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(i,j)-ξ-Open Sets in Bitopological Spaces

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Abstract

The aim of this paper is to introduce a new type of sets in bitopological spaces which is conditional ξ-open set in bitopological spaces called (i,j)-ξ-open set and we study its basic properties, and also we introduce some characterizations of this set.

Keywords: *ξ-open, (i,j)-ξ-open, semi-open, regular-closed*
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1 Introduction

In 1963 Kelley J. C. [7] was first introduced the concept of bitopological spaces, where X is a nonempty set and τ_1, τ_2 are topologies on X. In 1963 Levine [8] introduced the concept of semi-open sets in topological spaces. By using this concept, several authers defined and studied stronger or weaker types of topological concept.

In this paper, we introduce the concept of a conditional ξ-open set in a bitopological space, and we study their basic properties and relationships with other concepts of sets. Throughout this paper, (X,τ_1,τ_2) is a bitopological space, and if $A \subseteq Y \subseteq X$, then i-$Int(A)$ and i-$Cl(A)$ denote respectively the interior and closure of A with respect to the topology τ_i on X and i-$Int_Y(A)$, i-$Cl_Y(A)$ denote respectively the interior and the closure of A with respect to the induced toplogy on Y.

2 Preliminaries

We shall give the following definitions and results.

Definition 2.1 *A subset A of a space (X, τ) is called:*

1. *preopen [9], if $A \subseteq Int(Cl(A))$*
2. *semi-open [8], if $A \subseteq Cl(Int(A))$*
3. *α-open [11], if $A \subseteq Int(Cl(Int(A)))$*
4. *regular open [5], if $A = Int(Cl(A))$*
5. *regular semi-open [1], if $A = sInt(sCl(A))$*

The complement of a preopen (resp., semi-open, α-open, regular open, regular semi-open) set is said to be preclosed (resp., semi-closed, α-closed, regular closed, regular semi-closed). The intersection of all preclosed (resp., semi-closed, α-closed) sets of X containing A is called preclosure (resp., semi-closure, α-closure) of A. The union of all preopen (resp., semi-open, α-open) sets of X contained in A called preinterior (resp., semi-interior, α-interior) of A.

A subset A of a space X is called δ-open [15], if for each $x \in A$, there exists an open set G such that $x \in G \subseteq Int(Cl(G)) \subseteq A$. A subset A of a space X is called θ-semi-open [6] (resp., semi-θ-open [2]) if for each $x \in A$, there exists a semi-open set G such that $x \in G \subseteq Cl(G) \subseteq A$ (resp., $x \in G \subseteq sCl(G) \subseteq A$. A subset A of a topological space (X, τ) is called η-open [13], if A is a union of δ-closed sets. The complement of η-open sets is called η-closed.

Definition 2.2 *A topological space X is called,*

1. *Extermally disconnected [2], if $Cl(U) \in \tau$ for every $U \in \tau$.*
2. *Locally indiscrete [4], if every open subset of X is closed.*

From the above definition we obtain:

Remark 2.3 *If X is locally indiscrete space, then every semi-open subset of X is closed and hence every semi-closed subset of X is open.*

Theorem 2.4 *[9] A space X is semi-T_1 if and only if for any point $x \in X$ the singleton set $\{x\}$ is semi-closed.*

Theorem 2.5 *[10] For any space (X, τ) and (Y, τ) if $A \subseteq X$, $B \subseteq Y$ then:*

1. $pInt_{X\times Y}(A \times B) = pInt_X(A) \times pInt_Y(B)$
2. $sCl_{X\times Y}(A \times B) = sCl_X(A) \times sCl_Y(B)$

Theorem 2.6 *[10] For any topological space the following statements are true:*

1. *Let (Y, τ_Y) be a subspace of a space (X, τ), if $F \in SC(X)$ and $F \subseteq Y$ then $F \in SC(Y)$.*
2. *Let (Y, τ_Y) be a subspace of a space (X, τ), if $F \in SC(Y)$ and $Y \in SC(X)$ then $F \in SC(X)$*
3. *Let (X, τ) be a topological space, if Y is an open subset of a space X and $F \in SC(X)$, then $F \cap Y \in SC(X)$*

