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Classification of Kerr-Newman's Geodesics

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Abstract

An application of differential geometry together with Lie group method in theoretical physics is considered. In this paper we study the Kerr-Newman geometry as a geometric structure of a rotating charged black hole to classify all geodesic curves by using the set of Lie point symmetries of the associated system of differential equations.

Keywords: *Kerr-Newman geometry, symmetry groups, Riemannian geometry.*

1 Introduction

It is well known fact that gravitational mass can alter the space time structure. Charged rotation body can also alter the space-time similar to gravitational mass. We know that, the solution of Einsteins equations describing the exterior of an isolated, spherically symmetric objectis quite simple. Indeed, it has been found immediately after the derivation of Einsteins equation. In the case of a rotating body, instead, the problem is much more difficult: we do not know any analytic, exact solution describing the exterior of a rotating star. But we know the exact solution describing a rotating, stationary, axially symmetric black hole. It is the Kerr solution, derived in 1963 by R. Kerr [8].

Suppose we have a spherical symmetric mass with a given charge and rotation.The Kerr-Newman metric describes the geometry of spacetime in the vicinity of a rotating mass M with charge Q. The formula for this metric depends upon what coordinates or coordinate conditions are selected. In the mathematical description of general relativity, the Boyer-Lindquist coordinates are a generalization of the coordinates used for the metric of a Schwarzschild black hole that can be used to express the metric of a Kerr black hole. The Boyer-Lindquist coordinates written in the coordinate (t, r, θ, ϕ) in Boyer-Linquist form [2], the Kerr-Newman metric [3, 6, 8], is a Riemannian metric of the form

$$g = -\frac{\Delta}{\rho^2}[dt - a\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - adt] + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (1)$$

where $\Delta \equiv r^2 - 2Mr + a^2 + Q^2, \rho^2 \equiv r^2 + a^2\cos^2\theta$ and $a \equiv S/M$ is angular momentum per unit mass [8]. In these equations M and Q are mass and charge respectively. As we see the Kerr-Newman geometry has a horizon, and therefore describes a black hole, if and only if $M^2 \geq Q^2 + a^2$.

The present paper is organized as follows: The second chapter introduces a very impoprtant concept called *symmetries of differential equations* which is described in so many literatures [9, 10]. The third chapter involves the equations of Kerr-Newman's geodesics constructed from the Riemannian metric (1). The last chapter is devoted to classify the geodesic curves obtained in the third chapter by using the symmetries of the geodesic's system

2 Symmetries of Differential Equations

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [5] are solved with the aid of this method. There are still many authors using this method to find the exact solutions [9] of non-linear differential equations. It is also a powerful tool for finding exact solutions of non-linear problems [9, 10]. Many eaxmples of applications to physical problems have been demonstrated in a huge number of papers and a lot of excellent books. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [1, 5, 9, 10]) and in numerous papers in the literature (e.g. [4]).

Consider a system of differential equations (PDE or ODE) in the dependent variables $u^\alpha (1 \leq \alpha \leq m)$ and dependent variables $x^i (1 \leq i \leq n)$ of the form:

$$\Delta^s(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, ...) = 0, \quad 1 \leq s \leq k, \quad (2)$$

where the subscripts denote partial derivatives (e.g. $u^\alpha_i = \partial u^\alpha / \partial x^i$). To determine continuous symmetries of (2), it is useful to consider infinitesimal Lie transformations of the form:

$$\tilde{x}^i = x^i + \varepsilon\xi^i + O(\varepsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \varepsilon\eta^\alpha + O(\varepsilon^2), \quad (3)$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\eta^\alpha = \eta^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α. Generalized Lie symmetries are obtained in the case when the transformations (3) also depend on the derivatives or integrals of the u^α.

