v) = \sum_{n=0}^{\infty} C_n \qquad g(u, v) = \sum_{n=0}^{\infty} B_n \tag{2.20}$$

where A_n, B_n & C_n are Adomian polynomials that can be generated for any form of non linearity. Applying the decomposition method, the system (2.18) can be written as

$$Lu = u(\tau) - k_1 f(u) - k_2 g(u, v)$$

$$Lv = -k_0 v(\tau) + k_3 g(u, v) - k_4 h(v) \tag{2.21}$$

where $L = \dfrac{\partial}{\partial \tau}$, the linear differential operator.. Applying the integration inverse

operator $L^{-1} = \int_0^\tau (.) \, d\tau$ to equation (2.21),

$$u(\tau) = u(0) + L^{-1} u(\tau) - k_1 L^{-1} f(u(\tau)) - k_2 L^{-1} g(u(\tau), v(\tau))$$

$$v(\tau) = v_0 - k_0 L^{-1} v(\tau) + k_3 L^{-1} g(u(\tau), v(\tau) - k_4 L^{-1} h(v(\tau)) \tag{2.22}$$

Using equations (2.19) and (2.20),

$$\sum_{n=0}^{\infty} u_n = u_0 + L^{-1} \sum_{n=0}^{\infty} u_n - k_1 L^{-1} \sum_{n=0}^{\infty} A_n - k_2 L^{-1} \sum_{n=0}^{\infty} B_n$$

$$\sum_{n=0}^{\infty} v_n = v_0 - k_0 L^{-1} \sum_{n=0}^{\infty} v_n + k_3 L^{-1} \sum_{n=0}^{\infty} B_n - k_4 L^{-1} \sum_{n=0}^{\infty} C_n \tag{2.23}$$

Now, the iterates are determined in the following recursive way.

$$u(0) = u_0$$
$$u_{n+1} = L^{-1} u_n - k_1 L^{-1} A_n - k_2 L^{-1} B_n, n = 0, 1, 2.....$$
$$v(0) = v_0$$
$$v_{n+1} = - k_0 L^{-1} v_n + k_3 L^{-1} B_n - k_4 L^{-1} C_n, \quad n = 0, 1, 2..... \tag{2.24}$$

We then define the solution of the initial value problem (2.18) as

$$(u, v) = \left(\lim_{n \to \infty} \sum_{k=0}^{n} u_k, \ \lim_{n \to \infty} \sum_{k=0}^{n} v_k \right) \tag{2.25}$$

3   Application of Adomian Decomposition Method

Example 3.1

We first consider (2.1) with initial values $u(0) = u_0$, $v(0) = v_0$ and proceeding as in Section 2.1.
We take $f(u) = u^2$, $g(u, v) = uv$ (3.1)
The Adomian polynomials can be derived as follows:

$$A_n = \frac{1}{n} \frac{dn}{dx^n} \left(f\left(\sum_{n=0}^{\infty} \lambda^n u_n \right) \right)_{\lambda=0} \qquad n \geq 0 \tag{3.2}$$

$$f(u) = u^2 = \sum_{n=0}^{\infty} A_n = \left(u_0 + u_1 + u_2 + ... \right)^2$$

$$= (u_0)^2 + (2u_0 u_1) + (2u_0 u_2 + u_1^2) + (2u_0 u_3 + 2u_1 u_2) + (2u_0 u_4 + 2u_1 u_3 + u_2^2)$$
$$+ (2u_0 u_5 + 2u_1 u_4 + 2u_2 u_3) + ... \tag{3.3}$$

Therefore, we get the following Adomian polynomials

$$A_0 = u_0^2$$
$$A_1 = 2u_0 u_1$$
$$A_2 = 2u_0 u_2 + u_1^2$$
$$A_3 = 2u_0 u_3 + 2u_1 u_2$$
$$A_4 = 2u_0 u_4 + 2u_1 u_3 + u_2^2$$
$$A_5 = 2u_0 u_5 + 2u_1 u_4 + 2u_2 u_3 \text{ and so on.} \tag{3.4}$$

$$g(u, v) = uv = \sum_{n=0}^{\infty} B_n = \left(\sum_{n=0}^{\infty} u_n \right) \left(\sum_{n=0}^{\infty} v_n \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} u_k \, v_{n-k} \right) \tag{3.5}$$