Definition 2.7 *[12] A space X is said to be semi-regular if for any open set U of X and each point $x \in U$, there exists a regular open set V of X such that $x \in V \subseteq U$.*

3 Basic Properties

In this section, we introduce and define a new type of sets in bitopological spaces and find some of its properties

Definition 3.1 *A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-ξ-open, if A is a j-open set and for all x in A, there exist an i-semi-closed set F such that $x \in F \subseteq A$. A subset B of X is called (i, j)-ξ- closed if B^c is (i, j)-ξ- open.*
The family of (i, j)-ξ-open (resp., (i, j)-ξ- closed) subset of x is denoted by (i, j)-$\xi O(X)$ (resp.,(i, j)-$\xi C(X)$).

From the above definition we obtain:

Corollary 3.2 *A subset A of a bitopological space X is (i, j)-ξ-open, if A is j-open set and it is a union of i-semi-closed sets. This means that $A = \cup F_\alpha$, where A is a j-open and F_α is an i-semi-closed set for each α.*

It is clear from the definition that every (i, j)-ξ-open set is j-open, but the converse is not true in general as shown in the following example.

Example 3.3 *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{c\}, \{a, b\}, X\}$, *then* (i, j)-$\xi O(X)$= $\{\phi, \{c\}, X\}$.*It is clear that* $\{a, b\}$ *is* $j-$*open but not* (i, j)-ξ-*open.*

Proposition 3.4 *Let* (X, τ_1, τ_2) *be a bitopological space if* (X, τ_1) *is a semi-*T_1-*space , then* (i, j)-$\xi O(X)$= $\tau_j(X)$.

Proof. Let A be any subset of a space X and A is j-open set, if $A = \phi$, then $A \in (i, j)$-$\xi O(X)$, if $A \neq \phi$, now let $x \in A$, since (X, τ_1) is semi-T_1-space, then by Theorem 2.4 every singleton is i-semi-closed set , and hence $x \in \{x\} \subseteq A$, therefore $A \in (i, j)$-$\xi O(X)$, hence $\tau_j(X) \subseteq (i, j)$-$\xi O(X)$ but (i, j)-$\xi O(X) \subseteq \tau_j(X)$ generally, thus (i, j)-$\xi O(X) = \tau_j(X)$.

Proposition 3.5 *Let* (X, τ_1, τ_2) *be a bitopological space and* A *be a subset the space* X. *If* $A \in j$-$\delta O(X)$ *and* A *is an* i-*closed set, then* $A \in (i, j)$-$\xi O(X)$

Proof. If $A = \phi$, then $A \in (i, j)$- $\xi O(X)$, if $A \neq \phi$, let $x \in A$ since $A \in j$-$\delta O(X)$ and j-$\delta O(X) \subseteq \tau_j(X)$ in general so $A \in \tau_j(X)$, and since A is i-closed so A is i-semi-closed and $x \in A \subseteq A$, and hence $A \subseteq (i, j)$-$\xi O(X)$.

From Proposition 3.5 we obtain the following:

Corollary 3.6 *Let* (X, τ_1, τ_2) *be a bitopological space, if a subset* A *of* X *is* i-*regular closed and* j-*open then* $A \in (i, j)$-$\xi O(X)$

Theorem 3.7 *In a bitopological space* (X, τ_1, τ_2) *if a space* (X, τ_i) *is locally indiscrete then* (i, j)-$\xi O(X) \subseteq \tau_i$.

Proof. Let $V \in (i, j)$-$\xi O(X)$, then $V \in \tau_j(X)$ and for each $x \in V$, there exist i-semi-closed F in X such that $x \in F \subseteq V$, by Remark 2.3, F is i-open, it follows that $V \in \tau_i$, and hence (i, j)-$\xi O(X) \subseteq \tau_i$.

