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Mathematical Model of the Role of Vaccination and Treatment on the Transmission Dynamics of Tuberculosis

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Abstract

In this study the role of vaccination of new born babies against tuberculosis and treatment of both latently and activity infected individuals in controlling the spread of tuberculosis was mathematically modelled based on the standard SEIR model. The disease - free equilibrium state of the model was established and its stability analyzed using the Routh-Hurwitz theorem. The result of the analysis of the stability of the disease-free equilibrium state shows that tuberculosis can totally be eradicated if effort is made to ensure that the sum of the rate of recovery of the latent class, the rate at which latently infected individuals become actively infected and the rate of natural death , must have a lower bound.

Keywords: *Latent TB infection, Active TB infection, Disease-free equilibrium state; Endemic equilibrium state; Stability analysis.*

1 Introduction

Tuberculosis or TB (short for tubercles bacillus) is a highly infectious diseases caused by infection with the bacteria mycobacterium tuberculosis (Cohen et al.2004). The disease is airborne and so it is primarily transmitted through the respiratory route. When people who are infected with the disease cough, sneeze, spit or talk, they propel TB germs, (in mucus droplets), known as bacilli, into the air. A previously uninfected person needs only to inhale a small number of these germs to be infected. (Cohen et al., 2004).

Once infected, an individual enters a period of latency during which he exhibits no symptoms of the disease and is not infectious to others. Such a person is said to have a Latent TB infection. This latent period can be of extremely variable length of time. A great majority of those infected (≈ 90 %) may live with the disease as long as possible without it degenerating or progressing into Active TB. However, a small proportion of those infected (≈ 10 %) will progress from the latent TB to Active TB**,** falling ill within months or several years after infection (Colijn et al., 2006). Some may be asymptotically infected for decades before they become sick. Once ill and infectious, individuals may recover without treatment, may be cured with antibiotics or may die from the disease. Recovered individuals may relapse to disease or be re-infected. The degree of protection afforded by a previous infection and the mechanism by which individuals with partial immunity are protected are controversial. The highest risk group to acquire TB when exposed to it are children under five years old, persons who are immuno-compromised (i.e. have weakened immunity), especially those who are HIV-Positive, persons who have diabetes or kidney failure, people that take excessive alcohol and drugs, those with poor nutrition and lack of food, those suffering from stress and those living in poorly ventilated rooms (Sanga, 2008).Tuberculosis usually attacks the lungs but can also attack other parts of the body like the kidney, Spine, brain, bones, joints etc. The classic symptoms of TB of the lungs are a chronic cough which may result in blood-tinged sputum, fever, night sweats, loss of appetite, weight loss and fatigue. Infection of other organs causes a wide range of symptoms. Pneumonia, lung collapse and enlarged lymph nodes may also occur. (WHO; 2007). Two forms of tuberculosis that become life- threatening are Miliary TB, which means the bacteria have spread throughout the lungs and into the bloodstream and TB meningitis (infection of the covering of the spinal cord and /or brain by TB bacteria). Diagnosis relies on radiology (commonly chest X-ray), a tuberculin skin text, blood tests, as well as microscopic examination and microbiological culture of bodily fluids (such as sputum).

The introduction of Directly Observed Treatment Short (DOTS) Course has helped in the control and management of tuberculosis. DOTS, the internationally recommended strategy for TB control cures patients, saves lives, prevents the development and spread of drugs resistance, and reduces disease transmission. DOTS makes sure that TB diagnosis and medicine are available for all TB patients free-of-charge. Prevention relies on screening programmes and vaccination, usually with Bacillus calmette-Guérin (BCG) vaccine (Colditz et al.; 1994).

TB and HIV are the leading causes of death from infectious diseases among adults globally and the number of TB cases has risen significantly since, the start of the HIV epidemic, particularly in sub-Saharan Africa where the HIV epidemic is most severe (Dye: 2006).

TB progression from latent infection to active disease varies greatly. For instance, people with AIDS are more likely to develop to Active TB after infection. A patient with AIDS who becomes infected with Mycobacterium tuberculosis has a 50% chance of developing Active tuberculosis within 2 months and a 5 to 10% chance of developing Active disease each year thereafter. According to the World Health Organization (WHO), infants and young children infected with mycobacterium tuberculosis are also more likely to develop Active TB than older people since their immune system are not yet well developed. (Okyere, 2007).