Therefore $B_n = \sum_{k=0}^{n} u_k v_{n-k}$, $n = 0, 1, 2, ...$ (3.6)

We have Adomian Polynomials

$B_0 = u_0 v_0$
$B_1 = u_0 v_1 + u_1 v_0$
$B_2 = u_0 v_2 + u_1 v_1 + u_2 v_0$
$B_3 = u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0$
$B_4 = u_0 v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_0$ and so on. (3.7)

For numerical purpose, we take $a_1 = a_2 = b_1 = 1$, $b_2 = 2$, $c_1 = \dfrac{1}{4}$, $c_2 = \dfrac{1}{2}$ in (2.1) and initial values $u_0 = 1$, $v_0 = 2$. Therefore,

$u_0 = 1$
$v_0 = 2$
$u_1 = L^{-1} u_0 - \beta \, L^{-1} A_0 - (1 - \beta) \, L^{-1} B_0 \ = \tau$
$v_1 = \alpha \, L^{-1} B_0 - \alpha \, L^{-1} v_0 \ = 0$
$u_2 = L^{-1} u_1 - \beta \, L^{-1} A_1 - (1 - \beta) \, L^{-1} B_1 \ = -0.5\tau^2$
$v_2 = \alpha \, L^{-1} B_1 - \alpha \, L^{-1} v_1 \ = \tau^2$
$u_3 = L^{-1} u_2 - \beta \, L^{-1} A_2 - (1 - \beta) \, L^{-1} B_2 \ = -0.167\tau^3$
$v_3 = \alpha \, L^{-1} B_3 - \alpha \, L^{-1} v_2 \ = -0.334\tau^3$ (3.8)
$u_4 = L^{-1} u_3 - \beta \, L^{-1} A_3 - (1 - \beta) \, L^{-1} B_3 \ = 0.708\tau^4$
$v_4 = \alpha \, L^{-1} B_3 - \alpha \, L^{-1} v_3 \ = 0.167\tau^4$
$u_5 = L^{-1} u_4 - \beta \, L^{-1} A_4 - (1 - \beta) \, L^{-1} B_4 \ = -0.241\tau^5$
$v_5 = \alpha \, L^{-1} B_4 - \alpha \, L^{-1} v_4 \ = 0.116\tau^5$

The rest of the terms of the decomposition series have been calculated using Mathcad 7. Substituting these terms into (2.5), we obtain

$u(\tau) = u_0 + u_1(\tau) + u_2(\tau) + u_3(\tau) + u_4(\tau) + u_5(\tau) + ...$
$u(\tau) = 1 + \tau - 0.5\tau^2 - 0.167\tau^3 + 0.708\tau^4 - 0.241\tau^5 + ...$ (3.9)
$v(\tau) = v_0(\tau) + v_1(\tau) + v_2(\tau) + v_3(\tau) + v_4(\tau) + v_5(\tau) + ...$
$v(\tau) = 2 + \tau^2 - 0.334\tau^3 + 0.167\tau^4 + 0.116\tau^5 + ...$ (3.10)

Example 3.2

We now consider (2.23) with initial values $u(0) = u_0$, $v(0) = v_0$ and proceeding as in section 2.2.