The converse of Theorem 3.7, is not true in general, as shown in the following example:

Example 3.8 *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ *and* τ_2=$\{\phi, \{b, c\}$,$X\}$, *then* (i, j)-$\xi O(X) = \{\phi, \{b, c\}, X\}$ *and it is clear that* (X, τ_1) *is locally indiscrete but* τ_1 *is not a subset of* (i, j)-$\xi O(X)$

Theorem 3.9 *Let* X_1, X_2 *be two bitopological space and* $X_1 \times X_2$ *be the bitopological product , let* $A_1 \in (i, j)$-$\xi O(X_1)$ *and* $A_2 \in (i, j)$-$\xi O(X_2)$ *then* $A_1 \times A_2 \in (i, j)$-$\xi O(X_1 \times X_2)$

Proof. Let $(x_1, x_2) \in A_1 \times A_2$ then $x_1 \in A_1$ and $x_2 \in A_2$, and since $A_1 \in (i,j)$-$\xi O(X_1)$ and $A_2 \in (i,j)$-$\xi O(X_2)$, then $A_1 \in j$-$\xi O(X_1)$ and $A_2 \in j$-$\xi O(X_2)$, there exist $F_1 \in i$-$SC(X_1)$ and $F_2 \in i$-$SC(X_2)$ such that $x_1 \in F_1 \subseteq A_1$ and $x_2 \in F_2 \subseteq A_2$. Therefore $(x_1, x_2) \in F_1 \times F_2 \subseteq A_1 \times A_2$, and since $A_1 \in j$-$\xi O(X_1)$ and $A_2 \in j$-$\xi O(X_2)$, then by Theorem 2.5 part(1) $A_1 \times A_2 = j$-$\xi Int_{x_1}(A_1) \times j$-$\xi Int_{x_2}(A_2) = j$-$\xi Int_{x_1 \times x_2}(A_1 \times A_2)$, hence $A_1 \times A_2 \in j$-$\xi O(X_1 \times X_2)$ and since $F_1 \in i$-$SC(X_1)$ and $F_2 \in i$-$SC(X_2)$ then by Theorem 2.5 part (2) we get $F_1 \times F_2 = i$-$sCl_{x_1}(F_1) \times i$-$sCl_{x_2}(F_2) = i$-$sCl_{x_1 \times x_2}(F_1 \times F_2)$, hence $F_1 \times F_2 \in i$-$SC(X_1 \times X_2)$, therefore $A_1 \times A_2 \in (i,j)$-$\xi O(X)$.

Theorem 3.10 *For any bitopological space (X, τ_1, τ_2), if $A \in \tau_j(X)$ and either $A \in i$-$\eta O(X)$ or $A \in i$-$S\theta O(X)$, then $A \in (i,j)$-$\xi O(X)$*

Proof. Let $A \in i$-$\eta O(X)$ and $A \in \tau_j(X)$, if $A = \phi$, then $A \in (i,j)$-$\xi O(X)$, if $A \neq \phi$, since $A \in i$-$\eta O(X)$, then $A = \cup F_\alpha$, where $F_\alpha \in i$-$\delta C(X)$ for each α, and since i-$\delta C(X) \subseteq i$-$SC(X)$, so $F_\alpha \in i$-$SC(X)$ for each α, and $A \in \tau_j(X)$ so by Corollary 3.2 $A \in (i,j)$-$\xi O(X)$.
On the other hand, suppose that $A \in i$-$S\theta O(X)$ and $A \in \tau_j(X)$, if $A = \phi$, then $A \in (i,j)$-$\xi O(X)$, if $A \neq \phi$, since $A \in i$-$S\theta O(X)$, then for each $x \in A$, there exist i-semi-open set U such that $x \in U \subseteq i$-$sCl(U) \subseteq A$, this implies that $x \in i$-$sCl(U) \subseteq A$ and $A \in \tau_j(X)$, therefore by Corollary 3.2 $A \in (i,j)$-$\xi O(X)$.

