Gen. Math. Notes, Vol. 9, No. 2, April 2012, pp. 19-31
ISSN 2219-7184; Copyright © ICSRS Publication, 2012
www.i-csrs.org
Available free online at http://www.geman.in

Absolute Banach Summability Of a Factored Conjugate Series

S.K. Paikray[1], U.K. Misra[2] and N.C. Sahoo[3]

[1]Dept. of Mathematics, Ravenshaw University, Cuttack, Odisha, India
E-mail: spaikray2001@yahoo.com
[2]Dept. of Mathematics, Berhampur University, Bhanjabihar, Berhampur,Odisha
E-mail: umakanta_misra@yahoo.com
[3]Dept. of Mathematics, S.B. Women's College, Cuttack, Odisha
E-mail:nch.sahoo@gmail.com

(Received: 12-3-12/ Accepted: 28-3-12)

Abstract

A theorem for a factored conjugate series is proved in absolute Banach Summability.

Keywords: *Banach Summability, Absolute Banach Summability, Conjugate series.*

1 Introduction

Let Ω and l_∞ denote respectively the linear spaces of all sequences and bounded sequences on R. A linear functional l and l_∞ is called a limit functional if and only if l satisfies:

L_1 : For e = (1, 1, 1,…..)
L(e) = 1; …(1.1)

L_2 : For every $x \geq 0$, that is to say,
$x_n \geq 0, \forall n \in N, x \in l_\infty, l(x) \geq 0;$ …(1.2)

L_3 : For every $x \in \{x_n\} \in l_\infty$ …(1.3)
$l(x) = l(T(x))$
Where T is the shift operator on l_∞ such that $T(x) = \{x_{n+1}\}$

If l is a linear functional on l_∞, then for every $x \in l_\infty$, l is called a "Banach limit" of x.[2]
A sequence $x \in l_\infty$ is said to be Banach summable if all the Banach limits of x are same. Similarly, a series $\sum u_n$ with the sequence of partial sums $\{s_n\}$ is said to be Banach summable if and only if $\{s_n\}$ is Banach summable.
Let the sequence $\{t_k(n)\}$ be defined by

$$t_k(n) = \frac{1}{k}\sum_{v=0}^{k-1} s_{n+1},\ k \in N \qquad \text{…(1.4)}$$

Then $t_k(n)$ is called the k-th element of the Banach transformed sequence.
If $\lim_{k\to\infty} t_k(n) = s$, a finite number, uniformly for all $n \in N$, then $\sum u_n$ is said to be Banach summable [4], thus, if

$$\sup_n \left|t_k(n) - s\right| \to 0, \text{ as } k \to \infty$$

Then $\sum u_n$ is Banach summable to s. Further, if the series

$$\sum_{k=1}^{\infty} | t_k(n) - t_{k+1}(n) | < \infty$$

uniformly for all $n \in N$, for $t_k(n)$ as defined the (1.4), then the series $\sum u_n$ is called absolutely Banach summable or |B|-summable.

2 Definition and Notations

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Riemann over $(-\pi, \pi)$. Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t)$$

be the Fourier series of $f(t)$. Then the series

$$\sum_{n=1}^{\infty}\left(b_n \cos nt - a_n \sin nt\right) \equiv \sum_{n=1}^{\infty} B_n(t)$$

is called the conjugate series of the Fourier series.
We use the following notations:

$$\psi(t) = \frac{1}{2}\left\{f(x+t) - f(x-t)\right\};$$

$$\Psi_0(t) = \psi(t)$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\psi(u)\,du,\ \alpha > 0$$

$$\Psi_\alpha(t) = \Gamma(\alpha+1)\,t^{-\alpha}\ \Psi_\alpha(t),\ \alpha \geq 0$$

$$g'(k,t) = \frac{2}{\pi}\,\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{(n+v)\log(n+v+1)}\cos(n+v)\,t$$

$$J'(k,u) = \frac{1}{\Gamma(1-\alpha)}\int_u^\pi \frac{d}{dt}\,g(k,t)\,(t-u)^{-\alpha}\,dt$$

$[x] =$ greatest integer not exceeding x ;

$$\tau = \left[\frac{1}{t}\right]$$

and $$U = \left[\frac{1}{u}\right]$$

3 Theorems

In 1937, *Bosanquet* and *Hyslop* [3] proved the following theorem for absolute Cesàro summability of the conjugate series of a Fourier series.