We take $f(u) = u^2$, $g(u, v) = uv$, $h(v) = v^2$ (3.11)

The Adomian polynomials for $f(u) = u^2$, $g(u, v) = uv$ are just same as obtained in (3.4), (3.7) in example (3.1) and

$$h(v) = v^2 = \sum_{n=0}^{\infty} C_n = \left(v_0 + v_1 + v_2 + \ldots\ldots\right)^2 \tag{3.12}$$

$$= v_0^2 + (2v_0v_1) + (2v_0v_1) + (2v_0v_2 + v_1^2) + (2v_0v_3 + 2v_1v_2)$$
$$+ (2v_0v_4 + 2v_1v_3 + v_2^2) + (2v_0v_5 + 2v_1v_4 + 2v_2v_3) + \ldots\ldots.$$

Therefore, we get the following Adomian polynomials

$C_0 = v_0^2$
$C_1 = 2v_0v_1$
$C_2 = 2v_0v_2 + v_1^2$
$C_3 = 2v_0v_3 + 2v_1 v_2$
$C_4 = 2v_0v_4 + 2v_1v_3 + v_2^2$
$C_5 = 2v_0v_5 + 2v_1v_4 + 2v_2v_3$ and so on   (3.13)

For numerical evolution, we take $u(0) = 1$, $v(0) = 2$ and

$a_1 = a_2 = b_1 = 1$, $b_2 = 2$, $c_1 = \dfrac{1}{4}$, $c_2 = \dfrac{1}{2}$ in (2.21) and proceeding as in section 2.2,

we obtain

$u_0 = 1$
$v_0 = 2$
$u_1 = L^{-1} u_0 - k_1 \ L^{-1}A_0 - k_2 \ L^{-1}B_0 \ = -0.25\tau$
$v_1 = -k_0 \ L^{-1} v_0 + k_3 \ L^{-1} B_0 - k_4 \ L^{-1}C_0 \ = -\tau$
$u_2 = L^{-1} u_1 - k_1 \ L^{-1}A_1 - k_2 \ L^{-1}B_1 \ = -0.25\tau^2$
$v_2 = -k_0 \ L^{-1} v_1 + k_3 \ L^{-1}B_1 - k_4 \ L^{-1} C_1 = 0.3750 \ \tau^2$
$u_3 = L^{-1} u_2 - k_1 \ L^{-1}A_2 - k_2 \ L^{-1}B_2 \ = -0.1510\tau^3$
$v_3 = -k_0 \ L^{-1} v_2 + k_3 \ L^{-1}B_2 - k_4 \ L^{-1} \ C_2 = 0.0208\tau^3$ (3.14)
$u_4 = L^{-1} u_3 - k_1 \ L^{-1}A_3 - k_2 \ L^{-1}B_3 \ = 0.0814\tau^4$
$v_4 = -k_0 \ L^{-1} v_3 + k_3 \ L^{-1}C_3 - k_4 \ L^{-1}C_3 = -0.1563\,\tau^4$
$u_5 = L^{-1} u_4 - k_1 \ L^{-1}A_4 - k_2 \ L^{-1}B_4 \ = -0.0411\tau^5$
$v_5 = -k_0 \ L^{-1} v_4 + k_3 \ L^{-1}B_4 - k_4 \ L^{-1} \ C_4 = 0.1577\tau^5$ and so on.

The rest of the terms of the decomposition series have been calculated using Mathcad 7. Substituting these terms into (2.17)

$$u(\tau) = u_0 + u_1(\tau) + u_2(\tau) + u_3(\tau) + u_4(\tau) + u_5(\tau) + \ldots\ldots\ldots$$

$$u(\tau) = 1 - 0.25\tau - 0.25\,\tau^2 - 0.510\tau^3 + 0.0814\,\tau^4 - 0.0411\tau^5 + \ldots\ldots \tag{3.15}$$

$$v(\tau) = v_0(\tau) + v_1(\tau) + v_2(\tau) + v_3(\tau) + v_4(\tau) + v_5(\tau) + \ldots\ldots$$

$$= 2 - \tau + 0.3750\tau^2 + 0.0208\,\tau^3 - 0.1563\tau^4 + 0.1577\tau^5 \ldots\ldots \tag{3.16}$$

4 Homotopy Perturbation Method

The combination of the peturbation method and homotpy method is called the homotopy perturbation method which has eliminated the limitations of the traditional perturbation method. Consider the nonlinear differential equation.