Theorem 3.11 *Let Y be a subspace of a bitopological space (X, τ_1, τ_2), if $A \in (i,j)$-$\xi O(X)$ and $A \subseteq Y$, then $A \in (i,j)$-$\xi O(Y)$*

Proof. Let $A \in (i,j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and for each $x \in A$, there exists i-semi-closed set F in X such that $x \in F \subseteq A$, since $A \in \tau_j(X)$ and $A \subseteq Y$, then by Theorem 2.6 $A \in \tau_j(Y)$, and since $F \in i$-$SC(X)$ and $F \subseteq Y$, then by Theorem 2.6 $F \in i$-$SC(Y)$, hence $A \in (i,j)$-$\xi O(Y)$.

From the above theorem we obtain:

Corollary 3.12 *Let X be a bitopological space , A and Y be two subsets of X such that $A \subseteq Y \subseteq X$, $Y \in RO(X, \tau_j)$, $Y \in RO(X, \tau_i)$, then $A \in (i,j)$-$\xi O(Y)$ if and only if $A \in (i,j)$-$\xi O(X)$*

Proposition 3.13 *Let Y be a subspace of a bitopological space (X, τ_1, τ_2), if $A \in (i,j)$-$\xi O(Y)$ and $Y \in i$-$SC(X)$, then for each $x \in A$, there exists an i-semi-closed set F in X such that $x \in F \subseteq A$.*

Proof. Let $A \in (i,j)$-$\xi O(Y)$, then $A \in \tau_j(Y)$ and for each $x \in A$ there exist an i-semi-closed set F in Y such that $x \in F \subseteq A$, and since $Y \in i$-$SC(X)$ so by Theorem 2.6 $F \in i$-$SC(X)$, which completes the proof.

Proposition 3.14 *Let A and Y be any subsets of a bitopological space X, if $A \in (i,j)$-$\xi O(X)$ and $Y \in RO(X, \tau_j)$ and $Y \in RO(X, \tau_i)$ then $A \cap Y \in (i,j)$-$\xi O(X)$*

Proof. Let $A \in (i,j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$,where $F_\alpha \in i$-$SC(X)$ for each α, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since $Y \in RO(X,\tau_j)$, then Y is j-open, by Theorem 2.6 $A \cap Y \in \tau_j(X)$ and since $Y \in RO(X,\tau_i)$ then $Y \in i$-$SC(X)$ and hence $F_\alpha \cap Y \in i$-$SC(X)$, for each α, therefore by Corollary 3.2 , $A \cap Y \in (i,j)$-$\xi O(X)$.

Proposition 3.15 *Let A and Y be any subsets of a bitopological space X, if $A \in (i,j)$-$\xi O(X)$ and Y is regular semi-open in τ_i and τ_j, then $A \cap Y \in (i,j)$-$\xi O(Y)$*

Proof. Let $A \in (i,j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$ where $F_\alpha \in i$-$SC(X)$ for each α, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since $Y \in RSO(X,\tau_j)$, then $Y \in j$-$SO(X)$ and by Theorem 2.6, $A \cap Y \in \tau_j(Y)$ and since $Y \in RSO(X,\tau_i)$ then $Y \in i$-$SC(X)$ and hence $F_\alpha \cap Y \in i$-$SC(X)$ for each α, since $F_\alpha \cap Y \subseteq Y$ and $F_\alpha \cap Y \in i$-$SC(X)$ for each α, then by Theorem 2.6, $F_\alpha \cap Y \in i$-$SC(Y)$ therefore by Corollary 3.2 $A \cap Y \in (i,j)$-$\xi O(Y)$.