Theorem-A

If $0 < \alpha < 1,\ \Psi_\alpha(+0) = 0$ *and* $\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha} < \infty$ *then* $\sum_{n=1}^{\infty} B_n(t)$ *is summable* $|C,\beta|$ *at* $t = x,\ \beta > \alpha$.

In 1980, Swamy [7] extended the above result to generalized absolute Cesàro summability. He proved.

Theorem-B

If $0 < \alpha < 1,\ \Psi_\alpha(+0) = 0$ *and* $\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha} < \infty$ *then the conjugate series of the Fourier series of* $f(t)$ *is summable* $|C,\delta,\beta|$ *at* $t = x,\ \delta > \alpha$.

In 2002, an analogue theorem has been established by Misra and Sahoo [5] in Banach summability. They Proved:

Theorem-C

If $0<\alpha<1,\ \Psi_\alpha(+0)=0$ *and* $\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha}<\infty$ *then the conjugate series of the Fourier series of* $f(t)$ *is* $|B|$*-summable at* $t=x$.

In 2011, an analogue theorem has been established by Paikray, Misra and Sahoo [6] in Banach summability. They Proved:

Theorem-D

Let $\phi_\alpha(+0)=0$, *for* $0<\alpha<1$, *such that*

(3.1) $$\int_0^\pi \frac{d\phi_\alpha(t)}{t^\alpha \log(n+U)}<\infty . If$$

(3.2) $$\sum_{k<\frac{1}{u}} \log(n+U)\, k^{\alpha-1} = O\left(U^\alpha \log\log(n+2)\right),$$

(3.3) $$\sum_{k\geq\frac{1}{u}} k^{\alpha-2}(n+k+1)\log(n+k) = O\left(\frac{(\log\log(n+2))^2}{u^{\alpha-1}\log(n+U)}\right),$$

(3.4) $$\sum_{k\geq\frac{1}{u}} \frac{k^{\alpha-2}(2-k)}{(n+k-1)} = O\left(\frac{\{\log\log(n+2)\}^2(n+1)}{u^{\alpha-1}\log(n+U)}\right),$$

and

(3.5) $$\sum_{k\geq\frac{1}{u}} \frac{u^{\alpha-2}}{\log\log(n+k-1)} = O\left(\frac{1}{u^{\alpha-1}\log(n+U)}\right),$$

then the series $\sum_{n=1}^\infty \frac{A_n(t)}{\log\log(n+1)}$ *is* $|B|$*-summable.*

In this chapter we prove:

Theorem

Let $\Psi_\alpha(+0)=0,\ 0<\alpha<1,$ *and*

$$\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha \log(n+k)}<\infty .$$

Then the series $\sum_{n=1}^\infty \frac{B_n(t)}{\log(n+1)}$ *is* $|B|$*-summable at* $t=x$ *if*

(3.6.) $\sum_{k\le\frac{1}{u}} \log(n+U)\ k^{\alpha-1} = O\left(U^{\alpha}\ \log(n+2)\right).$

4 Lemmas

We require following lemmas for the proof of our theorem.

Lemma-1

Let $\sum u_n$ *be an infinite series with sequence of partial sums* $\{s_n\}$*. If*

$$t_k(n) = \frac{1}{k}\sum_{V=0}^{k-1} s_{n+v}\ ,$$

then

$$t_k(n) - t_{k+1}(n) = \frac{-1}{k(k+1)}\sum_{v=1}^{k} v\ u_{n+v}\ .$$

Proof. We have

$$t_k(n) - t_{k+1}(n) = \frac{s_n + s_{n+1} + \cdots + s_{n+k-1}}{k} - \frac{s_n + s_{n+1} + \cdots + s_{n+k}}{k+1}$$

$$= \frac{1}{k(k+1)}\left\{\left(s_n + s_{n+1} + \ldots + s_{n+k-1}\right) - k\ s_{n+k}\right\}$$