The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formulae:

$$\tilde{u}_i^\alpha = u_i^\alpha + \varepsilon\zeta_i^\alpha, \quad \tilde{u}_{ij}^\alpha = u_{ij}^\alpha + \varepsilon\zeta_{ij}^\alpha, \tag{4}$$

where

$$\zeta_i^\alpha = D_i\hat{\eta}^\alpha + \xi^s u_{si}^\alpha, \quad \zeta_{ij}^\alpha = D_i D_j\hat{\eta}^\alpha + \xi^s u_{sij}^\alpha. \tag{5}$$

Here

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^s u_s^\alpha, \tag{6}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon\hat{\eta}^\alpha$. The symbol D_i in (5) denotes the total derivative operator with respect to x^i. Similar formulae to (5) apply for the transformation of the higher order derivatives.

The condition for invariance of the system of differential equations (2) to $O(\varepsilon^2)$ under the Lie transformation (3) can be expressed in the form:

$$\mathcal{L}_{\mathbf{v}}\Delta^s \equiv \tilde{\mathbf{v}}(\Delta^s) = 0 \quad \text{whenever} \quad \Delta^s = 0, \quad 1 \le s \le k, \tag{7}$$

where

$$\tilde{\mathbf{v}} = \mathbf{v} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{8}$$

is the prolongation of the vector field

$$\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{9}$$

associated with the infinitesimal transformation (3). The symbol $\mathcal{L}_{\mathbf{v}}\Delta^s$ in (7) denotes the Lie derivative of Δ^s with respect to the vector field $\mathbf{v}$ (i.e. $\mathcal{L}_{\mathbf{v}}\Delta^s = \frac{\mathrm{d}\Delta^s}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$). The system of linear PDEs (7) is called *determinig equations* for the system (2).

3 Kerr-Newman's Geodesic

Let M be a Riemannian manifold with Riemannian metric g. Suppose γ is a smooth curve with domain I defined on the open subset $U \subset M$. γ is a geodesic curve [7] if and only if its component functions $\gamma(s) = (x^1(s), \cdots, x^n(s)$ satisfies the geodesic equation:

$$\ddot{x}^k(s) + \dot{x}^i(s)\dot{x}^j(s)\Gamma^k_{ij}(x(s)) = 0, \qquad i, j, k = 1, ..., n, \tag{10}$$

where Γ_{ij} is the Christofel symbol of the second kind [7], and $x = (x^1, ..., x^n)$ is a coordinate chart on U. The geodesics of the metric (1) could be found from the system (10) as the following complicated system:

