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On 2-Skew-Commuting Mappings of Prime Rings

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Abstract

Brešar proved in [13] that if R is a prime ring of characteristic not 2, C is an extended centroid of R and an additive mapping $f : R \to R$ satisfies the functional identity $f(x)x^2 - x^2f(x) = 0$ $\forall x \in R$, then there exits $\lambda \in C$ and an additive mapping $\xi : R \to C$ such that $f(x) = \lambda x + \xi(x)$ $\forall x \in R$. The purpose of this paper is to prove that if R is a prime ring with characteristic of not 2 and 3 and if an additive mapping $f : R \to R$ satisfies functional identity $f(x)x^2 + x^2f(x) = 0$ $\forall x \in I$, then $f(x) = 0$ $\forall x \in I$. We will conclude this paper by introducing a conjecture about the solution of an important functional identity.

Keywords: *Additive mapping, Commutng mapping, Derivation, n-skew-commuting mapping, Prime ring.*

1 Introduction

Throughout, R will denote an associative ring with center $Z(R)$ and extended centroid C (for definition of external centroid, see [3, Chapter 2]). For any $x, y \in R$, the Lie bracket is defined as $[x, y] = xy - yx$. A ring R is called prime if $aRb = 0$ for all $a, b \in R$ implies either $a = 0$ or $b = 0$ and semiprime if $aRa = 0$ implies $a = 0$. For equivalent definitions of prime and semiprime rings, we refer [18]. An additive mapping $d : R \to R$ is called derivation on R if $d(xy) = d(x)y + xd(y)$ $\forall x, y \in R$. A mapping $f : R \to R$ is said to be commuting (centralizing) on R if $[f(x), x] = 0$ $\forall x \in R$ ($[f(x), x] \in Z(R)$

$\forall x \in R$). Clearly every commuting mapping is centralizing but the converse is not always true. However under certain mild conditions both mappings are same [11]. These mappings were firstly studied by E. Posner in [24]. He proved that a centralizing derivation on a noncommutative prime ring becomes zero. Many generalizations of this result have been established in the course of time. For more details, we refer [2,4,5,6,8,11,15,21,22,23,27]. Brešar proved in [10] that commuting additive mapping f on a prime ring R is of form $\lambda x+\xi(x)$, where $\lambda \in C$ and $\xi : R \to C$ is an additive mapping. This result has been generalized in semiprime ring by Ara and Mathieu in [1] and then in a different way by Brešar in [12]. Based on these results, he initialized the theory of functional identities. Roughly speeking, a functional identity on a ring is functional relation involving some arbitrary maps of ring and this relation is satisfied by all elements from ring. The basic purpose of the theory is either to find the forms of mappings involved or structure of ring satisfying functional identity. If solution maps of functional identities are trivial, these maps are called standard solutions and if solution maps have some special forms, these maps are called nonstandard solutions of the identity. Commuting and centralizing maps are most basic and important examples of functional identifies. For further details, we refer [14]. Motivated by the concept of commuting maps, Brešar in [9] introduced skew-commuting (skew-centralizing) mapping on R as a mapping $f : R \to R$ such that $f(x)x + xf(x) = 0 \ \forall x \in R$ ($f(x)x + xf(x) \in Z(R) \ \forall x \in R$) and proved that prime ring of charateristic of not 2 which satisfies skew-commuting additive mapping gives trivial solution mapping i.e. $f = 0$. He also extended this result to semiprime rings. Bell and Lucier in [7] replaced primeness by the left identity of ring and proved the same result. Deng and Bell in [16] extended the notion of commuting mappings to n-commuting (n-centralizing) on R as an mapping $f : R \to R$ such that $[x^n, f(x)] = 0 \ \forall x \in R$ ($[f(x), x^n] \in Z(R) \ \forall x \in R$). They showed that if d is a derivation of R and U is a nonzero left ideal of R and $d(U) \neq 0$, then condition $[d(x), x^n] \in Z(R) \ \forall x \in R$ forces $n!$-torsion free semiprime R to have a nonzero central ideal. Lee, Jung and Chang in [20] proved that if D and G are derivations on $n!$-torsion free semiprime R such that $[D^2(x)+G(x), x^n] = 0$ $\forall x \in R$, then $[D(x), x] = [G(x), x] = 0 \ \forall x \in R$. N. Rehman and D. Filippis in [25] extended the result to generalized derivations. A map $f : R \to R$ such that $f(x)x^n + x^n f(x) = 0 \ \forall x \in R$ ($[f(x), x^n] \in Z(R) \ \forall x \in R$) is called n-skew-commuting (n-skew-centralizing). It can be observed that every n-skew-commuting mapping becomes $2n$-commuting mapping. Bell and Lucier in [7] proved that if R is an n-torsion free ring with left identity e and f is additive n-skew-commuting on an additive subgroup H containing e, then $f(x) = 0 \ \forall x \in H$. In [26] R. K. Sharma and B. Dhara improved this result by introducing multi-additive mappings.