$$= \frac{1}{k(k+1)}\left\{s_n + \left(\ (s_n + u_n) + 1\right) + \ldots + (s_n + u_{n+1}\ \ldots + u_{n+k-1})\right)$$

$$- k\left(s_n + u_{n+1}\ \ldots + u_{n+k}\ \right)\}$$

$$= -\frac{1}{k(k+1)}\left\{ku_{n+k} + (k-1)\ u_{n+k-1} \ldots + u_{n+1}\right\}$$

$$= -\frac{1}{k(k+1)}\sum_{v=1}^{k} v\,u_{n+1}\ . \qquad \ldots(4.1)$$

Lemma-2

Let

(4.2) $$g'(k,t) = \frac{2}{\pi}\ \frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{(n+v)\ \log(n+v+1)}\ \cos\ (n+v)t\ .$$

Then

$$\text{(4. 3)} \qquad |g'(k,t)| = \begin{cases} O\left(\dfrac{1}{(k+1)\log(n+2)}\right), & for\ all\ t \\ O\left(\dfrac{\tau}{k(k+1)}\left(\dfrac{1}{\log(n+k)} + \dfrac{1}{(n+k)\log^2(n+k)}\right)\right), & t \ge \dfrac{1}{k} \end{cases}$$

$$\left|\frac{d}{dt} g'(k,t)\right| = \begin{cases} O\left(\dfrac{1}{\log(n+2)}\right), & for\ all\ t \\ O\left(\dfrac{\tau}{k+1}\left(\dfrac{1}{\log(n+k)} + \dfrac{1}{(n+k)\log^2(n+k)}\right)\right), & t \ge \dfrac{1}{k} \end{cases}$$

Proof.
For all t,

$$\begin{aligned} |g'(k,t)| &= \left|\frac{2}{\pi}\ \frac{1}{k(k+1)} \sum_{v=1}^{k} \frac{v\cos(n+v)t}{(n+v)\log(n+v+1)}\right| \\ &\le \left|\frac{1}{k(k+1)} \sum_{v=1}^{k} \frac{v.1}{(n+v)\log(n+v+1)}\right| \\ &\le \frac{1}{k(k+1)} \cdot \frac{1}{\log(n+2)} \cdot \sum_{k=1}^{k} \frac{v}{n+v} \\ &\le \frac{1}{k(k+1)}\ \frac{1}{\log(n+2)} \cdot \sum_{v=1}^{k} 1 \\ &= \frac{1}{k(k+1)\log(n+2)} \end{aligned}$$

for $t \ge \dfrac{1}{k}$,

$$\begin{aligned} |g'(k,t)| &= \left|\frac{2}{\pi}\ \frac{1}{k(k+1)} \cdot \sum_{v=1}^{k} \frac{v\cos(n+v)t}{(n+v)\log(n+v+1)}\right| \\ &\le \frac{1}{k(k+1)} \cdot 1 \cdot \sum_{v=1}^{k} \frac{\cos(n+v)t}{\log(n+v+1)}, \left(\frac{v}{n+v} < 1\right) \\ &= \frac{1}{k(k+1)} \sum_{p=n+1}^{n+k} \frac{\cos pt}{\log(p+1)} \qquad (\text{since } p = n+v) \\ &= \frac{1}{k(k+1)} \left[\sum_{p=n+1}^{n+k-1} \left\{\sum_{r=1}^{p} \cos rt\right\} \cdot \Delta_p \left(\frac{1}{\log(p+1)}\right) \right. \\ &\qquad \left. + \left(\sum_{r=n+1}^{n+k} \cos rt\right) \frac{1}{\log(n+k+1)} \right] \end{aligned}$$

$$= \frac{1}{k(k+1)}\left[\sum_{p=n+1}^{n+k-1} O(\tau)\,\Delta_p\left(\frac{1}{\log(p+1)}\right) + O(\tau)\,\frac{1}{\log(n+k+1)}\right]$$

$$= \frac{O(\tau)}{k(k+1)}\left[\sum\nolimits_1 + \sum\nolimits_2\right]. \qquad \ldots(4.6)$$