Proposition 3.16 *If Y is an i-open and j-open subspace of a bitopological space X and $A \in (i,j)$-$\xi O(X)$, then $A \cap Y \in (i,j)$-$\xi O(Y)$*

Proof. Let $A \in (i,j)$-$\xi O(X)$, then $A \in \tau_j(X)$ and $A = \cup F_\alpha$ where $F_\alpha \in i$-$SC(X)$ for each α, then $A \cap Y = \cup F_\alpha \cap Y = \cup(F_\alpha \cap Y)$, since Y is j-open subspace of X then $Y \in j$-$SO(X)$ and hence by Theorem 2.6 $A \cap Y \in \tau_j(Y)$, and since Y is an i-open subspace of X then by Theorem 2.6 $F_\alpha \cap Y \in i$-$SC(Y)$ for each α then by Corollary 3.2 $A \cap Y \in (i,j)$-$\xi O(Y)$.

Frome the above proposition we obtain the following corollary:

Corollary 3.17 *If either $Y \in RSO(X,\tau_j)$ and $Y \in RSO(X,\tau_i)$ or Y is an i-open and j-open subspace of a bitopological space X, and $A \in (i,j)$-$\xi O(X)$, then $A \cap Y \in (i,j)$-$\xi O(Y)$*

The following result shows that any union of (i,j)-$\xi O(X)$ sets in bitopological space (X,τ_1,τ_2) is (i,j)-$\xi O(X)$.

Proposition 3.18 *Let $\{A_\lambda : \lambda \in \Delta\}$ be family of (i,j)-ξ-open sets in bitopological space (X,τ_1,τ_2), then $\cup \{A_\lambda : \lambda \in \Delta\}$ is an (i,j)-ξ-open set.*

Proof. Let $\{A_\lambda : \lambda \in \Delta\}$ be family of (i,j)-ξ-open sets in bitopological space (X,τ_1,τ_2). Since A_λ is j-open for each $\lambda \in \Delta$ then $\cup \{A_\lambda : \lambda \in \Delta\}$ is j-open set in a space X.
Suppose that $x \in \cup A_\lambda$, this implies that there exist $\lambda o \in \Delta$ such that $x \in A_{\lambda o}$ and since $A_{\lambda o}$ is an (i,j)-ξ-open set, so there exists i-semi-closed set F in X such that $x \in F \subseteq A_{\lambda o} \subseteq \cup A_\lambda$ for all $\lambda \in \Delta$. Therefore, $\cup \{A_\lambda : \lambda \in \Delta\}$ is an (i,j)-ξ-open set.

The following result shows that finite intersection of (i,j)-$\xi O(X)$ sets in bitopological space (X,τ_1,τ_2) is (i,j)-$\xi O(X)$.

Proposition 3.19 *Any finite intersection of (i,j)-ξ-open sets in bitopological space (X, τ_1, τ_2), is an (i,j)-ξ-open set.*

Proof. Let A_i be (i,j)-ξ-open for i =1,2,. . .,n, in bitopological space (X, τ_1, τ_2). Then $\cap A_i$ is j-open in a space X. Let $x \in \cap A_i$, then $x \in A_i$ for i =1,2,. . .,n, but A_i is (i,j)-ξ-open, so there exists semi-closed F_i for each i =1,2,. . .,n, such that $x \in F_i \subseteq A_i$. This implies that $x \in \cap F_i \subseteq \cap A_i$. Therefore, $\cap A_i$ is an (i,j)-ξ-open set. Hence, the family (i,j)-ξ-open subset of (X, τ_1, τ_2) forms a bitopology on X.

4 On (i,j)-ξ- operators

Definition 4.1 *A subset N of a bitopological space (X, τ_1, τ_2) is called (i,j)-ξ-neighbourhood of a subset A of X if there exists an (i,j)-ξ-open set U such that $A \subseteq U \subseteq N$. When $A = \{x\}$, we say that N is (i,j)-ξ- neighbourhood of x.*

Definition 4.2 *A point $x \in X$ is said to be an (i,j)-ξ-interior point of A if there exists an (i,j)-ξ- open set U containing x such that $U \subseteq A$. The set of all (i,j)-ξ- interior points of A is said to be (i,j)-ξ-interior of A and it is denoted by (i,j)-$\xi Int(A)$*

Proposition 4.3 *Let X be a bitopological space and $A \subseteq X$, $x \in X$, then x is (i,j)-ξ-interior of A if and only if A is an (i,j)-ξ-neighbourhood of x.*

