Gen. Math. Notes, Vol. 18, No. 1, September, 2013, pp.10-15
ISSN 2219-7184; Copyright ©ICSRS Publication, 2013
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The Relationship between M-Weakly Compact Operator and Order Weaky Compact Operator

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(Received: 12-6-13 / Accepted: 21-7-13)

Abstract

In this note, we will show that the class of order weakly compact operators bigger than the class of M-weakly compact operators. Under a new condition, we will show that each M-weakly compact operator is an order weakly compact operator. We will show that, if Banach lattice E be an AM-space with unit and has the property (b), then the class of M-weakly compact operators from E into Banach space Y coincides with that of order weakly compact operators from E into Y. Also we establish some relationship between M-weakly compact operators and weakly compact operators and b-weakly compact operators and order weakly compact operators.

Keywords: *Banach lattice, order weakly compact operator, M-weakly compact operator, b-weakly compact operator, AM-space.*

1 Introduction

The class of order weakly compact operators bigger than the class of M-weakly compact operators. In this note by combining Theorems 3.1 and 3.2, we will show that, if Banach lattice E is an AM-space with unit and has the property (b), then the class of M-weakly compact operators on E coincides with that of order weakly compact operators on E.

A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A sequence $\{x_n\}$ in a vector lattice E is said to be disjoint whenever

$n \neq m$ implies $| x_n | \wedge | x_m |= 0$. A vector lattice E is called σ -Dedekind complete whenever every countable subset that is bounded from above has a supremum. A subset B of a vector lattice E is said to be solid if it follows from $| y |\leq| x |$ with $x \in B$ and $y \in E$ that $y \in B$. A solid vector subspace of a vector lattice E is refferd to as an ideal. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. If E is a vector lattice, we denote by $E^{\sim}$ its order dual. Recall from [2] that a subset A of a vector lattice E is called b-order bounded in E if it is order bounded in the order bidual $(E^{\sim})^{\sim}$. A vector lattice E is said to have property (b) if $A \subset E$ is order bounded whenever A is b-order bounded in E. A Banach lattice is a Banach space $(E, \| \cdot \|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $| x |\leq| y |$, we have $\| x \|\leq\| y \|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\| \cdot \|$ of a Banach lattice E is order continuous if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, the net (x_α) converges to 0 for the norm $\| \cdot \|$. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$ we have $\| x + y \|= \max\{\| x \|, \| y \|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space. A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent.

We will use the term operator $T : E \to F$ between two Banach lattices to mean a linear mapping.

2 Main Result of Relationship

Definition 2.1 *Let $T : X \to Y$ be an operator between two Banach spaces. Then, T is said to be weakly compact whenever T carries the closed unit ball of X onto a relatively weakly compact subset of Y, the collection of weakly compact operators will be denoted by W(X,Y).*

Definition 2.2 *A continuous operator $T : E \to Y$ from a Banach lattice into a Banach space is said to be M-weakly compact whenever* $\lim_n \| Tx_n \|= 0$ *holds for every norm bounded disjoint sequence $\{x_n\}$ of E, denoted by $W_M(E, Y)$.*

Definition 2.3 *A continuous operator $T : E \to Y$ from a Banach lattice into a Banach space is said to be b-weakly compact whenever T carries each b-order bounded subset of E into relatively weakly compact subset of Y, denoted by $W_b(E, Y)$.*

Definition 2.4 *Finaly, a continuous operator $T : E \to Y$ from a Banach lattice into a Banach space is order weakly compact whenever $T\,[0, x]$ is a relatively weakly compact subset of Y for each $x \in E^+$, denoted by $W_o(E, Y)$.*

Theorem 2.5 *For a Banach lattice E, the following statements are equivalent:*
(1) *E has order continuous norm.*
(2) *If $0 \leq x_n \uparrow \leq x$ holds in E, then $\{x_n\}$ is norm couchy sequence.*
(3) *E is σ-Dedekind complete, and $x_n \downarrow 0$ in E implies $\| x_n \| \downarrow 0$.*
(4) *E is an ideal of E''.*
(5) *Each order interval of E is weakly compact.*
Proof. (1) $\Rightarrow$ (2) *Let $0 \leq x_\alpha \uparrow \leq x$ hold in E, and let $\varepsilon > 0$. By Lemma 12.8 of* [*1*] *there exists a net $(y_\lambda) \subseteq E$ with $y_\lambda - x_\alpha \downarrow 0$. Thus, there exists λ_0 and α_0 such that $\| y_\lambda - x_\alpha \| < \varepsilon$ holds for all $\lambda \geq \lambda_0$ and $\alpha \geq \alpha_0$. From the inequality*