Now,

$$\sum\nolimits_1 = \sum_{p=n+1}^{n+k-1} \Delta_p\left(\frac{1}{\log(p+1)}\right).$$

We have

$$\Delta_p\left(\frac{1}{\log(p+1)}\right) = \frac{1}{\log(p+1)} - \frac{1}{\log(p+2)}$$

$$= \frac{\log(p+2) - \log(p+1)}{\log(p+2)\,\log(p+1)}$$

$$= \frac{1}{\log(p+2)\,\log(p+1)} \cdot \log\left(\frac{p+2}{p+1}\right)$$

$$= \frac{1}{\log(p+2)\log(p+1)} \cdot \left(\log\left(1 + \frac{1}{p+1}\right)\right)$$

$$= \frac{1}{\log(p+2)\,\log(p+1)} \cdot \left\{\frac{1}{p+1} - \frac{1}{2(p+1)^2} + \ldots\right\}$$

$$\leq O\left(\frac{1}{(p+1)\cdot\log(p+1)\,\log(p+2)}\right)$$

$$\leq O\left(\frac{1}{(p+1)\,\log^2(p+1)}\right)$$

Hence

$$\sum\nolimits_1 < \sum_{p=n+1}^{m+k-1}\left(\frac{1}{(p+1)\cdot\log^2(p+1)}\right)$$

$$\leq \sum_{m=2}^{n+k} \frac{1}{m\log^2 m} \quad (m = p+1).$$

using Euler’s summation formula [1]

$$\sum_{m=2}^{n+k}\frac{1}{m\log^2 m} = \int_2^{n+k} \frac{1}{t(\log t\ )^2}\,dt + \int_2^{n+k}\left(t-[t]\left(-2\left(\log t^{-3}\right)\frac{1}{t^2} - (\log t)^{-2}\right)\frac{1}{t^2}\right)dt$$

$$+ \frac{1}{(n+k)\log\,(n+k)}\cdot 0 - \frac{1}{2\log\,2}\cdot 0$$

$$= \int_2^{n+k} \frac{1}{t(\log t)^2}\, dt + \int_2^{n+k} \frac{t-[t]}{t^2}\left(-\frac{1}{(\log t)^2} - 2\frac{1}{(\log t)^3}\right) dt$$

$$= I_1 + I_2$$

$$I_1 = \int_2^{n+k} \frac{1}{t(\log t)^2}\, dt = \left[\frac{1}{-\log t}\right]_2^{n+k} = -\left[\frac{1}{\log(n+k)}\ \frac{1}{\log 2}\right]$$

$$|I_1| \le \left|\frac{1}{\log(n+k)}\right|$$

$$|I_2| = \left|\int_2^{n+k} \frac{t-[t]}{t^2}\left(-\frac{1}{(\log t)^2} - \frac{2}{(\log t)^3}\right) dt\right|$$

$$\le \left|\int_2^{n+k} -\frac{1}{t^2}\ \frac{1}{(\log t)^2}\, dt - \int_2^{n+k} \frac{1}{t^2}\cdot\frac{2}{(\log t)^3}\, dt\right|$$

$$I_{21} = \int \frac{1}{t^2}\ \frac{1}{(\log t)^2}\, dt,$$

$$= \int e^{-u}\ \frac{1}{u^2}\, du = \int e^{-u}\ u^{-2}\, du$$

$$= u^{-2}\cdot e^{-u} - \int -2\cdot u^{-3}\ e^{-u}\cdot(-1)\, du$$

$$= -u^{-2}\ e^{-u} - \int 2u^{-3}\ e^{-u}\, du$$

$$= -\frac{1}{(\log t)^2}\ \frac{1}{t} - \int 2\frac{1}{(\log t)^3}\cdot\frac{1}{t}\cdot\frac{1}{t}\, dt$$

$$|I_2| \le \left|\left[-\frac{1}{t(\log t)^2}\right]_2^{n+k} + \int_2^{n+k} \frac{2}{t^2(\log t)^3}\, dt - \int_2^{n+k} \frac{2}{t^2(\log t)^3}\, dt\right|$$

$$\le \frac{1}{(n+k)\,(\log(n+k))^2}$$

$$\Sigma_1 \le \left(\frac{1}{\log(n+k)} + \frac{1}{(n+k)\,(\log(n+k))^2}\right)$$

$$\Sigma_2 = \frac{1}{\log(n+k+1)}$$

$$\le \frac{1}{\log(n+k)}.$$

From (4.6)

$$|g'(k,t)| = O\left(\frac{\tau}{k(k+1)}\left(\frac{1}{\log(n+k)} + \frac{1}{(n+k)(\log(n+k))^2}\right)\right)$$