Treatment for tuberculosis uses antibiotics to kill the bacteria. Effective TB treatment is difficult, due to the unusual structure and chemical composition of the mycobacterium cell wall, which makes many antibiotics ineffective and hinders the entry of drugs. (Migliore et al., 1966;, Acharya et al. 1970 and Brennan and Nikaido, 1995).

The two antibiotics most commonly used are Rifampicin and Isoniazid. However, instead of the short course of antibiotics typically used to cure other bacterial infections, TB requires much longer periods of treatment (around 6 to 24 months) to entirely eliminate mycobacterium from the body. (Center for Disease Control and Prevention (CDC), 2000).

Latent TB treatment usually uses a single antibiotic, while Active TB infection is best treated with combinations of several antibiotics, to reduce the risk of the bacteria developing antibiotic resistance. (O'Brien,1994). People with latent infections are treated to prevent them from progressing to Active TB disease later in life.

2 The Global Perspective of Tuberculosis

Tuberculosis (TB) remains a leading cause of infectious mortality in the world despite many decades of study, the widespread availability of vaccines, an arsenal of anti-microbial drugs and more recently, a highly visible World Health Organization (WHO) effort to promote a unified global control strategy. The world health organization declared TB a global emergency in 1993. It has been approximated that one third of the world's population is infected with mycobacterium tuberculosis. (Cohen et al. 2004). Data released by the health protection Agency in 2000 shows that 8 million new incidences of TB occur per year. 2 million persons die from the disease per year and 80% of new incidence lives in high burden countries like Zimbabwe, Kenya, Uganda, United Republic of Congo and India (Blower et al.; 1995).

3 Methodology

Mathematical models have played a key role in the formulation of TB control strategies and the establishment of interim goals for intervention programme.

Many types of mathematical models exist. They include: the stochastic model ,the deterministic (compartmental) model such as:the SIR, SIS, SIRS, SEIS, SEIR, MSIR, MSEIR, and the MSEIRS models. (Where S = Susceptible class; I = Infective class; M = passively immune infants; E = Exposed class; and R = Removed or Recovered class) etc.
Our model is a deterministic or compartmental, SEIR type model where the population is partitioned into components or classes based on the epidemiological state of individuals, and it is assumed that the population size in a compartment is differentiable with respect to time and that the epidemic process is deterministic. Therefore, the TB transmission dynamics between the compartments shall be described by a system of differential equation which shall be solved to obtain the disease-free equilibrium state.
The stability analysis of the disease-free equilibrium state shall be carried out using the Routh-Hurwitz theorem.

4 Assumption of the Model

The model is based on the following assumptions.

1. That the population is heterogeneous. That is, the individuals that make up the population can be grouped into different compartments or groups according to their epidemiological state.
2. That the population size in a compartment is differentiable with respect to time and that the epidemic process is deterministic. In other words, that the changes in population of a compartment can be calculated using only history to develop the model.
3. That a proportion of the population of newborns is immunized against TB infection through vaccination.
4. That the immunity conferred on individuals by vaccination expires after some time at a given rate.
5. That the population mixes homogeneously. That is, all susceptible individuals are equally likely to be infected by infectious individuals in case of contact.
6. That the infection does not confer immunity to the cured and recovered individuals and so they go back to the susceptible class at a given rate.
7. That people in each compartment have equal natural death rate of β.
8. That all newborns are previously uninfected by TB and therefore join either the immunized compartment or the susceptible compartment depending on whether they are vaccinated or not.
9. That there are no immigrants and emigrants. The only way of entry into the population is through new – born babies and the only way of exit is through death from natural causes or death from TB-related causes.

5 Model Variables and Parameters

The following variables and parameters shall be used in this model.

M(t): the number of individuals who are immunized against TB through Vaccination at time t

S(t): the number of susceptible individuals at time t
L(t): the number of latently infected / exposed individuals at time t.
I(t): the number of infectious individuals at time t.
R(t): the number of individuals who have been treated and have recovered from the infection at time t.
φ: the rate of expiration of vaccine efficacy.
κ: the rate at which susceptible individuals become latently infected by TB.
μ: the rate at which latently infected individuals become actively infected.
ψ: the rate at which actively infected individuals recover from TB infection.
q: the rate at which individuals who are latently infected recover from TB through treatment.
π: the rate at which recovered individuals become susceptible to TB Infection again. η: the tuberculosis-induced mortality / death rate.
β: the natural mortality / death rate
P: population of new births joining the population N.
cP: the proportion of new births that have been immunized through Vaccination.
N: the total population size

6 Model Description

Based on the standard SEIR model, the population is partitioned into 5 compartments or classes namely: Immunized M(t), Susceptible S(t), Latent L(t), Infectious I(t) and Recovered R(t) compartments .