$$\begin{aligned}
\mathbf{R}_1 \;=\; & \ddot{r}(s) - \frac{\Delta\, r(2\,\alpha^2+1)\sin^2\theta(s)\dot{\phi}(s^2)}{\rho^4} = 0,\\
\mathbf{R}_2 \;=\; & \ddot{\phi}(s) - 4[\Delta\,(-\alpha^2-\rho^2+\Delta)\alpha\, c\cos\theta(s)\rho^4\dot{t}(s)\dot{\theta}(s)]\\
& [-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8 + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2\\
& - \cos^2\theta(s)\alpha^6 \; - 2\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10} - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s) - \Delta^2\cos^2\theta(s)\alpha^2\\
& - \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10} - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10}\\
& + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2(\theta(s))\alpha^4 r^2 + 2\,\Delta\,\cos^2(\theta(s))\alpha^4).\sin\theta(s)]^{-1}\\
& - 2[r(2\,\alpha^2+1)(\alpha^2\cos^2\theta(s) - \alpha^2 + \Delta\dot{r}(s)\dot{\phi}(s)]\\
& [-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8 + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2.\\
& - \cos^2\theta(s)\alpha^6 - 2\,\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10} - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s) - \Delta^2\cos^2\theta(s)\alpha^2\\
& - \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10} - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10}\\
& + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2 + 2\,\Delta\,\cos^2\theta(s)\alpha^4]^{-1}\\
& + 2[\cos\theta(s)(-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8.\\
& + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + 2\,\Delta^2\alpha^2 - \cos^2\theta(s)\alpha^6 - 3\,\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10}\\
& - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - 2\,\Delta\,\alpha^4\cos^4\theta(s) - 2\,\Delta^2\cos^2\theta(s)\alpha^2 - \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10}\\
& - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10}\\
& + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2 + 4\,\Delta\,\cos^2\theta(s)\alpha^4\dot{\theta(s)}(s)\dot{\phi}(s)]\\
& [(-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8 + \alpha^6.\\
& + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2 - \cos^2\theta(s)\alpha^6 - 2\,\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10} - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s)\\
& - \Delta^2\cos^2\theta(s)\alpha^2 - \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10} - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2\\
& + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10}.\\
& + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2 + 2\,\Delta\,\cos^2\theta(s)\alpha^4)\sin\theta(s)]^{-1} = 0,\\
\mathbf{R}_3 \;=\; & \ddot{t}(s) - 2[\alpha^2\sin\theta(s)\cos\theta(s)(4\,\alpha^4\rho^8 + 8\,\alpha^2\rho^{10} + 4\,\rho^{12} - 8\,\Delta\,\alpha^2\rho^8 - 8\,\Delta\,\rho^{10}\\
& + 4\,\Delta^2\rho^8 - \Delta\,\cos^2\theta(s)\alpha^2 - \alpha^4 - 2\,r^2\alpha^2 + \Delta\,\alpha^2 - r^2)\dot{t}(s)\dot{\theta(s)}(s)][-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8\\
& - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8 + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2 - \cos^2\theta(s)\alpha^6 - 2\,\Delta\,\alpha^4\\
& - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10} - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s) - \Delta^2\cos^2\theta(s)\alpha^2 - \cos^2\theta(s)\alpha^2 r^2\\
& + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10} - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10}\\
& + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2 + 2\,\Delta\,\cos^2\theta(s)\alpha^4]^{-1}\\
& - 4[(-\alpha^2 - \rho^2 + \Delta)\alpha\,\rho^4 r(2\,\alpha^2+1)\sin^2\theta(s)\dot{r}(s)\dot{\phi}(s)]\\
& [c(-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10} + 4\,\Delta^2\cos^2(\theta(s))\alpha^2\rho^8 + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2]\\
& - \cos^2\theta(s)\alpha^6 - 2\,\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10} - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s) - \Delta^2\cos^2\theta(s)\alpha^2 -\\
& \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10} - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8\\
& + 8\,\cos^2\theta(s)\alpha^4\rho^{10} + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2 + 2\,\Delta\,\cos^2\theta(s)\alpha^4)]^{-1} + 4[\sin^3\theta(s)\\
& \cos\theta(s)\Delta\,\alpha^3(-\alpha^2 - \rho^2 + \Delta)\rho^4\dot{\theta}(s)\dot{\phi}(s)][c(-8\,\Delta\,\cos^2\theta(s)\alpha^4\rho^8 - 8\,\Delta\,\cos^2\theta(s)\alpha^2\rho^{10}\\
& + 4\,\Delta^2\cos^2\theta(s)\alpha^2\rho^8 + \alpha^6 + r^2\alpha^2 - \Delta\, r^2 + \Delta^2\alpha^2 - \cos^2\theta(s)\alpha^6 - 2\,\Delta\,\alpha^4 - 4\,\alpha^6\rho^8 - 8\,\alpha^4\rho^{10}\\
& - 4\,\alpha^2\rho^{12} + 2\,\alpha^4 r^2 - \Delta\,\alpha^4\cos^4\theta(s) - \Delta^2\cos^2\theta(s)\alpha^2 - \cos^2\theta(s)\alpha^2 r^2 + 8\,\Delta\,\alpha^4\rho^8 + 8\,\Delta\,\alpha^2\rho^{10}\\
& - 4\,\Delta^2\alpha^2\rho^8 - 2\,\Delta\,\alpha^2 r^2 + 4\,\cos^2\theta(s)\alpha^6\rho^8 + 8\,\cos^2\theta(s)\alpha^4\rho^{10} + 4\,\cos^2\theta(s)\alpha^2\rho^{12} - 2\,\cos^2\theta(s)\alpha^4 r^2\\
& + 2\,\Delta\,\cos^2\theta(s)\alpha^4)]^{-1} = 0,
\end{aligned}$$