$$M(u) - g(s) = 0, \; s \in S \tag{4.1}$$

with boundary conditions

$$N\left(u, \frac{\partial u}{\partial n}\right) = 0, \; s \in \Gamma$$

where M is a differential operator, N is a boundary operator, g(s) is a known analytic function. Γ is the boundary of the domains. Divide the operator M into linear operator R and nonlinear operator Q. Therefore (4.1) can be expressed as

$$R(u) + Q(u) = g(s) \tag{4.2}$$

By the homotopy technique, constructing a homotopy
v(s,p): S x [0,1] → R, which satisfies

$$L(v,\,p) = (1-p)\,[R(u) - R(u_0) + p\,(M\,(v) - g(s)] = 0 \; p \in (0, 1),\, s \in S. \tag{or}$$

$$L(v,p) = R(u) - R(u_0) + pR(u_0) + p\,[M(v) - g(s)] = 0 \tag{4.3}$$

where p ε [0,1] is an embedding parameter, u_0 is an initial approximation of (4.1) which satisfies the boundary conditions.

$$L(v,0) = R(u) - R(u_0) = 0 \tag{4.4}$$

$$H(v,1) = M\,(v) - g(s)] = 0 \tag{4.5}$$

Changing p from zero to unity is similar to that of changing $v[s,p]$ from $u_0(s)$ to $u(s)$. By the method of HPM, we use the embedding parameter p as a small parameter and assume that the solutions of (4.3) and (4.4) can be written as a power series of p.

$$v = v_0 + p\,v_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{4.6}$$

Setting p → 1 results inn the approximating solutions of (4.1) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{4.7}$$

The series (4.7) is convergent for most cases. However, the convergent rate depends on the nonlinear operator M(v).

5   Application of Homotopy Perturbation Method

Example 5.1

We now solve (2.3) using HPM with initial conditions $u(0)=1$, $v(0)=2$ as chosen in example 3.1. We rewrite (2.3) as

$$\frac{du}{d\tau} = -p\left[u(\tau) - \beta u^2(\tau) - (1-\beta)u(\tau)\,v(\tau)\right]$$

$$\frac{dv}{d\tau} = -p\left[\alpha\,v(\tau) - \alpha\,u(\tau)\,v(\tau)\right] \qquad (5.1)$$

where $p \in (0,1)$ is an embedding parameter. The solutions of (5.1) when expressed as a power series is in the following form

$$u = u_0 + pu_1 + p^2 u_2 + \ldots\ldots$$

$$v = v_0 + pv_1 + p^2 v_2 + \ldots\ldots \qquad (5.2)$$

$$(5.3)$$

Substituting (5.2), (5.3) in (5.1) and equating the like coefficients of p, we obtain the following differential equations

$$p^0 : \begin{bmatrix} u_0' = 0 \\ v_0' = 0 \\ u(0)=u_0,\ v(0)=v_0. \end{bmatrix}$$

$$p^1 : \begin{bmatrix} u_1' = u_0 - \beta u_0^2 - (1-\beta)u_0 v_0 \\ v_1' = -\alpha\,v_0 + \alpha\,u_0 v_0 \\ u_1(0)=0,\ v_1(0)=0 \end{bmatrix}$$

$$p^2 : \begin{bmatrix} u_2' = u_1 - (1-\beta)\,(u_0 v_1 + u_1 v_0) - 2\beta\,u_0 u_1 \\ v_2' = \alpha\,(u_1 v_0 + u_0 v_1) - \alpha v_1 \\ u_2(0)=0,\ \ v_2(0)=0 \end{bmatrix}$$

$$p^3 : \begin{cases} u_3{}' = u_2 - (1-\beta)(u_0v_2 + u_1v_1 + u_2v_0) - 2\beta u_0u_2 - \beta u_1{}^2 \\ v_3{}' = \alpha(u_0v_2 + u_1v_1 + u_2v_0) - \alpha v_2 \\ u_3(0) = 0,\ v_3(0) = 0 \end{cases}$$