Again,

$$\frac{d}{dt}\,g'(k,t)=\frac{d}{dt}\left\{\frac{2}{\pi}\,\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{(n+v)\,\log(n+v+1)}\cdot\cos(n+v)\,t\right\}$$

$$=-\frac{2}{\pi}\,\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{\log(n+v+1)}\,\sin\,(n+v)\,t\,.$$

Now for all *t*,

$$\left|\frac{d}{dt}\,g'(k,t)\right|\le\left|\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{\log}\,\frac{\sin(n+v)t}{\log(n+v+1)}\right|$$

$$=\left|\frac{1}{k(k+1)}\cdot\frac{1}{\log(n+2)}\sum_{v=1}^{k}v\right|$$

$$=\left|\frac{1}{k(k+1)}\,\frac{1}{\log(n+2)}\cdot\frac{k(k+1)}{2}\right|$$

$$=O\left(\frac{1}{\log(n+2)}\right).$$

For $t\ge\frac{1}{k}$,

$$\left|\frac{d}{dt}\,g'(k,t)\right|=\left|\frac{2}{\pi}\,\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{\log(n+v+1)}\,\sin(n+v)\,t\right|$$

$$\le\frac{1\cdot k}{k(k+1)}\sum_{v=1}^{k}\frac{\sin(n+v)t}{\log(n+v+1)}$$

$$=O\left(\frac{1}{(k+1)}\left\{\frac{1}{\log(n+k)}+\frac{1}{(n+k)\log^2(n+k)}\right\}\right)$$

Lemma-3

Let

$$J'(k,u)=\frac{1}{\Gamma(1-\alpha)}\int_a^{\pi}\frac{d}{dt}\,g(k,t)\,(t-u)^{-\alpha}\,dt\,.$$

Then

$$\text{(4.7)}\qquad |J'(k,u)|=\begin{cases}O\left(\dfrac{k^{\alpha-1}}{\log(n+2)}\right) & \textit{for all } u\\[2ex] O\left(Uk^{\alpha-2}\cdot\dfrac{1}{\log(n+k)}\right) & \textit{for } u\ge\dfrac{1}{k}\end{cases}$$

Proof. We have, for all *u*

$$|J'(k,u)|=\left|\left(\int_u^{u+\frac{1}{k}}+\int_{u+\frac{1}{k}}^{\pi}\right)\frac{d}{dt}\,g'(k,t)(t-a)^{\alpha}\,dt\right|$$

$$\leq |J_1'| + |J_2'| \text{, say}$$

$$|J_1'| = \left| \int_u^{u+\frac{1}{k}} \frac{d}{dt}\, g'(k,t)\, (t-u)^{-\alpha}\; dt \right|$$

$$\leq \frac{1}{\log(n+2)} \left| \int_u^{u+\frac{1}{k}} (t-u)^{-\alpha}\; dt \right| \text{ using (4.4)}$$

$$= O\left(\frac{1}{\log(n+2)} \cdot k^{\alpha-1} \right).$$

$$|J_2'| = \left| \int_{u+\frac{1}{k}}^{\pi} \frac{d}{dt}\, g'(k,t)\, (t-u)^{-\alpha}\; dt \right|$$

$$= O(k^{\alpha}) \int_{u+\frac{1}{k}}^{\pi} \frac{d}{dt}\, g'(k,t)\; dt$$

$$= O(k^{\alpha}) \cdot O\left(\frac{1}{(k+1)\, \log(n+2)} \right) \text{ using (4.2)}$$