Proposition 4.4 *A subset G of a bitopological space X is (i,j)-ξ-open if and only if it is an (i,j)-ξ-neighbourhood of each of its points .*

Proposition 4.5 *Let A be any subset of a bitopological space X. If a point x in the (i,j)-ξ-$Int(A)$, then there exists a i-semi-closed set F of X containing x and $F \subseteq A$.*

Proof. Suppose that $x \in (i,j)$-ξ-$Int(A)$, then there exists an (i,j)-ξ-open set U of X containing x such that $x \in U \subseteq A$. Since U is an (i,j)-ξ-open set, so there exists an i-semi-closed set F such that $x \in F \subseteq U \subseteq A$. Hence, $x \in F \subseteq A$.

Some properties of (i,j)-ξ-interior operators on a set are given in the following:

Theorem 4.6 *For any subsets A and B of a bitopological space X, the following statements are true:*

1. *The (i,j)-ξ-interior of A is the union of all (i,j)-ξ-open sets contained in A.*

2. (i, j)-ξ-$Int(A)$ *is an* (i, j)-ξ-*open set in* X *contained in* A.

3. (i, j)-ξ-$Int(A)$ *is the largest* (i, j)-ξ-*open set in* X *contained in* A.

4. A *is an* (i, j)-ξ-*open set if and only if* $A = (i, j)$-ξ-$Int(A)$

5. (i, j)-ξ-$Int(\phi) = \phi$.

6. (i, j)-ξ-$Int(X) = X$

7. (i, j)-ξ-$Int(A) \subseteq A$.

8. *If* $A \subseteq B$, *the* (i, j)-ξ-$Int(A) \subseteq (i, j)$-ξ-$Int(B)$.

9. (i, j)-ξ-$Int(A) \cap (i, j)$-ξ-$Int(B) = (i, j)$-ξ-$Int(A \cap B)$.

10. (i, j)-ξ-$Int(A) \cup (i, j)$-ξ-$Int(B) \subseteq (i, j)$-ξ-$Int(A \cup B)$.

Proof. Straightforward.

In general (i, j)-$\xi Int(A)$ $\cup (i, j)$-$\xi Int(B) \neq (i, j)$-$\xi Int(A \cup B)$ as it shown in the following example:

Example 4.7 *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}$ *and* $\tau_2 = \{\phi, \{b, c\}, X\}$, *then* (i, j)-$\xi O(X) = \{\phi, \{b, c\}, X\}$ *if we take* $A = \{a, b\}$ *and* $B = \{b, c\}$, *then* (i, j)-$\xi Int(A) = \phi$, *and* (i, j)-$\xi Int(B) = \{b, c\}$, *and* (i, j)-$\xi Int(A) \cup (i, j)$-$\xi Int(B) = \{b, c\}$, (i, j)-$\xi Int(A \cup B) = (i, j)$-$\xi Int(X) = X$.

In general (i, j)-$\xi Int(A) \subseteq j$-$Int(A)$, but (i, j)-$\xi Int(A) \neq j$-$Int(A)$, which is shown in the following example:

Example 4.8 *Let* $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}$ *and* $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$, *then* (i, j)-$\xi O(X) = \{\phi, \{b, c\}, X\}$, *if we take* $A = \{a\}$,*then* (i, j)-$\xi Int(A) = \phi$, *but* j-$Int(A) = A$. *Hence* (i, j)-$\xi Int(A) \neq j$-$Int(A)$.

Definition 4.9 *The intersection of all* (i, j)-ξ-*closed set containing* F *is called the* (i, j)-ξ-*closure of* F *and we denoted it by* (i, j)-$\xi Cl(F)$

Corollary 4.10 *Let* F *be any subset of a space* X. *A point* $x \in X$ *is in the* (i, j)-ξ-*closed of* F *if and only if* $F \cap U \neq \phi$ *for every* (i, j)-ξ-*open set* U *contaning* x.