$$p^4 : \begin{cases} u_4{}' = u_3 - (1-\beta)(u_0v_3 + u_1v_2 + u_2v_1 + u_3v_0) - 2\beta u_0u_3 - 2\beta u_1u_2 \\ v_4{}' = \alpha(u_0v_3 + u_2v_1 + u_1v_2 + u_3v_0) - \alpha v_3 \\ u_4(0) = 0,\ v_4(0) = 0 \end{cases}$$

$$(5.4)$$

$$p^5 : \begin{cases} u_5{}' = u_4 - \beta(2u_0u_4 + 2u_1u_3 + u_2{}^2) - (1-\beta)(u_ov_4 + u_1v_3 + u_2v_2 + u_3v_1 + u_4v_0) \\ v_4{}' = \alpha(u_0v_4 + u_1v_3 + u_2v_2 + u_3v_1 + u_4v_0) - \alpha v_4 \\ u_5(0) = 0,\ v_5(0) = 0 \end{cases}$$

where the coefficients of p denote differentiation with respect to p. Solving the above system of equations for u_i, v_i (i = 1,2,3,4,5)

$$
\begin{array}{ll}
u_0 = 1 & v_0 = 2 \\
u_1 = \tau & v_1 = 0 \\
u_2 = -0.5\tau^2 & v_2 = \tau^2 \\
u_3 = -0.167\tau^3 & v_3 = -0.334\tau^3 \\
u_4 = 0.708\tau^4 & v_4 = 0.167\tau^4 \\
u_5 = -0.241\tau^5 & v_5 = 0.116\tau^5
\end{array}
$$

$$(5.5)$$

Substituting these u_i, v_i (i = 1,2.....5) into (5.2) and (5.3) respectively, we have

$$u = 1 + p\tau - 0.5 p^2\tau^2 - 0.167 p^3\tau^3 + 0.708 p^4\tau^4 - 0.241p^5\tau^5 - ...... \qquad (5.6)$$

$$v = 2 + p^2\tau^2 - 0.334 p^3\tau^3 + 0.167 p^4\tau^4 + 0.116 p^5\tau^5 + ... \qquad (5.7)$$

letting p → 1

$$u = 1 + \tau - 0.5\tau^2 - 0.167\tau^3 + 0.708\tau^4 - 0.241\tau^5 + ... \qquad (5.8)$$

$$v = 2 + \tau^2 - 0.334\tau^3 + 0.167\tau^4 + 0.116\tau^5 + ... \qquad (5.9)$$

which are exactly the same solutions obtained in (3.9) and (3.10) respectively

Example 5.2

We now solve (2.24) using HPM with initial conditions u(0) =1, v(0) = 2 as taken in example 3.2. We rewrite (2.24) as

$$\frac{du}{d\tau} = p\left[u(\tau) - k_1 u^2(\tau) - k_2 u(\tau)v(\tau)\right]$$

$$\frac{dv}{d\tau} = p\left[k_0 v(\tau) + k_3 u(\tau)v(\tau) - k_4 v^2(\tau)\right] \tag{5.10}$$

where $p \in (0, 1)$ is an embedding parameter and the solutions of (5.10) are expressed as a power series as in (5.2) & (5.3).

Substituting (5.2), (5.3) in (5.10) and collecting the like coefficients p, we obtain the following differential equations

$$p^0 : \begin{cases} u_0{}' = 0 \\ v_0{}' = 0 \\ u(0)=1,\ v(0)=2. \end{cases}$$

$$p^1 : \begin{cases} u_1{}' = u_0 - k_1 u_0{}^2 - k_2 u_0 v_0 \\ v_1{}' = -k_0\, v_0 + k_3\, u_0 v_0 - k_4\, v_0{}^2 \\ u_1(0)=0,\ v_1(0)=0 \end{cases}$$