$$= O\left(\frac{k^{\alpha-1}}{\log(n+2)} \right).$$

For $u \geq \dfrac{1}{k}$,

$$|J'(k,u)| = \left| \left(\int_u^{u+\frac{1}{k}} + \int_{u+\frac{1}{k}}^{\pi} \right) \frac{d}{dt}\, g'(k,t)\, (t-u)^{-\alpha}\; dt \right|$$

$$= I_1 + I_2$$

$$|I_1| = \left| \int_u^{u+\frac{1}{k}} \frac{d}{dt}\, g'(k,t)\, (t-u)^{-\alpha}\; dt \right|$$

$$\leq \left(\frac{U}{(k+1)\log(n+k)} + \frac{U}{(k+1)(n+k)\, \log^2(n+k)} \right) \int_a^{u+\frac{1}{k}} (t-u)^{-\alpha} dt$$

$$= O\left(\frac{U}{(k+1)\log(n+k)} + \frac{U}{(k+1)(n+k)\log^2(n+k)} \right) \cdot k^{\alpha-1}$$

$$= O\left(U \cdot k^{\alpha-2} \left(\frac{1}{\log(n+k)} + \frac{1}{(n+k)\, \log^2(n+k)} \right) \right)$$

$$= O\left(U \cdot k^{\alpha-2} \cdot \frac{1}{\log(n+k)}\right).$$

$$|I_2| = \left| \int_{u+\frac{1}{k}}^{\pi} \frac{d}{dt} g'(k,t)\,(t-u)^{-\alpha}\,dt \right|$$

$$= O\left(k^{\alpha}\right) \cdot \left| \int_{u+\frac{1}{k}}^{\pi} \frac{d}{dt} g'(k,t) \right|$$

$$= O\left(k^{\alpha}\right)\left(\frac{U}{k(k+1)} \cdot \frac{1}{\log(n+k)} + \frac{U}{k\cdot(k+1)}\,\frac{1}{(n+k)\,\log^2(n+k)}\right)$$

using (4.3)

$$= O\left(Uk^{\alpha-2} \cdot \frac{1}{\log(n+k)} + U \cdot k^{\alpha-2} \cdot \frac{1}{(n+k)\,\log^2(n+k)}\right)$$

$$= O\left(U \cdot k^{\alpha-2}\,\frac{1}{\log(n+k)}\right).$$

5 Proof of the Theorem

By Lemma-1,

$$t_k(n) - t_{k+1}(n) = \frac{-1}{k(k+1)} \sum_{v=1}^{k} \frac{v\,B_{n+v}(t)}{\log(n+v+1)}$$

Using

$$v\,B_{n+v}(t) = \frac{2}{\pi}\int_0^{\pi} v\,\psi(t)\sin\,(n+v)\,t\,dt$$

$$t_k(n) - t_{k+1}(n) = -\frac{1}{k(k+1)} \sum_{v=1}^{k} \frac{2}{\pi}\int_0^{\pi} \frac{v\psi(t)\sin\,(n+v)\,t}{\log(n+v+1)}\,dt$$

$$= \frac{-2}{\pi}\int_0^{\pi}\psi(t)\left[\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{\log(n+v+1)} \cdot \sin(n+v)\,t\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\psi(t)\left[\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{\log(n+v+1)(n+v)} \cdot \frac{d}{dt}\cos(n+v)t\right]dt$$

$$= \int_0^{\pi}\psi(t)\,\frac{d}{dt}\left[\frac{2}{\pi}\,\frac{1}{k(k+1)}\sum_{v=1}^{k}\frac{v}{(n+v)\log(n+v+1)} \cdot \cos(n+v)t\right]dt$$

$$= \int_0^{\pi}\psi(t)\,\frac{d}{dt}\,g'(k,t)$$

$$= \int_0^{\pi} \frac{d}{dt}\, g^{'}(k,t) \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^{t} (t-u)^{-\alpha}\ d\Psi_{\alpha}(u) \right\} dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^{\pi} d\Psi_{\alpha}(u) \int_a^{\pi} \frac{d}{dt}\, g'(k,t)\ (t-u)^{-\alpha}\ dt$$

$$= \int_0^{\pi} d\Psi_{\alpha}(u) \left\{ \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi} \frac{d}{dt}\, g'(k,t)\ (t-a)^{-\alpha}\ dt \right\}$$