$$\mathbf{R}_4 = \ddot{\theta}(s) - \frac{c^2\alpha^2\sin\theta(s)\cos\theta(s)\dot{t}(s)^2}{\rho^4} - 4\,c(-\alpha^2-\rho^2+\Delta)\alpha\,\sin\theta(s)\cos\theta(s)\dot{t}(s)\dot{\phi}(s)$$
$$-[\sin\theta(s)\cos\theta(2\,\Delta\,\cos^2\theta\alpha^2+\alpha^4+2\,r^2\alpha^2-2\,\Delta\,\alpha^2+r^2\dot{\phi}(s)^2][\rho^4]^{-1}=0$$

The infinitesimal Lie transformations for the system above are of the form:

$$\begin{aligned}&\tilde{s}=s+\varepsilon\xi^s, \quad \tilde{t}=t+\varepsilon\eta^t, \quad \tilde{r}=r+\varepsilon\eta^r, \quad \tilde{\theta}=\theta+\varepsilon\eta^\theta,\\ &\tilde{\phi}=\phi+\varepsilon\eta^\phi.\end{aligned} \tag{11}$$

The corresponding canonical symmetry generators $\hat{\eta}^t, \hat{\eta}^r, \hat{\eta}^\theta$ and $\hat{\eta}^\varphi$ are given by the formulae analogous to (4). Thus

$$\hat{\eta}^\alpha = \eta^\eta - \xi^s\eta_s, \tag{12}$$

relates the canonical symmetry generator $\hat{\eta}^\alpha$ to η^α, where α can be any of the dependent variables t, r, θ and ϕ.

If

$$\begin{aligned}\mathbf{v} = \xi(s,t,r,\theta,\phi)\frac{\partial}{\partial s} + \eta_t(s,t,r,\theta,\phi)\frac{\partial}{\partial t} + \eta_r(s,t,r,\theta,\phi)\frac{\partial}{\partial r}\\ +\eta_\theta(s,t,r,\theta,\phi)\frac{\partial}{\partial\theta} + \eta(s,t,r,\theta,\phi)\frac{\partial}{\partial\phi},\end{aligned} \tag{13}$$

be the general form of a symmetry for geodesics, using a software such as `Maple` we derive the six-dimensional Lie algebra of the symmetries spanned by the following vector fields:

$$\mathbf{v}_1 = \frac{\partial}{\partial s}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = \frac{\partial}{\partial\phi}, \quad \mathbf{v}_4 = s\frac{\partial}{\partial s},$$
$$\mathbf{v}_5 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\phi}, \quad \mathbf{v}_6 = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}.$$

The commutator table of the Lie algebra $\mathcal{G}$ spanned by the vector fields $\mathbf{v}_i$'s are given in table (1).

4 Classification of Geodesics

A straight forward calculation shows that if $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a geodesic curve of metric (1), then so are:

$$\tilde{\gamma}(s) = \gamma(t(s+\varepsilon), r(s+\varepsilon), \theta(s+\varepsilon), \phi(s+\varepsilon)), \tag{14}$$
$$\tilde{\gamma}(s) = \gamma(t(s)+\varepsilon, r(s), \theta(s), \phi(s)), \tag{15}$$
$$\tilde{\gamma}(s) = \gamma(t(s), r(s), \theta(s), \phi(s)+\varepsilon), \tag{16}$$
$$\tilde{\gamma}(s) = \gamma(t(e^\varepsilon), r(e^\varepsilon), \theta(e^\varepsilon), \phi(e^\varepsilon)), \tag{17}$$
$$\tilde{\gamma}(s) = \gamma(t(s), r(s), \varepsilon\sin\phi(s), \arcsin(\varepsilon\theta(s)) + \varepsilon\sqrt{1-\varepsilon^2\theta^2(s)}\cot\theta(s)), \tag{18}$$
$$\begin{aligned}\tilde{\gamma}(s) = {}&\gamma(t(s), r(s), -\cos\phi(s),\\ &\arccos(-\varepsilon\theta(s)) + \varepsilon\sqrt{1-\varepsilon^2\phi^2(s)}\cot\theta(s)),\end{aligned} \tag{19}$$

Table 1: Commutators Table of $\mathcal{G}$

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	0	0	0	0
$\mathbf{v}_2$	0	0	0	0	0	0
$\mathbf{v}_3$	$-\mathbf{v}_1$	0	0	0	$-\mathbf{v}_6$	$\mathbf{v}_5$
$\mathbf{v}_4$	0	0	0	0	0	0
$\mathbf{v}_5$	0	0	$\mathbf{v}_6$	0	0	$-\mathbf{v}_3$
$\mathbf{v}_6$	0	0	$-\mathbf{v}_5$	0	$\mathbf{v}_3$	0

The equations (14)–(19) are the flows of the basis vector fileds $\{\mathbf{v}_1, ..., \mathbf{v}_6\}$. The equation (14) and (17) demonstrate the time, radius, colatitude and longitude angle invariance of the system, (18) and (19) show genuinely local group of transformations.

We know that evaluation of the flow of vector fields in $\mathcal{G}$ serves to define the exponential map $\exp : \mathcal{G} \to G$. Since $\exp(0) = e$, $d\exp(0) = I_d$, the exponential map defines a local diffeomorphism in a neighborhood of $0 \in \mathcal{G}$. Consequently, all Lie groups having the same Lie algebra looks locally the same in a neighborhood of the identity; only the global topological properties are different. Globally, the exponential map is not necessarily one-to-one nor onto. However, if a Lie group is connected, it can be completely recovered by successive exponentiations.

The most general one parameter group of symmetries is obtained by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_6\mathbf{v}_6$ of the given vector fields; the explicit formulae for the group transformations are very complicated. In particular if g is near the identity, it can be represented uniquely in the form

$$g = \exp(\varepsilon_6\mathbf{v}_6) \circ \cdots \circ \exp(\varepsilon_1\mathbf{v}_1). \tag{20}$$

For instance if $\varepsilon_1 = \varepsilon_4 = 0$, then the most general Lie group action with respect to (1) is

$$g = \Bigg(t + \varepsilon_2, r, \varepsilon_5 - cos\Big(\arcsin(\varepsilon_5\theta) + \varepsilon_5\sqrt{1 - \varepsilon_5^2\theta^2}\cot\theta\Big), \sqrt{1 - \varepsilon_5^2\varepsilon_6^2\sin^2\phi} + \varepsilon_6\sqrt{1 - \varepsilon_6\Big[\arcsin(\varepsilon_5\theta) + \varepsilon_5\sqrt{1 - \varepsilon_5\theta^2}\cot\theta\Big]^2}\cot(\varepsilon_5\sin\phi) + \varepsilon_3\Bigg).$$

5 Conclusion

In this paper we presented an application of differential geometry in theoretical physics. First we introduce the Kerr-Newman geometry as a kind of

Riemannian geometry. Then, we calculated the geodesics of the black hole as the geodesics of a geometric structure defind on the Kerr-Newman black hole. Finally, we classified the geodesics of the Kerr-Newman black hole by using the flow of the symmetries of the geodesics. It is noteworthy that some packages such as `DifferentialGeometry` and `PDEtools` of `Maple` is used to find the symmetries and geodesic equations.

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