$$\| x_\alpha - x_\beta \| \leq \| x_\alpha - y_{\lambda_0} \| + \| x_\beta - y_{\lambda_0} \|,$$

we see that $\| x_\alpha - x_\beta \| < 2\varepsilon$ holds for all $\alpha, \beta \geq \alpha_0$. Hence, (x_α) is a norm couchy net.
(2) $\Rightarrow$ (3) *It follows immediately from Theorem 11.2(2) of* [*1*].
(3) $\Rightarrow$ (1) *Let $x_\alpha \downarrow 0$. If (x_α) is not a norm Cauchy net, then there exist some $\varepsilon > 0$ and a sequence $\{\alpha_n\}$ of indices with $\alpha_n \uparrow$, and $\| x_{\alpha_n} - x_{\alpha_{n+1}} \| > \varepsilon$ for all n. Since E is σ-Dedekind complete, there exists some $x \in E$ with $x_{\alpha_n} \downarrow x$. Now from our hypothesis, we see that $\{x_{\alpha_n}\}$ is a norm Cauchy sequence, which contradicts $\| x_{\alpha_n} - x_{\alpha_{n+1}} \| > \varepsilon$. Thus, (x_α) is a norm Cauchy net, and so (x_α) is norm convergent to some $y \in E$. By Theorem 11.2(2) of* [*1*] *we see that $y = 0$, and so $\| x_\alpha \| \downarrow 0$ holds.*
The other equivalences follow easily from Theorems 11.13 and 11.10 of [*1*].

Theorem 2.6 *Let E be a Banach lattice. E is a KB-space if and only if $I : E \to E$ is a b-weakly compact operator.*

Proof. Let E be KB-space and A be an b-order bounded subset of E. Since E by Proposition 2.1 of [*2*] *has property (b), A is an order bounded subset of E and thus there exists some $x \in E^+$ for which $A \subset [-x, x]$. Then, by Theorem 2.5, $[-x, x]$ and hence A is a relatively weakly compact subset of E.*
Conversely, let $I : E \to E$ be b-weakly compact and $\{x_n\}$ be an increasing, norm bounded sequence in E^+. We wish to show $\{x_n\}$ is norm convergent. Let us define $x'' : (E^+)' \to R$ by $x''(f) = \lim_n f(x_n)$ for each $f \in (E^+)'$. x'' is additive on $(E^+)'$ and extends to an element of $(E^+)''$ which we shall also denote by x''. We have $0 \leq x_n \leq x''$ in E'' for each n. Therefore, $\{x_n\}$ is an b-order bounded subset of E. By b-weak compactness of I, we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ in $\sigma(E, E')$ for some $x \in E$.

Since $\{x_n\}$ *is increasing,* $x = \sup_k x_{n_k}$ *and we have* $x = \sup_n x_n$*. Thus* $x_n \to x$ *in* $\sigma(E, E')$. $x - x_n \downarrow 0, x - x_n \to 0$ *in* $\sigma(E, E')$ *now yield* $x - x_n \to 0$ *in the norm topology.*

Theorem 2.7 *M-weakly compact operators are weakly compact operators.*
Proof. Assume first that $T : E \to Y$ *is an M-weakly compact operator. Denot by U and V the Closed unit balls of E and Y, respectively, and let* $\varepsilon > 0$*. By Theorem 18.9(1) of [1], there exists some* $u \in E^+$ *such that* $\| T(|x| - u)^+ \| < \varepsilon$ *holds for all* $x \in U$*, and consequently from the identity* $| x | = | x | \wedge u + (|x| - u)^+$ *we see that*

$$T(U^+) \subseteq T\,[0, u] + \varepsilon V. \qquad (*)$$

On the other hand, if $\{u_n\}$ *is disjoint sequence of* $[0, u]$*, then it follows from our hypothesis that* $\lim \| Tu_n \| = 0$*, and thus by Theorem 18.1 of [1] the set* $T\,[0, u]$ *is relatively weakly compact. Now (*) combined with Theorem 10.17 of [1] shows that* $T(U^+)$*(and hence* $T(U)$*) is relatively weakly compact, and so T is a weakly compact operator.*