The Immunized component increases due to the coming in of the immunized newborns into the population, where we assumed that a proportion, cP, of the incoming individuals are immunized through vaccination. The component reduces due to the expiration of the duration of vaccine efficacy at the rate of φ and also as a result of natural death at the rate of β.

The susceptible component of the population grows due to the coming in of new-born babies not immunized against TB infection into the population at the rate of (1-c)P, the coming in of some recovered individuals due to the fact that the infection does not confer immunity to recovered individuals, at the rate of π and as a result of the expiration of the efficacy of the vaccine, at the rate of φ. This component decreases due to the latent infection of individuals at the rate of κ and due to death from natural causes at the rate of β.

The population of the latent component grows as a result of infection of individuals in the susceptible class at the rate of κ. This class reduces due to the progression of latently infected individuals to active TB infection at the rate of μ, the successful treatment and cure of latent TB patients at the rate of q and as a result of death from natural causes at the rate of β.

The infectious compartment increases due to the progression of latently infected individuals to active TB infection at the rate of μ .The component reduces as a result of successful cure of infectious TB patients at the rate of Ψ , death as a result of active TB infection at the rate of η and also due to death from natural causes at the rate of β.

Lastly, the recovered component grows as a result of successful treatment and cure of latent TB patients at the rate of q and that of the infectious TB patient at rate of ψ and decrease due to the fact that recovered individuals are not immune against the infection and so they return to the susceptible class at the rate of π and also as a result of death from natural cause at the rate of β (Enagi et. al.; 2011).

The model can schematically be presented as shown below

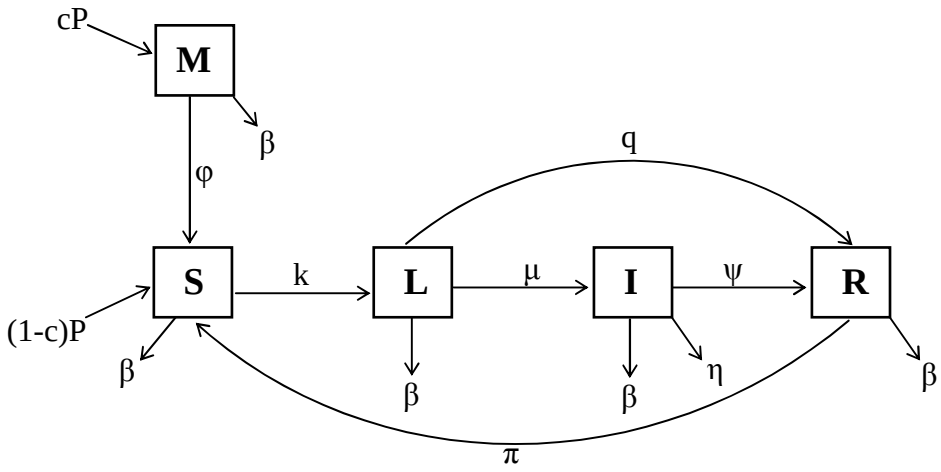

Fig 1: Schematic Presentation of the model

7 The Model Equations

Applying the assumptions and the inter-relations between the variables and the parameters as described above the role of vaccination and treatment on tuberculosis transmission dynamics can be describe by the following differential equations:

$$\frac{dM}{dt} = \text{cP} - \varphi\text{M} - \beta\text{M} = \text{cP} - (\varphi - \beta)\text{M} \tag{1}$$

$$\frac{dS}{dt} = (1-c)P + \varphi M + \pi R - \kappa SI - \beta S = (1-c)P + \varphi M + \pi R - (\kappa I + \beta)S \tag{2}$$

$$\frac{dL}{dt} = \kappa SI - qL - \mu L - \beta L = \kappa SI - (q + \mu + \beta)L \tag{3}$$

$$\frac{dI}{dt} = \mu L - \psi I - \eta I - \beta I = \mu L - (\psi + \eta + \beta) I \quad (4)$$

$$\frac{dR}{dt} = qL + \psi I - \pi R - \beta R = qL + \psi I - (\pi + \beta) R \quad (5)$$

$$N(t) = M(t) + S(t) + L(t) + I(t) + R(t) \quad (6)$$

8 Results and Discussion

8.1 Equilibrium Solutions

Let E (M, S, L, I, R) be the equilibrium point of the system described by the equations (1) – (6). At the equilibrium state, we have

$$\frac{dM}{dt} = \frac{dS}{dt} = \frac{dL}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