Proposition 4.11 *Let* A *be any subset of a bitopological space* X. *If a point* x *in the* (i, j)-ξ-*closure of* A, *then* $F \cap A \neq \phi$ *for every* i-*semi-closed set* F *of* X *containing* x.

Proof. Suppose that $x \in (i,j)$-$\xi cl(A)$, then by Corollary 4.10, $A \cap U \neq \phi$ for every (i,j)-ξ-open set U of X containing x. Since U is an (i,j)-ξ-open set, so there exists an i-semi-closed set F containing x, such that $F \subseteq U$. Hence, $F \cap A \neq \phi$.

Some properties of (i,j)-ξ-closure operators on a set are given.

Theorem 4.12 *For any subsets A and B of a bitopological space X, the following statements are true:*

1. *The (i,j)-ξ-closure of A is the intersection of all (i,j)-ξ-closed sets containing A.*
2. *(i,j)-ξ-$cl(A)$ is an (i,j)-ξ-closed set in X containing A.*
3. *(i,j)-ξ-$cl(A)$ is the smallest (i,j)-ξ-closed set in X containing A.*
4. *A is an (i,j)-ξ-closed set if and only if $A = (i,j)$-ξ-$cl(A)$*
5. *(i,j)-ξ-$cl(\phi) = \phi$.*
6. *(i,j)-ξ-$cl(X) = X$*
7. *$A \subseteq (i,j)$-ξ-$cl(A)$.*
8. *If $A \subseteq B$,then (i,j)-ξ-$cl(A) \subseteq (i,j)$-ξ-$cl(B)$.*
9. *(i,j)-ξ-$cl(A) \cap (i,j)$-ξ-$cl(B) \subseteq (i,j)$-ξ-$cl(A \cap B)$.*
10. *(i,j)-ξ-$cl(A) \cup (i,j)$-ξ-$cl(B) = (i,j)$-ξ-$Int(A \cup B)$.*

Proof. Directly from Definition 4.9.

Corollary 4.13 *For any subset A of a bitopological space X, then the following statements are true:*

1. *$X \setminus ((i,j)$-$\xi Cl(A)) = (i,j)$-$\xi Int(X \setminus A)$*
2. *$X \setminus ((i,j)$-$\xi Int(A)) = (i,j)$-$\xi Cl(X \setminus A)$*
3. *(i,j)-$\xi Int(A) = X \setminus ((i,j)$-$\xi Cl(X \setminus A))$*

It is clear that j-$Cl(F) \subseteq (i,j)$-$\xi Cl(F)$, the converse may be false as shown in the following example:

Example 4.14 *Considering a space X as defned in Example 3.3, if we take $F = \{a,b\}$, then j-$Cl(F) = \{a,b\}$, and (i,j)-$\xi Cl(F) = X$, this shows that (i,j)-$\xi Cl(F)$ is not a subset of j-$Cl(F)$.*

Corollary 4.15 *If A is any subset of a bitopological space X, then (i,j)-$\xi Int(A) \subseteq j$-$Int(A) \subseteq A \subseteq j$-$Cl(A) \subseteq (i,j)$-$\xi Cl(A)$.*

Definition 4.16 *Let A be a subset of a bitopological space X, A point $x \in X$ is said to be (i,j)-ξ-limit point of A if for each (i,j)-ξ-open set U containing x, $U \cap (A \setminus \{x\}) \neq \phi$, The set of all (i,j)-ξ-limit point of A is called (i,j)-ξ-derived set of A and is denoted by (i,j)-$\xi D(A)$*

In general It is clear that(i,j)-$\xi D(A) \subseteq j$-$D(A)$, but the converse may not be true as shown in the following example:

Example 4.17 *Considering the space X as defined in Example 3.3 if we take $A = \{a,c\}$, So (i,j)-$\xi D(A)$ =$\{a,b\}$ and j-$D(A) = \{b\}$, hence (i,j)-$\xi D(A)$ is not a subset of j-$D(A)$*