$$p^2 : \begin{cases} u_2{}' = u_1 - 2k_1 u_0 u_1 - k_2(u_0 v_1 + u_1 v_0) \\ v_2{}' = -k_0\, v_1 + k_3(u_0 v_1 + u_1 v_0) - 2k_4\, v_0 v_1 \\ u_2(0)=0,\ v_2(0)=0 \end{cases}$$

$$p^3 : \begin{cases} u_3{}' = u_2 - k_1(u_1{}^2 + 2u_0 u_2) - k_2(u_0 v_2 + u_1 v_1 + u_2 v_0) \\ v_3{}' = -k_0\, v_2 + k_3(u_0 v_2 + u_1 v_1 + u_2 v_0) - k_4(2v_0 v_2 + v_1{}^2) \\ u_3(0)=0,\ v_3(0)=0 \end{cases}$$

$$p^4 : \begin{cases} u_4{}' = u_3 - k_1(2u_0 u_3 + 2u_1 u_2) - k_2(u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0) \\ v_4{}' = -k_0 v_3 + k_3(u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3\, v_0) - k_4(2v_0\, v_3 + 2v_1 v_2) \\ u_4(0)=0,\ v_4(0)=0 \end{cases} \tag{5.11}$$

$$p^5 : \begin{cases} u_5{}' = u_4 - k_1(2u_4 u_0 + 2u_3 u_1 + u_2{}^2) - k_2(u_o v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_0) \\ v_5{}' = -k_0 v_4 + k_3(u_0 v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_0) - k_4(2v_0 v_4 + 2v_1 v_3 + v_2{}^2) \\ u_5(0)=0,\ v_5(0)=0 \end{cases}$$

Thus, solving the above system of equations yields

$$u_0 = 1 \qquad\qquad v_0 = 2$$
$$u_1 = -0.25\tau \qquad\qquad v_1 = -\tau$$
$$u_2 = -0.25\tau^2 \quad v_2 = 0.3750\tau^2$$
$$u_3 = 0.1510\tau^3 \quad v_3 = 0.0208\tau^3 \tag{5.12}$$
$$u_4 = 0.0814\tau^4 \qquad v_4 = -0.1563\tau^4$$
$$u_5 = -0.0411\tau^5 \qquad v_5 = 0.1577\tau^5$$

Substituting these u_i, v_i (1 = 1,2…..5) into (5.2) and (5.3) respectively, we have

$$u(\tau) = 1 - 0.25p\tau - 0.25p^2\tau^2 - 0.1510p^3\tau^3 + 0.0814p^4\tau^4 - 0.0411p^5\tau^5 + ... \tag{5.13}$$
$$u(\tau) = 2 - p\tau + 0.3750p^2\tau^2 + 0.0208p^3\tau^3 - 0.1563p^4\tau^4 + 0.1577p^5\tau^5 + ... \tag{5.14}$$

letting p → 1, we get

$$u(\tau) = 1 - 0.25\tau - 0.25\tau^2 - 0.1510\tau^3 + 0.0814\tau^4 - 0.0411\tau^5 + ... \tag{5.15}$$
$$u(\tau) = 2 - \tau + 0.3750\tau^2 + 0.0208\tau^3 - 0.1563\tau^4 + 0.1577\tau^5 + ... \tag{5.16}$$

which are exactly the same solutions as those obtained in (3.15) and (3.16) in example 3.2, section 3.

6   Conclusion

In each of the examples described in the paper for the purpose of validation of the work, we have solved the same problem of finding the series solutions of the two species Lotka–Volterra prey-predator interaction species using two different methods i.e. the Adomian Decomposition method & the Homotopy Perturbation method. Though we obtain the same solutions for both methods, the advantage of the Homotopy Perturbation method is its straightforwardness and calculation procedure is simple when compared to the Adomian Decomposition method, where the calculation of Adomian polynomials is tedious. Hence, we conclude that the Homotopy Perturbation method is much easier and a more efficient method than the Adomian Decomposition method. These are the main results of the paper.

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