That is,

$$cP - (\varphi + \beta) M = 0 \quad (7)$$

$$(1 - c) P + \varphi M + \pi R - (\kappa I + \beta) S = 0 \quad (8)$$

$$\kappa SI - (q + \mu + \beta) L \quad (9)$$

$$\mu L - (\psi + \eta + \beta) I = 0 \quad (10)$$

$$qL + \psi I - (\pi + \beta) R = 0 \quad (11)$$

In order to obtain the disease-free equilibrium state we solve equations (7) – (11) simultaneously.

8.2 The Existence of a Trivial Equilibrium State

Let E_o (M_o, S_o, L_o, I_o, R_o) be the trivial equilibrium state of the model. There is no trivial equilibrium state for the model since the population cannot be extinct so long as new babies are born into the population. In other words, so long as the recruitment terms cρ and (1 – c)ρ are not zero, the population will never be extinct.

That is, E_o, (M_o , S_o , L_o , I_o , R_o) ≠ (0,0,0,0,0)

8.3 The Disease-Free Equilibrium State

The disease-free equilibrium state is the state of total eradication of the disease. Let E^o (M^o, S^o, L^o, I^o, R^o) be the disease-free equilibrium state. For disease-free

equilibrium state, both the infectious class and the latently infectious class must be zero. That is, for disease-free equilibrium state

$I^o = L^o = 0$ (12)

Substituting I = L = 0 into Equations (7) – (11) and solving simultaneously we have:

From Equation (7)

$cP - (\varphi + \beta)M = 0$

$$M^\circ = \frac{cP}{(\varphi + \beta)} \quad (13)$$

From Equation (8)

$$(1 - c)P + \frac{\varphi cP}{\varphi + \beta} + \pi R - \beta S = 0 \quad (14)$$

From Equation (11)

$qL + \psi I - (\pi + \beta)R = 0$

$\rightarrow \quad (\pi + \beta)R = 0,$ (since L = I = 0)

$\rightarrow$ Either $(\pi + \beta) = 0$ (15)

Or $R = 0$ (16)

Since π and β are positive constants, $(\pi + \beta) \neq 0$

Therefore, $R^o = 0$

If R = 0, Equation (14) becomes

$$(1 - c)P + \frac{\varphi cP}{\varphi + \beta} - \beta S = 0$$

$$\rightarrow S^\circ = \frac{(\varphi + \beta)(1 - c)P + \varphi cP}{\beta(\varphi + \beta)}$$

$$\text{or } S^\circ = \frac{(\varphi + \beta - c\beta)P}{\beta(\varphi + \beta)} \quad (17)$$

Therefore the disease -free equilibrium state of the model is

$$E^\circ(M^\circ, S^\circ, L^\circ, I^\circ, R^\circ) = \left(\frac{cP}{\varphi + \beta}, \quad \frac{(\varphi + \beta - c\beta)P}{\beta(\varphi + \beta)}, \quad 0, \quad 0, \quad 0\right)$$

8.4 Stability Analysis of the Disease-Free Equilibrium State

To determine the stability or otherwise of the disease-free equilibrium state E^o, we examine the behaviour of the model population near this equilibrium solution. Here we determine the condition(s) that must be met for the disease -free equilibrium state to be stable. In other words, we determine the condition(s) that must be met if the disease is to be totally eradicated from the population.