Theorem 4.18 *Let X be a bitopological space and A be a subset of X, then $A \cup (i,j)$-$\xi D(A)$ is (i,j)-ξ- closed.*

Proof. Let $x \notin A \cup (i,j)$-$\xi D(A)$. This implies that $x \notin A$ and $x \notin (i,j)$-$\xi D(A)$. Since $x \notin (i,j)$-$\xi D(A)$, then there exists an (i,j)-ξ-open U of X which contains no point of A other than x, but $x \notin A$, so U contains no point of A, which implies that $U \subseteq X \setminus A$.Again, U is an (i,j)-ξ-open setfor each of its points.But as U does not contain any point of A, no point of U can be (i,j)-ξ-limit point of A. Therefore, no point of U can belong to(i,j)-$\xi D(A)$. This implies that $U \subseteq X \setminus (i,j)$-$\xi DA$. Hence, it follows that $x \in X \setminus A \cap (X \setminus (i,j)$-$\xi D(A))$ $= X \setminus (A \cup (i,j)$-$\xi D(A))$, Therefore $A \cup (i,j)$-$\xi D(A)$ is an (i,j)-ξ-closed. Hence (i,j)-$\xi cl(A) \subseteq A \cup (i,j)$-$\xi D(A)$. .

Corollary 4.19 *If a subset A of a bitopological space X is (i,j)-ξ-closed, then A contains the set of all of its (i,j)-ξ-limit points.*

Theorem 4.20 *Let A be any subset of a bitopological space X, then the following statements are true:*

1. $((i,j)$-$\xi D((i,j)$-$\xi D(A))) \setminus A \subseteq (i,j)$-$\xi D(A)$
2. (i,j)-$\xi D(A \cup (i,j)$-$\xi D(A)) \subseteq A \cup (i,j)$-$\xi D(A)$

Proof. Obvious .

Theorem 4.21 *Let X be a bitopological space and A be a subset of X, then:(i,j)-$\xi Int(A) = A \setminus ((i,j)$-$\xi D(X \setminus A))$*

Proof. Obvious .

Definition 4.22 *If A is a subset of a bitopological space X, then (i,j)-ξ-boundary of A is (i,j)-$\xi Cl(A) \cap ((i,j)$-$\xi Int(A))^c$, and denoted by (i,j)-$\xi Bd(A)$*

Theorem 4.23 *For any subset A of a bitopological space X, the following statements are true:*

1. (i,j)-$\xi Bd(A) = (i,j)$-$\xi Bd(X \setminus A)$
2. $A \in (i,j)$-$\xi O(X)$ *if and only if* (i,j)-$\xi Bd(A) \subseteq X \setminus A$, *that is* $A \cap (i,j)$-$\xi Bd(A) = \phi$.
3. $A \in (i,j)$-$\xi C(X)$ *if and only if* (i,j)-$\xi Bd(A) \subseteq A$.
4. (i,j)-$\xi Bd((i,j)$-$\xi Bd(A)) \subseteq (i,j)$-$\xi Bd(A)$
5. (i,j)-$\xi Bd((i,j)$-$\xi Int(A)) \subseteq (i,j)$-$\xi Bd(A)$
6. (i,j)-$\xi Bd((i,j)$-$\xi Cl(A)) \subseteq (i,j)$-$\xi Bd(A)$
7. (i,j)-$\xi Int(A)$=$A \setminus ((i,j)$-$\xi Bd(A))$

Proof. Directly from Definition 4.22.

Theorem 4.24 *Let A be a subset of a bitopological space X, then (i,j)-$\xi Bd(A) = \phi$ if and only if A is both (i,j)-ξ- open and (i,j)-ξ-closed set.*

Proof. Lte A be (i,j)-ξ- open and (i,j)-ξ-closed, then $A = (i,j)$-$\xi Int(A) = (i,j)$-$\xi cl(A)$, hence by Definition4.22 $A = (i,j)$-$\xi Cl(A)$-$((i,j)$-$\xi Int(A))^c = \phi$.

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