Brešar in [13] proved that 2-commuting additive map on a prime ring R has form $f(x)= \lambda x+\xi(x)$, where $\lambda \in C$ and $\xi : R \to C$ is an additive map. In this paper, we study the form of 2-skew-commuting additive map on prime ring R. Specifically, we prove that if R is a nonzero prime ring with characteristic of not 2 and 3 and an additive mapping $f : R \to R$ satisfies $f(x)x^2 + x^2 f(x) = 0$ $\forall x \in I$, then this identity has trivial solution map i.e. $f(x) = (0)$ $\forall x \in I$. As with many other functional identities, this identity might be useful in operator theory, functional analysis and Lie type theory etc.

2 Preliminaries

Following are important results which we modify according to our need.

Lemma 2.1 (9) *Let R be a nonzero prime ring and I be any ideal of R. Set $I_n = \{x^n | x \in I\}$, where n is any positive number. If $I_n a = 0$ (or $aI_n = 0$) for $a \in R$, then $a = 0$.*

Lemma 2.2 (17) *Let R be a prime ring with char $R \neq 2, 3$ and I be any ideal of R . Suppose that an additive mapping $f : R \to R$ satisfies $[f(x), x^4] = 0$ $\forall x \in I$, then f is commuting on I.*

Lemma 2.3 (19) *Let R be a prime ring and I be any ideal of R such that that $x^n = 0$ $\forall x \in I$ for a fixed positive integer n, then $I = (0)$.*

Now we are ready to prove our main theorem. In the proof we will use some ideas similar to those used in [9].

3 Main Result

Theorem 3.1 *Let R be a nonzero prime ring with char. $R \neq 2, 3$ and I be any ideal of R. Suppose that an additive mapping $f : R \to R$ satisfies $x^2 f(x) + f(x)x^2 = 0$ $\forall x \in I$, then $f(I) = (0)$.*

Proof. Let we have

$$f(x)x^2 + x^2 f(x) = 0 \, \forall x \in I. \tag{1}$$

Multiplying the above relation from left by x^2, we get

$$x^2 f(x)x^2 + x^4 f(x) = 0. \tag{2}$$

Multiplying (1) from right by x^2, we have

$$f(x)x^4 + x^2 f(x)x^2 = 0. \tag{3}$$

From (2) and (3), we conclude

$$[f(x), x^4] = 0. \tag{4}$$

By using Lemma Lemma 2.2, we get

$$f(x)x - xf(x) = 0. \tag{5}$$

The above relation can be written as

$$f(x)x = xf(x)\,\forall x \in I. \tag{6}$$

Using the last relation, (1) reduces to $2f(x)x^2 = 0$, this becomes

$$f(x)x^2 = 0\,\forall x \in I. \tag{7}$$

Using (6), the last relation can be written as

$$x^2 f(x) = 0\,\forall x \in I. \tag{8}$$

Replace x by $x + y$ in (7) to get

$$f(x)y^2 + f(x)(xy + yx) + f(y)x^2 + f(y)(xy + yx) = 0\,\forall x, y \in I. \tag{9}$$

Now replace x by $x - y$ in equation (7) to have

$$f(x)y^2 - f(x)(xy + yx) - f(y)x^2 + f(y)(xy + yx) = 0. \tag{10}$$

Adding (9) and (10), we get

$$2f(x)y^2 + 2f(y)(xy + yx) = 0. \tag{11}$$

As R is of $\mathrm{char}(R) \neq 2$, so last relation becomes $f(x)y^2 + f(y)(xy + yx) = 0$. Multiplying this by y^2 from left and using (8), we get

$$y^2 f(x)y^2 = 0\,\forall x, y \in I. \tag{12}$$

This can be written as

$$uf(x)u = 0\,\forall u \in I_2. \tag{13}$$

replace x by $x - u$ in (5) to get

$$f(x)u - uf(x) + f(u)x - xf(u) = 0\,\forall x \in I,\, \forall u \in I_2. \tag{14}$$

Multiplying the last equation by u from right and using (13), we get

$$f(x)u^2 + f(u)xu - xf(u)u = 0\,\forall x \in I,\, \forall u \in I_2. \tag{15}$$

As $I_2 \subseteq I$, this implies $f(x)u^2 + f(u)xu - xf(u)u = 0 \, \forall x, u \in I_2$.