Recall that the system of equations in this model at equilibrium state is:

$$cP - (\varphi + \beta)\, M = 0$$
$$(1 - c)P + \varphi M + \pi R - (\kappa I + \beta)\, S = 0$$
$$\kappa SI - (q + \mu + \beta)\, L = 0$$
$$\mu L - (\psi + \beta + \eta)\, I = 0$$
$$qL + \psi I - (\pi + \beta)\, R = 0$$

We now linearize the system of equations to get the Jacobian Matrix J.

$$\mathbf{J} = \begin{pmatrix} -(\varphi+\beta) & 0 & 0 & 0 & 0 \\ \varphi & -(\kappa I^{\circ} + \beta) & 0 & -\kappa S^{\circ} & \pi \\ 0 & \kappa I^{\circ} & -(q+\mu+\beta) & \kappa S^{\circ} & 0 \\ 0 & 0 & \mu & -(\psi+\beta+\eta) & 0 \\ 0 & 0 & q & \psi & -(\pi+\beta) \end{pmatrix} \quad (18)$$

At the disease-free equilibrium, E^o (M^o, S^o, L^o, I^o, R^o), the Jacobian Matrix becomes

$$J_0 = \begin{pmatrix} -(\varphi+\beta) & 0 & 0 & 0 & 0 \\ \varphi & -\beta & 0 & -\kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & \pi \\ 0 & \kappa I^{\circ} & -(q+\mu+\beta) & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & 0 \\ 0 & 0 & \mu & -(\psi+\beta+\eta) & 0 \\ 0 & 0 & q & \psi & -(\pi+\beta) \end{pmatrix} \quad (19)$$

The characteristic equation $|J_o - I\lambda| = 0$ is obtained from the Jacobian determinant with the Eigen values λ_i (i=1, 2, 3, 4, 5)

That is;

$$|J_0 - I\lambda| = \begin{vmatrix} -(\varphi+\beta)-\lambda & 0 & 0 & 0 & 0 \\ \varphi & -\beta-\lambda & 0 & -\kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & \pi \\ 0 & 0 & -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & 0 \\ 0 & 0 & \mu & -(\psi+\beta+\eta)-\lambda & 0 \\ 0 & 0 & q & \psi & -(\pi+\beta)-\lambda \end{vmatrix} = 0$$

$$= (-(\varphi+\beta)-\lambda) \begin{vmatrix} -\beta-\lambda & 0 & -\kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & \pi \\ 0 & -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & 0 \\ 0 & \mu & -(\psi+\beta+\eta)-\lambda & 0 \\ 0 & q & \psi & -(\pi+\beta)-\lambda \end{vmatrix} = 0$$

$$= (-(\varphi+\beta)-\lambda)(-\beta-\lambda) \begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} & 0 \\ \mu & -(\psi+\beta+\eta)-\lambda & 0 \\ q & \psi & -(\pi+\beta)-\lambda \end{vmatrix} = 0$$

$$= (-(\varphi+\beta)-\lambda)(-\beta-\lambda)(-(\pi+\beta)-\lambda) \begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} \\ \mu & -(\psi+\beta+\eta)-\lambda \end{vmatrix} = 0$$

$$= (\varphi+\beta+\lambda)(\beta+\lambda)(-\pi-\beta-\lambda) \begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} \\ \mu & -(\psi+\beta+\eta)-\lambda \end{vmatrix} = 0$$

$$= \left(\lambda^2 + (\varphi+2\beta)\lambda + (\varphi\beta+\beta^2)\right)(-\pi-\beta-\lambda) \begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} \\ \mu & -(\psi+\beta+\eta)-\lambda \end{vmatrix} = 0$$

From Equation (20)

Either $\left(\lambda^2 + (\varphi+2\beta)\lambda + (\varphi\beta+\beta^2)\right)(-\pi-\beta-\lambda) = 0$ (21)

Or $$\begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} \\ \mu & -(\psi+\beta+\eta)-\lambda \end{vmatrix}=0 \quad (22)$$

From Equation (21)

$$\lambda_1=-(\pi+\beta) \quad (23)$$

$$\lambda_2=-\beta \quad (24)$$

and $$\lambda_3=-(\varphi+\beta) \quad (25)$$

From Equation (22)

$$(-q-\mu-\beta-\lambda)(-\psi-\beta-\eta-\lambda)-\kappa\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}=0$$

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta+\lambda)-\kappa\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}=0$$

Let $$A=\begin{vmatrix} -(q+\mu+\beta)-\lambda & \kappa\dfrac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)} \\ \mu & -(\psi+\beta+\eta)-\lambda \end{vmatrix}$$

For the disease-free equilibrium to be asymptotically stable, trace (A) < 0 and det A > 0.

$$\det A=(q+\mu+\beta+\lambda)(\psi+\beta+\eta+\lambda)-\kappa\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}$$

And the trace of A is :

$$\text{Trace}(A)=-(q+\mu+\beta+\lambda)-(\psi+\beta+\eta+\lambda)$$

It is clear that trace (A) < 0 since all the parameters q, t, β, ψ, β and η are positive.