$$= \int_0^{\pi} d\Psi_{\alpha}(u)\ J'(k,u)$$

$$= \int_0^{\pi} \frac{d\Psi_{\alpha}}{u^{\alpha}\ \log(n+U)} \cdot u^{\alpha}\ \log(n+U) \cdot J'(k,u)$$

$$= \int_0^{\pi} \frac{d\Psi_{\alpha}(u)}{u^{\alpha} \log(n+U)} \cdot W(k,u)$$

where

$$W(k,u) = u^{\alpha}\ \log(n+U)\ J^{'}(k,u).$$

Now,

$$\sum_{k=1}^{\infty} \left| t_k(n) - t_{k+1}(n) \right| = \int_0^{\pi} \frac{d\Psi_{\alpha}(u)}{u^{\alpha} \log(n+U)} \sum_{k=1}^{a} \left| W(k,u) \right|$$

Since

$$\int_0^{\pi} \frac{d\Psi_{\alpha}(u)}{u^{\alpha} \log(n+U)} < \infty$$

then it remains to prove that

$$\sum_{k=1}^{\infty} \left| W(k,u) \right| < \infty,$$

uniformly for all $u, 0 < u < \pi$.

$$\sum_{k=1}^{\infty} \left| W(k,u) \right| = \sum_{k \le \frac{1}{u}} \left| W(k,u) \right| + \sum_{k > \frac{1}{u}} \left| W(k,u) \right|$$

$$= \Sigma_1 + \Sigma_2 .$$

$$\Sigma_1 = \sum_{k \le \frac{1}{u}} \left| W(k,u) \right|$$

$$= \sum_{k \le \frac{1}{u}} O\left(u^{\alpha}\ \log(n+U)\ \frac{k^{\alpha}}{\log(n+2)} \right)$$

$$= O\left(\frac{u^{\alpha}}{\log(n+2)} \right) \sum_{k \le \frac{1}{u}} \log(n+U)\ k^{\alpha-1}$$

$$= O\left(\frac{u^{\alpha}}{\log(n+2)}\right) \cdot O\left(u^{\alpha}\log(n+2)\right) \text{ using (3.1)}$$
$$= O(1)$$
$$\Sigma_2 = \sum_{k>\frac{1}{u}} \left|W(k,u)\right|$$
$$= \sum_{k>\frac{1}{u}} \left|u^{\alpha}\ \log(n+U)\cdot J(k,u)\right|$$
$$= \sum_{k>\frac{1}{u}} u^{\alpha}\ \log(n+U)\ O\left(Uk^{\alpha-2}\ \frac{1}{\log(n+k)}\right)$$
$$= u^{\alpha-1}\ \frac{\log(n+U)}{\log(n+U)} \cdot O\left(\sum_{k>\frac{1}{u}}\left(k^{\alpha-2}\right)\right)$$
$$= O\left(u^{\alpha-1}\ U^{\alpha-1}\right)$$
$$= O(1).$$

This completes the proof of the theorem.

References

[1] M.T. Apostol, *Introduction to Analytic Number Theory*, Springer International Student Edition.

[2] S. Banach, Theorie des operations lineaires, *Monoyrafie Mathematyezne*, Warsaw, 1(1932).

[3] L.S. Bosanquet and J.M. Hyslop, On the absolute summability of the allied series of a Fourier series, *Mathematische zeitschrift*, 42(1937), 489-512.

[4] G.G. Lorentz, A contribution to the theory of divergent sequences, *Acta. Math*., 80(1948), 167-190.

[5] U.K. Misra and N.C. Sahoo, Absolute Banach Summability of the conjugate series of a Fourier series, *Journal of Indian Academy of Mathematics*, 24(2) (2002), 307-318.

[6] S.K. Paikray, U.K. Misra and N.C. Sahoo, Absolute banach summability of a factored Fourier Series, *IJRRAS*, 7(3) (2011).

[7] N.N. Swamy, Absolute summability of the conjugate series of a Fourier series, *Ph.D. Thesis*, Utkal University, (1980), 95-105.