First left multiplying this by x^2, then using (8) and (13), we get $x^3 f(u)u = 0 \; \forall x \in I_2$. By Lemma 2.1, we get $f(u)u = 0$. This implies that

$$uf(u) = 0 \, \forall u \in I_2. \tag{16}$$

In view of last relation, (15) reduces to

$$f(x)u^2 + f(u)xu = 0 \, \forall x \in I, \forall u \in I_2. \tag{17}$$

Replacing x by xu in last relation to get

$$f(xu)u^2 + f(u)xu^2 = 0. \tag{18}$$

Multiplying (17) by u from right, we get

$$f(x)u^3 + f(u)xu^2 = 0. \tag{19}$$

Using (18) and (19), we get

$$f(xu)u^2 = f(x)u^3. \tag{20}$$

Multiplying (14) from left by u and using (13) and (16), we get

$$u^2 f(x) + uxf(u) = 0. \tag{21}$$

Replacing x by xu in the last relation, we get

$$u^2 f(xu) + uxuf(u) = 0. \tag{22}$$

Using (16), the last relation reduce to

$$u^2 f(xu) = 0 \quad \forall x \in I, \forall u \in I_2. \tag{23}$$

As a special case of (14), we have

$$f(x)zu - zuf(x) + f(zu)x - xf(zu) = 0 \quad \forall x, y, z \in I, \forall u \in I_2. \tag{24}$$

Multiplying the last relation by u^2 from left and right simultaneously

$$u^2 f(x)zu^3 - u^2 zuf(x)u^2 + u^2 f(zu)xu^2 - u^2 xf(zu)u^2 = 0 \quad \forall x, y, z \in I, \forall u \in I_2. \tag{25}$$

using (13), (20) and (23), the last relation becomes

$$u^2 f(x)zu^3 - u^2 xf(z)u^3 = 0. \tag{26}$$

Multiplying the last relation by u from left, we get $u^3f(x)zu^3 - u^3xf(z)u^3 = 0$. This can be written as

$$tf(x)zt - txf(z)t = 0 \quad \forall x, z \in I, \forall t \in I_6. \tag{27}$$

Replacing z by $ztf(v)$, where $\forall v \in I$, in last relation, we get

$$tf(x)ztf(v)t - txf(ztf(v))t = 0. \tag{28}$$

Using (13), we obtain

$$txf(ztf(v))t = 0. \tag{29}$$

As R is prime ring, so above relation, in view of Lemma 2.3, reduces to

$$f(ztf(v))t = 0. \tag{30}$$

Replacing y by $ztf(v)$ in (13), we get

$$f(x)ztf(v) - ztf(v)f(x) + f(ztf(v))x - xf(ztf(v)) = 0. \tag{31}$$

Multiplying last relation from right by t, using (13) and (30), we get

$$f(ztf(v))xt - ztf(v)f(x)t = 0. \tag{32}$$

Replacing z by rz in last relation, where $r \in I$, we get

$$f(rztf(v))xt - rztf(v)f(x)t = 0. \tag{33}$$

Left multiplying (32) by r, we get

$$rf(ztf(v))xt - rztf(v)f(x)t = 0. \tag{34}$$

Using (33) and (34), we get

$$f(rztf(v))xt = rf(ztf(v))xt. \tag{35}$$

Left multiplying the last relation by w, where $w \in I_2$, we have

$$wf(rztf(v))xt = wrf(ztf(u))xt \quad \forall x, r, v, z \in I, \forall w \in I_2, \forall t \in I_6. \tag{36}$$

Replacing x by wx in last relation, we get

$$wf(rztf(v))wxt = wrf(zutf(u))wxt. \tag{37}$$

By using (13), we get

$$wrf(ztf(v))wxt = 0. \tag{38}$$

As R is a prime ring, so by using Lemma 2.3, last relation reduces to

$$f(ztf(v))wxt = 0. \tag{39}$$

Again by primeness of R and Lemma 2.3, we get

$$f(ztf(v))w = 0. \tag{40}$$

In view of Lemma 2.1, we get

$$f(ztf(v)) = 0. \quad \forall v, z \in I, \forall t \in I_6 \tag{41}$$

Suppose $f(v) \neq 0$ for some v, otherwise the theorem is proved. Hence by Lemma 2.1, $tf(v) \neq 0$, so let $a = xtf(v) \neq 0$, where $x \in I$, then $L = Ra$ is a non-zero left ideal. From (41), we conclude $f(L) = 0$. By (13), we get $f(x)l - lf(x) = 0$, where $l \in L$. Replace l by rl, where $r \in R$, to get $f(x)rl - rlf(x) = 0$. Substitute r by x^2r and use (7) to obtain $x^2rlf(x) = 0$. This reduces to $lf(x) = 0$ due to primeness of R, this implies $f(x)l = 0$. Now replace l by rl to obtain $f(x)rl = 0$. As R is prime ring, so we get $f(x) = 0$. This completes the proof.

Remark 3.2 *Note that the condition of primeness of ring cannot be relaxed in the theorem. For let R be not a prime ring, then there exit nonzero ideals I and J such that $IJ = 0 = JI$ and let $f : R \to R$ be such that $f(I) \subseteq J$, then $f(x)x^2 + x^2f(x) = 0$ $\forall x \in I$, but it is possible that $f(I) \neq 0$.*

4 Conclusion

Motivated by a conjecture stated in paper [17], we conclude our paper by introducing following conjecture.

Conjecture: Let $n \geq 2$ be any fixed natural number and R be a prime ring with some suitable torsion restrictions. Suppose that an additive mapping $f : R \to R$ satisfies functional identity $f(x)x^n + x^nf(x) = 0$ $\forall x \in R$, then $f = 0$.

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