3 Main Result of Equality

Recall from [1] that Banach space X has the Dunford-pettis property whenever $x_n \to 0$ *in* $\sigma(X, X')$ *and* $x'_n \to 0$ *in* $\sigma(X', X'')$ *imply* $\lim x'_n(x_n) = 0$*, and we say that an operator* $T : X \to Y$ *between two Banach spaces is a Dunford-pettise operator whenever* $x_n \to 0$ *in* $\sigma(X, X')$ *implies* $\lim \| Tx_n \| = 0$.

Theorem 3.1 *Let T is an operator from AM-space with unit E into Banach space Y. Then the following assertion are equivalent:*
(1) *T is M-weakly compact.*
(2) *T is weakly compact.*
(3) *T is Dounford-pettis.*
(4) *T is b-weakly compact.*
Proof. (1) ⇒ (2) *Follows from Theorem 2.6.*
(2) ⇒ (3) *From Theorem 19.6 0f [1] E has the Duoford-pettis property. Then from Theorem 19.4 of [1] it follows that every weakly compact operators from E which has the Duonford-pettis property into an arbitary Banach space Y is a Duonford-pettis operator.*
(3) ⇒ (1) *E′ is an AL-space so it will be KB-space and then E′ has the order contiuous norm. Then from Theorem 3.7.10 of [5] every Duonford-pettis operator from E into Y is a M-weakly compact operator.*
(2) ⇒ (4) *Obvious.*
(4) ⇒ (2) *Since E is AM-space with unit so from Theorem 12.20 of [1] its closed unit ball is like an order interval. So we have the result.*

Theorem 3.2 *Let E is a Banach lattice with property (b). Then every order weakly compact operator from E into Banach space Y is a b-weakly compact operator.*
Proof. Let E has the property (b) and T from E into Banach space Y is order weakly compact operator and A is a b-order bounded subset of E. Since E has the property (b) we can choose $x \in E^+$ with $A \subseteq [-x, x]$. Therefore

$$\overline{T(A)}^w \subseteq \overline{T([-x, x])}^w.$$

Therefor by hypothesis, we will result.

4 Conclusion

In the following, we establish some relationships between some class of operators.

i) Each weakly compact operator from Banach lattice E into Banach space Y is b-weakly compact operator.

ii) Each b-weakly compact operator from Banach lattice E into Banach space Y is order weakly compact.

iii) Now by Theorem 2.7, i , ii, we will have

$$W_M(E, Y) \subset W(E, Y) \subset W_b(E, Y) \subset W_o(E, Y) \qquad (**)$$

iv) Since the norm of c_0 is order continuous, by Theorem 2.5, $[0, x]$ is weakly compact in c_0, then $I : c_0 \to c_0$ is order wekly compact. But c_0 is not KB-space, then by Theorem 2.6, $I : c_0 \to c_0$ is not b-weakly compact operator. Therefore, by (**) every order weakly compact operator is not M-weakly compact and weakly compact operator.

v) Since $L_1([0, 1])$ is a KB-space therefor $I : L_1([0, 1]) \to L_1([0, 1])$ is b-weakly compact operator. But its not weakly compact operator. By (**) every b-weakly compact operator is not M-weakly compact operator.

vi) By theorems 19.6 and 17.5 of [1], operator $T : l^1 \to l^\infty$ defined by

$$T(\alpha_1, \alpha_2, ...) = \left(\sum_{n=1}^{\infty} \alpha_n, \sum_{n=1}^{\infty} \alpha_n, ... \right) = \left[\sum_{n=1}^{\infty} \alpha_n \right] (1, 1, 1, ...)$$

is weakly compact. The sequence $\{e_n\}$ of the standard unit vectors is a norm bounded disjoint sequence of l^1 satisfying $Te_n = (1, 1, 1, ...)$ for each n. This follow that T is not M-weakly compact. Then every weakly compact is not M-weakly compact.

vii)If E is an AM-space with unit and has the property (b), by Theorems 3.1 and 3.2 we will have

$$W_o(E,Y) = W_b(E,y) = W_M(E,Y) = W(E,Y).$$

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