For the determinant of A to be positive (i.e. > 0), we must have

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta)-\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right)>0$$

or $$(q+\mu+\beta+\lambda)(\psi+\beta+\eta)>\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right) \quad (26)$$

From equation (23) – equation (25) we see that the first 3 Eigen values of equation 20 all have negative real parts. We now establish the necessary and sufficient conditions for the remaining two Eigen values of equation (20) to have negative real part. The remaining two Eigen values of equation (20) will have negative real part if and only if det A > 0. That is, if and only if

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta) > \kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right)$$

The Routh-Hurwitz theorem states that the equilibrium state will be asymptotically stable if and only if all the Eigen values of the characteristics equation $|J - I\lambda| = 0$ have negative real part.

Using this theorem we see that the disease-free equilibrium of this model will be asymptotically stable if and only if

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta) > \kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right)$$

or

$$\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right) < (q+\mu+\beta+\lambda)(\psi+\beta+\eta) \qquad (27)$$

The inequality (27) gives the necessary and sufficient condition for the disease-free equilibrium state of the model to be stable.

This means that the necessary and sufficient condition for the disease-free equilibrium state of this model to be asymptotically stable is that the product of total contraction and total breakdown of latent class given by $\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right)$ must be less than the total removal rate from both latent and infectious classes given by $(q+\mu+\beta)(\psi+\beta+\eta)$.

Alternatively, the inequality (26) and (27) can also be expressed as

$$(q+\mu+\beta+\lambda) > \kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)(\psi+\beta+\eta)}\right) \qquad (28)$$

The inequality (28) also gives the necessary and sufficient condition for the stability of the disease-free equilibrium state. It means that the necessary and sufficient condition for stability of the disease-free equilibrium state of this model is that the sum of the rate of recovery of latently infected people, the rate at which latently infected individuals progress to active infection and the rate of natural death of individuals in the population (ie total removal rate from the latent class) must have a lower bound given by

$$\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)(\psi+\beta+\eta)}\right)$$

9 Discussion

Presented in this project is a mathematical model of the role of vaccination and treatment on tuberculosis transmission dynamics.
The population was partitioned into five compartments namely: the Immunized, the Susceptible, the latently infected, the Infectious and the Recovered compartments or classes. It was assumed that the only entrants into the population are new born babies, proportion, cP of whom are immunized against TB infection

and the other proportion, (1 – c)p, are not immunized and so join the susceptible class. The population dynamics of the compartments was described using five differential equations.

The trivial equilibrium state E_o (M_o, S_o, L_o, I_o, R_o), that is, the state where there is no individual in the population, was found to be unstable since the recruitment terms cP and (1 – c)P cannot be zero. This is so because there must always be new born babies entering the population. That is, $E_o(M_o, S_o, L_o, I_o, R_o) \neq (0,0,0,0,0)$

The disease-free equilibrium state, E^o (M^o, S^o, L^o, I^o, R^o), was determined and its stability analysis conducted using the Routh-Hurwitz theorem. The analysis shows that the necessary and sufficient condition for the disease-free equilibrium state to be locally asymptotically stable is that the product of total contraction and total breakdown of the latent class should be less than the total removal rate from both the latent and the infectious classes. That is,

$$\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)}\right)<(q+\mu+\beta+\lambda)(\psi+\beta+\eta)$$

Put differently, the result of the stability analysis of the disease-free equilibrium state is that for the disease-free equilibrium state to be locally asymptotically stable, the sum of the rate of recovery of the latently infected individuals, the rate at which latently infected become actively infected by the TB disease and the rate of natural death of individuals in the population must have a lower bound. That is,

$$(q+\mu+\beta+\lambda)>\kappa\mu\left(\frac{(\varphi+\beta-c\beta)P}{\beta(\varphi+\beta)(\psi+\beta+\eta)}\right)$$

This establishes the condition under which tuberculosis can completely be eradicated in any population.

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